Recent topics on C^*-algebras (consistency and independency) and Kadison-Singer problem

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1 Introduction

The author (1968 [16], 1971 [17]) proved that any derivation on a simple C^*-algebra is induced by an element of its multiplier C^*-algebra and in particular, any derivation on a unital simple C^*-algebra is always inner. On the other hand, any $*$-automorphism on a separable simple C^*-algebra is induced by an unitary element of its multiplier C^*-algebra if and only if it is $*$-isomorphic to the C^*-algebra of all compact linear operators on a separable Hilbert space, and in particular, any $*$-automorphism on a separable, unital simple C^*-algebra is always inner if and only if it is $*$-isomorphic to a finite-dimensional full matrix algebra [18]. In the paper [18], the author asked whether one can extend this result to non-separable cases, and mentioned three outstanding problems of Naimark, Connes, Brown-Douglas-Fillmore as related problems. Recently all of these three problems have been solved in three different ways (Consistency [1], within ZFC [9] and Independency [14], [5]).

Since Naimark's problem was negatively solved by Akemann-Weaver [1], the author [19] has raised a new problem, which might substitute Naimark's problem.

In the present paper, we shall, at first, briefly explain the recent papers of Phillips-Weaver [14] and Farah [5] in which they prove that the statement of "The Calkin algebra has an outer $*$-automorphism" is undecidable within ZFC. Next we shall discuss the outstanding problem of Kadison-Singer [11]. We shall show that this problem is equivalent to a problem in the Calkin algebra and point out that this hard problem might be interested from the view point of the set theory in Operator algebras.

2 Automorphisms on the Calkin algebra

Brown-Douglas-Fillmore [3] asked whether there exists a $*$-automorphism of the Calkin algebra which sends the image of the unilateral shift to its adjoint.

The author [18] restated simply whether there exists an outer $*$-automorphism on the Calkin algebra, because even this simplified problem seemed to be difficult. Then, assuming the Continuum Hypothesis, Phillips-Weaver [14] has proved that the Calkin algebra has $2^{\aleph_1}$ outer $*$-automorphisms. Very recently, assuming the Open Coloring Axiom, Farah [5] proves that all $*$-automorphisms on the Calkin algebra are inner.

Therefore these two results imply that the statement of "The Calkin algebra has an outer $*$-automorphism" is undecidable within Zelmelo-Fraenkel set theory with the axiom of choice (ZFC). Consequently this gives a complete solution to Brown-Douglas-Fillmore problem.

Todorcevic's Open Coloring Axiom follows from the Proper Forcing Axiom in the set theory for negation of the Continuum Hypothesis.

In the first paper [5], Farah assumes also Martin's Axiom, but in the second paper [6] he assumes Open Coloring Axiom only.

3 On the problem of Kadison-Singer

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, $B(\mathcal{H})$ the W^*-algebra of all bounded linear operators on $\mathcal{H}$. Let (ξ_n) be a fixed orthonormal basis of $\mathcal{H}$ and let (p_n) be a family of mutually orthogonal one-dimensional projections such that $p_n\xi_n = \xi_n$ $(n = 1, 2, 3, \ldots)$. Let C be an atomic maximal commutative W^*-subalgebra of $B(\mathcal{H})$ generated by $\{p_n \mid n = 1, 2, 3, \ldots\}$. Let P be a projection of $B(\mathcal{H})$ onto C given by $P(a) = \displaystyle\sum_{n=1}^{\infty}(a\xi_n, \xi_n)p_n$ $(a \in B(\mathcal{H}))$.

Let $C = C(\beta\mathbb{N})$ be the Gelfand representation of C, where $\beta\mathbb{N}$ is the Stone-Čech compactification of all positive integers $\mathbb{N}$, and $C(\beta\mathbb{N})$ is the C^*-algebra of all complex valued continuous functions on $\beta\mathbb{N}$.

For $t \in \beta\mathbb{N}$, put $P(a)(t) = \varphi_t(a)$ $(a \in B(\mathcal{H}))$; then by Anderson's theorem [2], φ_t is a pure state on $B(\mathcal{H})$. If $t \in \mathbb{N}$, then the restriction $\varphi_t\,|\,C$ of φ_t to C

has a unique pure state extension φ_t. Kadison-Singer problem is as follows: For $t \in \beta\mathbb{N} \setminus \mathbb{N}$ (namely free ultrafilter), can we conclude that $\varphi_t \,|\, C$ has a unique pure state extension?

The proposers inclined to the view that the problem has a negative solution, when it was proposed in 1959. However most of the paper, which have been published, are intended to obtain the positive solution, though no one has succeeded. Even today, many researchers are very actively studying the problem, expecting probably the positive solution. The main reason is due to the fact that the problem is equivalent to many important open problems of several branches in mathematics, applied mathematics and engineering. Concerning these matters, one can consult with a nice survey by Casazza-Ficks-Tremain-Weber [4].

Also Reid [15] in 1971 proved the following interesting Theorem: Under the assumption of the Continuum Hypothesis, if a free ultrafilter $\mathcal{U}$ is rare, then for the corresponding $t \in \beta\mathbb{N} \setminus \mathbb{N}$, $\varphi_t \mid C$ has a unique pure state extension φ_t.

In this paper, we shall present another consideration on the problem, which might be interested from the view point of the set theory in operator algebras. In our discussion, the so-called extension property of Stonean spaces will play an important role. Since $\beta\mathbb{N}$ is a Stonean space, $C(\beta\mathbb{N})$ has the extension property (cf. Goodner [7], Nachbin [13] and Kelley [12] for real case and Hasumi [8] for complex case).

Namely, let E be a real (complex) Banach space and F be a closed linear subspace of E. Then any bounded linear mapping T of F into $C(\beta\mathbb{N})$ can be extended to a bounded linear mapping $\widetilde{T}$ of E to $C(\beta\mathbb{N})$ such that $\|\widetilde{T}\| = \|T\|$.

For a selfadjoint element $a \in B(\mathcal{H})$, let $L(a) = \{c \in C \mid a \leqq c\}$ and $U(a) = \{c \in C \mid c \leqq a\}$. Since the selfadjoint part $C_r(\beta\mathbb{N})$ of $C(\beta\mathbb{N})$ is a boundedly complete lattice, there exists $\inf\limits_{c \in L(a)} c$ and $\sup\limits_{d \in U(a)} d$.

Take arbitrary finite subset $\mathcal{F} = \{c_{\alpha_1}, c_{\alpha_2}, \ldots, c_{\alpha_n}\}$ of $L(a)$. Then $c_{\mathcal{F}}(t) = \left(\bigwedge\limits_{i=1}^{n} c_{\alpha_i}\right)(t) = \inf\limits_{1 \leqq i \leqq n} c_{\alpha_i}(t)$ $(t \in \beta\mathbb{N})$.

Since $C(\beta\mathbb{N})$ is a commutative W^*-algebra, a decreasing directed set $\{c_{\mathcal{F}}\}$ converges to an element in $C(\beta\mathbb{N})$ in the $\sigma(C(\beta\mathbb{N}), C(\beta\mathbb{N})_*)$-topology, where $C(\beta\mathbb{N})_*$ is the predual of $C(\beta\mathbb{N})$. This element is $\inf\limits_{c\in L(a)} c$. Analogously we have $\sup\limits_{c\in U(a)} c$.

Lemma 1 $\sup\limits_{c\in U(a)} c = \inf\limits_{d\in L(a)} d = P(a)$ *for all* $a \in B(\mathcal{H})^s$ *where* $B(\mathcal{H})^s$ *is the selfadjoint part of* $B(\mathcal{H})$.

Proof. For arbitrary $c_1 \in U(a)$, $d_1 \in L(a)$ we have $c_1 \leqq a \leqq d_1$; hence $c_1 \leqq d_1$ and so $\sup\limits_{\substack{c\leqq a\\ c\in C}} c \leqq d_1$ and $\sup\limits_{\substack{c\leqq a\\ c\in C}} c \leqq \inf\limits_{\substack{d\geqq a\\ d\in C}} d$. $a \leqq c$ implies $P(a) \leqq c$, and so $P(a) \leqq \inf\limits_{\substack{a\leqq d\\ d\in C}} d$. Analogously $\sup\limits_{\substack{c\leqq a\\ c\in C}} c \leqq P(a)$.

Now suppose that $P(a) < \inf\limits_{c\in L(a)} c$ and put $a_0 = \inf\limits_{c\in L(a)} c$. Then $-\|a+c\|1 - c \leqq (a+c)-c \leqq \|a+c\|1-c$; hence $-\|a+c\|1-c \leqq \sup\limits_{c\in U(a)} c \leqq \inf\limits_{c\in L(a)} c = a_0 \leqq \|a+c\|1-c$. Therefore $-\|a+c\|1 \leqq a_0 + c \leqq \|a+c\|1$ and so $\|a_0+c\| \leqq \|a+c\|$ $(c \in C(\beta\mathbb{N})^s)$. Define $T(\lambda a + c) = \lambda a_0 + c$ $(\lambda \in \mathbb{R}, c \in C(\beta\mathbb{N})^s)$; then $\|T\| = 1$; hence by the extension property, there is a bounded linear mapping $\widetilde{T}$ of $B(\mathcal{H})^s$ onto $C(\beta\mathbb{N})^s$ such that $\widetilde{T}(x) = T(x)$ for $x \in \mathbb{R}a + C(\beta\mathbb{N})^s$ and $\|\widetilde{T}\| = \|T\|$.

If $h > 0$, then $\|1 - \frac{h}{\|h\|}\| \leqq 1$ and so $\|\widetilde{T}(1 - \frac{h}{\|h\|})\| = \|1 - \widetilde{T}(\frac{h}{\|h\|})\| \leqq 1$; hence $\widetilde{T}(h) \geqq 0$.

Since $\widetilde{T}(a) = a_0 \neq P(a)$, there is an element ξ_{n_0} in the orthonormal basis $\{\xi_n\}$ such that $(a_0\xi_{n_0}, \xi_{n_0}) \neq (P(a)\xi_{n_0}, \xi_{n_0})$, because the predual of $C(\beta\mathbb{N}) = l^1(\mathbb{N})$.

Now define $\varphi(x) = (\widetilde{T}(x)\xi_{n_0}, \xi_{n_0})(x \in B(\mathcal{H})^s)$ and $\hat{\varphi}(x+iy) = \varphi(x) + i\varphi(y)$ $(x, y \in B(\mathcal{H})^s)$. then $\tilde{\varphi}(h^*h) \geqq 0$ and $\tilde{\varphi}(1) = 1$; hence $\tilde{\varphi}$ is a state on $B(\mathcal{H})$.

Since $\tilde{\varphi}(c) = (c\xi_{n_0}, \xi_{n_0})$ $(c \in C)$, by the unicity of the pure state extension of discrete pure states on C to $B(\mathcal{H})$, $\tilde{\varphi}(b) = (b\xi_{n_0}, \xi_{n_0}) = (P(b)\xi_{n_0}, \xi_{n_0})$ $(b \in B(\mathcal{H}))$, a contradiction. Hence $P(a) = \inf\limits_{c\in L(a)} c$ and analogously $P(a) = \sup\limits_{c\in U(a)} c$. $\square$

For $a \in B(\mathcal{H})$, let $W(a)$ be a $\sigma(B(\mathcal{H}), B(\mathcal{H})_*)$-closed convex subset of $B(\mathcal{H})$ generated by $\{vav^* \mid v \in C^u\}$, where $B(\mathcal{H})_*$ is the predual of $B(\mathcal{H})$ and C^u is the group of all unitary elements of C. Then by Markov-Kakutani fixed point theorem, there is a fixed point a_0 under $\{\mathrm{Ad}\, v \mid v \in C^u\}$; hence $va_0v^* = a_0$ $(v \in C^u)$. Since C is a maximal commutative W^*-subalgebra of $B(\mathcal{H})$, $a_0 \in C$.

Moreover, $\|vav^* + c\| = \|v(a+c)v^*\| = \|a+c\|$ $(c \in C)$; hence $\|a_0 + c\| \leqq \|a + c\|$ $(c \in C)$. Define $T(\lambda a + c) = \lambda a_0 + c$ $(\lambda \in \mathbb{C}, c \in C)$; then $\|T(\lambda a + c)\| \leqq \|\lambda a + c\|$; hence T can be extend to a bounded linear mapping $\widetilde{T}$ of $B(\mathcal{H})$ onto $C(\beta\mathbb{N})$ with $\|\widetilde{T}\| = \|T\|$. By a similar discussion with the proof of Lemma 1, we have

Lemma 2 *$\widetilde{T} = P$ on $B(\mathcal{H})$ and so $W(a) \cap C = \{P(a)\}$ $(a \in B(\mathcal{H}))$.*

Let $V(b)$ be the norm-closed convex subset of $B(\mathcal{H})$ generated by $\{vbv^* \mid v \in C^u\}$ for $b \in B(\mathcal{H})$ and $V(b)^{oo}$ (resp. C^{oo}) be the bipolar of $V(b)$ (resp. C) in $B(\mathcal{H})^{**}$, where $B(\mathcal{H})^{**}$ is the second dual of $B(\mathcal{H})$.

Since $V(b)^{oo}$ is $\sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^*)$-compact and $vV(b)^{oo}v^* \subset V(b)^{oo}(v \in C^u)$, $V(b)^{oo}$ is invariant under $\mathrm{Ad}\, v$ $(v \in C^u)$: hence by Markov-Kakutani fixed point theorem, $V(b)^{oo} \cap C' \neq \emptyset$, where C' is the commutant of C in the W^*-algebra $B(\mathcal{H})^{**}$. Then we have the following theorem.

Theorem 1 *The following properties are mutually equivalent.*

(1) Kadison-Singer problem is positive for all $t \in \beta\mathbb{N} \setminus \mathbb{N}$;

(2) for any commutative AW^*-subalgebra D of $B(\mathcal{H})^{**}$ such that $C \subset D \subset B(\mathcal{H})^{**}$, let Q be a norm-one projection of $B(\mathcal{H})^{**}$ onto D; then $Q(a) = P(a)$ for all $a \in B(\mathcal{H})$;

(3) $V(a)^{oo} \cap C' = \{P(a)\}$ $\quad (a \in B(\mathcal{H}))$;

(4) $V(a) \cap C = \{P(a)\}$ $\quad (a \in B(\mathcal{H}))$;

(5) $\sup\limits_{\substack{c \leqq b \\ c \in C^s}} c(t) = \inf\limits_{\substack{b \leqq d \\ d \in C^s}} d(t)$ $\quad$ for $t \in \beta\mathbb{N} \setminus \mathbb{N}$ and $b \in B(\mathcal{H})^s$;

(6) $\{c_{\mathcal{F}}\}$ (resp. $\{d_{\mathcal{F}}\}$) converges $P(b)$ uniformly on $\beta\mathbb{N}$, where $c_{\mathcal{F}} = \bigvee\limits_{i \in \mathcal{F}} c_{\alpha_i}$ with $(c_{\alpha_i})_{i \in \mathcal{F}} \subset U(b)$ (resp. $d_{\mathcal{F}} = \bigwedge\limits_{i \in \mathcal{F}} d_{\alpha_i}$ with $(d_{\alpha_i})_{i \in \mathcal{F}} \subset L(b)$) for every finite subset $\mathcal{F}$ and every $b \in B(\mathcal{H})^s$;

(7) $\{c_{\mathcal{F}}\}$ (resp. $\{d_{\mathcal{F}}\}$) converges to $P(b)$ in the $\sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^*)$-topology for all $b \in B(\mathcal{H})^s$.

Proof. (1) $\Rightarrow$ (2). Let Q be a norm-one projection of $B(\mathcal{H})^{**}$ onto D (it exists always, because the spectrum space of D is Stonean). Let $D = C(K)$ be the Gelfand representation, and suppose that $Q(a) \neq P(a)$ for some element $a \in B(\mathcal{H})$. Then there exists a point $t_0 \in K$ such that $Q(a)(t_0) \neq P(a)(t_0)$. On the other hand $Q(c)(t_0) = c(t_0) = P(c)(t_0)$ for $c \in C$. $|Q(x)(t_0)| \leqq \|Q(x)\| \leqq \|x\|$ ($x \in B(\mathcal{H})$) and $Q(1)(t_0) = 1$; hence $x \to Q(x)(t_0)$ is a state on $B(\mathcal{H})$. Since $c \to c(t_0)$ ($c \in C$) is a character of C, there exists a point s_0 in $\beta\mathbb{N}$ such that $c(t_0) = c(s_0)$ ($c \in C$); hence by the (1), $P(a)(t_0) = Q(a)(t_0)$, a contradiction.

(2) $\Rightarrow$ (3). For $b \in B(\mathcal{H})^s$, $V(b)^{oo} \cap C' \neq \emptyset$ by Markov-Kakutani fixed point theorem. Take $b_0 \in V(b)^{oo} \cap C'$; then $b_0 \in (B(\mathcal{H})^{**})^s$. Let D be a commutative W^*-subalgebra of $B(\mathcal{H})^{**}$ generated by b_0 and C. Since $\|vbv^* + d\| = \|v(b+d)v^*\| = \|b + d\|$ for $d \in D$ and $v \in C^u$, $\|b_0 + d\| \leqq \|b + d\|$ for $d \in D$. Now define $T(\lambda b + d) = \lambda b_0 + d$ ($\lambda \in \mathbb{C}, d \in D$); then T is a norm-one linear mapping of $\{\lambda b + D \mid \lambda \in \mathbb{C}\}$ onto D and so it can be extend to a norm-one projection $\widetilde{T}$ of $B(\mathcal{H})^{**}$ onto D. By (2), $\widetilde{T}(b) = b_0 = P(b)$; hence $V(b)^{oo} \cap C' = \{P(b)\}$. For $a \in B(\mathcal{H})$, let $a = a_1 + ia_2$ ($a_1, a_2 \in B(\mathcal{H}^s)$; then $a_0 \in V(a)^{oo} \cap C'$ implies $\dfrac{a_0 + a_0^*}{2} \in$ $(V(a_1)^{oo} \cap C$ and $\dfrac{ia_0 - ia_0^*}{2} \in V(a_2)^{oo} \cap C$; hence $a_0 = P(a)$.

(3) $\Rightarrow$ (4). since $P(a) \in V(a)^{oo}$ ($a \in B(\mathcal{H})$), there is a direct set of elements $\{x_\alpha\}$ in $V(a)$ such that $\sigma(B(\mathcal{H}), B(\mathcal{H})^*) - \lim_\alpha x_\alpha = P(a)$. Therefore by the convexity of $V(a)$, there is a sequence $\{y_n\}$ in $V(a)$ such that $\{y_n\}$ converges to $P(a)$ in norm; hence $V(a) \cap C = \{P(a)\}$.

(4) $\Rightarrow$ (1). Suppose that φ is a state on $B(\mathcal{H})$ such that $\varphi(c) = c(t)$ for $c \in C$ ($t \in \beta\mathbb{N}$). Then

$$|\varphi(a(c - c(t)1))| \leqq \varphi(aa^*)^{1/2}\varphi((c - c(t)1)^*(c - c(t)1))^{1/2} = 0;$$

hence $\varphi(ac) = c(t)\varphi(a)$ ($c \in C, a \in B(\mathcal{H})$). Hence

$$\varphi(c_1ac_2) = c_2(t)\varphi(c_1a) = c_2(t)\overline{\varphi(a^*c_1^*)} = c_2(t)c_1(t)\overline{\varphi(a^*)} = c_1(t)c_2(t)\varphi(a).$$

for $c_1, c_2 \in C$ and $a \in B(\mathcal{H})$). Hence $\varphi(uau^*) = \varphi(a)$ for $u \in C^u$, and so $\varphi(a) = \varphi(x)$ for $x \in V(a)$. Since $V(a) \cap C = \{P(a)\}$, $\varphi(a) = P(a)(t)$ ($a \in B(\mathcal{H})$.

(1) $\Rightarrow$ (5). Suppose that $\sup_{\substack{c \leqq b \\ c \in C^s}} c(t) < \inf_{\substack{b \leqq d \\ d \in C^s}} d(t)$ for some $t \in \beta\mathbb{N} \setminus \mathbb{N}$. Clearly

$\sup_{\substack{c \leqq b \\ c \in C^s}} c(t) \leqq P(b)(t) \leqq \inf_{\substack{b \leqq d \\ d \in C^s}} d(t)$. Suppose that $P(b)(t) < \inf_{\substack{b \leqq d \\ d \in C^s}} d(t)$ (put r). Then $-\|b+c\| - c(t) \leqq r \leqq \|b+c\| - c(t)$ $(c \in C^s)$. Hence $|r + c(t)| \leqq \|b+c\|$ $(c \in C^s)$. Define $f(b+c) = r + c(t)$; then $|f(b+c)| \leqq \|b+c\|$ $(c \in C^s)$ and so $|f(\lambda b + c)| \leqq \|\lambda b + c\|$ $(\lambda \in \mathbb{R}, c \in C^s)$. Therefore f can be extended to a real linear functional $\hat{f}$ on $B(\mathcal{H})^s$ such that $\|\hat{f}\| = 1$, and so $\hat{f}$ can be extended to a state φ on $B(\mathcal{H})$; then $\varphi(c) = c(t)$ $(c \in C)$, but $\varphi(b) \neq P(b)(t)$, a contradiction.

(5) $\Rightarrow$ (6). $\sup_{\substack{c \leqq b \\ c \in C^s}} c(t) = \inf_{\substack{b \leqq d \\ d \in C}} d(t) = P(b)(t)$ for all $t \in \beta\mathbb{N}$ implies that $P(b)$ is continuous on $\beta\mathbb{N}$. Since $c_{\mathcal{F}}(t) \uparrow P(b)(t)$, it converges to $P(b)$ uniformly on $\beta\mathbb{N}$. Also $d_{\mathcal{F}}(t) \downarrow P(a)(t)$ and so it converges to $P(a)(t)$ uniformly.

(6) $\Rightarrow$ (7). For $\mu \in C^*$, $\lim \int c_{\mathcal{F}}(t) d\mu(t) = \int P(b)(t) d\mu(t)$ and $\lim \int d_{\mathcal{F}}(t) d\mu(t) = \int P(b)(t) d\mu(t)$; hence $c_{\mathcal{F}} \to P(b)$ and $d_{\mathcal{F}} \to P(b)$ in the $\sigma(C, C^*)$-topology.

(7) $\Rightarrow$ (1). Put $\chi_t(c) = c(t)$ $(c \in C)$; then $\chi_t \in C^*$; hence $c_{\mathcal{F}}(t) \to P(b)(t)$ and $d_{\mathcal{F}}(t) \to P(b)(t)$ for $t \in \beta\mathbb{N}$; hence Kadison-Singer Problem is positive for all $t \in \beta\mathbb{N}$. $\square$

Remark 1 *From the proofs of (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) in Theorem 1, Kadison-Singer problem is positive for all $t \in \beta\mathbb{N} \setminus \mathbb{N}$ if and only if $P(a) \in V(a)^{oo}$ for all $a \in B(\mathcal{H})$. Therefore $P(a) \in V(a)^{oo}$ $(a \in B(\mathcal{H}))$ implies $V(a)^{oo} \cap C' = \{P(a)\}$ $(a \in B(\mathcal{H}))$.*

Theorem 1 (7) implies that Kadison-Singer problem is equivalent to a problem concerning the $\sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^*)$-topology. We shall discuss this topology in the following. Let $K(\mathcal{H})$ be the algebra of all compact linear operators on $\mathcal{H}$; then $K(\mathcal{H})^{oo} = B(\mathcal{H})^{**} z_0$, where z_0 is a central projection of $B(\mathcal{H})^{**}$. Moreover $B(\mathcal{H})^{**} z_0 = B(\mathcal{H}) z_0$ and $a \mapsto a z_0$ $(a \in B(\mathcal{H}))$ is a $*$-isomorphism of $B(\mathcal{H})$ onto $B(\mathcal{H}) z_0$. Therefore we have: $B(\mathcal{H})^{**} = B(\mathcal{H})^{**} z_0 \oplus B(\mathcal{H})^{**}(1 - z_0) = B(\mathcal{H}) z_0 \oplus B(\mathcal{H})^{**}(1 - z_0)$.

Put $C_0 = C \cap K(\mathcal{H})$; then $(C_0)^{oo} = C z_0$ and so $z_0 \in (C_0)^{oo}$. $B(\mathcal{H})(1 - z_0)$ is considered the Calkin algebra $B(\mathcal{H})/K(\mathcal{H})$ and so $B(\mathcal{H})^{**}(1 - z_0)$ is the second dual of $B(\mathcal{H})/K(\mathcal{H})$. Let $a_0 \in V(a)^{oo} \cap C'$ $(a \in B(\mathcal{H}))$; then $a_0 = a_0 z_0 + a_0(1 - z_0)$. Since $a_0 z_0 \in C' z_0$ and $C' z_0 = C z_0$, $a_0 z_0 \in V(a)^{oo} z_0 \cap C z_0$; hence $a_0 z_0 = P(a) z_0$ (Lemma

2). Since $z_0 \in (C_0)^{oo} \subset C^{oo}$, $P(a)z_0 \in C^{oo}$. Therefore if $a_0(1-z_0) \in C^{oo}(1-z_0)$, then $a_0 \in C^{oo}z_0 \oplus C^{oo}(1-z_0) = C^{oo}$; hence $a_0 \in V(a)^{oo} \cap C^{oo}$.

Lemma 3 *If $V(a)^{oo} \cap C^{oo} \neq \emptyset$, then $V(a)^{oo} \cap C^{oo} = \{P(a)\}$ ($a \in B(\mathcal{H})$).*

Proof. For $a_1 \in V(a)^{oo} \cap C^{oo}$, there is a directed set $\{a_\alpha\}$ in $V(a)$ such that $\sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^*)-\lim a_\alpha = a_1$. Let P^{**} be the second dual of P; then $P^{**}(\lim a_\alpha) = \lim P^{**}(a_\alpha) = \lim P(a_\alpha) = \lim P(a) = P(a) = P^{**}(a_1) = a_1$, because P^{**} is a projection of $B(\mathcal{H})^{**}$ onto C^{oo}. Hence $P(a) = a_1$. □

Therefore if $a_0(1-z_0) \in C^{oo}(1-z_0)$, then $a_0(1-z_0) = P(a)(1-z_0)$ and so $a_0 = P(a)$. Now we have the following theorems.

Theorem 2 *Kadison-Singer problem is positive for all $t \in \beta\mathbb{N} \setminus \mathbb{N}$ if and only if $V(a)^{oo} \cap C^{oo} \neq \emptyset$ for all $a \in B(\mathcal{H})$.*

This is clear from Theorem 1, Remark 1 and Lemma 3.

Theorem 3 *Suppose that $a_0 \in V(a)^{oo} \cap C'$ for $a \in B(\mathcal{H})$. Then $a_0 = P(a)$ if and only if $a_0(1-z_0) \in C^{oo}(1-z_0)$.*

Now we shall show that Kadison-Singer problem can be reduced to a problem on the Calkin algebra. By Johnson-Parrott theorem [10], $C + K(\mathcal{H})/K(\mathcal{H})$ is a maximal commutative C^*-subalgebra of $B(\mathcal{H})/K(\mathcal{H})$ and so $C(1-z_0)$ is a maximal commutative C^*-subalgebra of $B(\mathcal{H})(1-z_0)$. Hence $C'(1-z_0)$ in $B(\mathcal{H})(1-z_0) = C(1-z_0)$. Therefore we have the following theorem.

Theorem 4 *Kadison-Singer Problem is positive for all $t \in \beta\mathbb{N} \setminus \mathbb{N}$ if and only if $V(a)^{oo}(1-z_0) \cap C'(1-z_0) \cap B(\mathcal{H})(1-z_0) \neq \emptyset$ for all $a \in B(\mathcal{H})$.*

Proof. Suppose that $V(a)^{oo}(1-z_0) \cap C'(1-z_0) \cap B(\mathcal{H})(1-z_0) \neq \emptyset$ for all $a \in B(\mathcal{H})$; then $V(a)^{oo}(1-z_0) \cap C'(1-z_0) \cap B(\mathcal{H})(1-z_0) = V(a)^{oo}(1-z_0) \cap C(1-z_0) \neq \emptyset$. Let $b_0 \in V(a)^{oo}(1-z_0) \cap C(1-z_0)$; then $b_0 = x(1-z_0) = c(1-z_0)$, where $x \in V(a)^{oo}$ and $c \in C$. Hence $vx(1-z_0)v^* = vc(1-z_0)v^* = vcv^*(1-z_0) = c(1-z_0)$ for $v \in C^u$. Since $vx(1-z_0)v^* = vxv^*(1-z_0)$, there is an element $a_0 \in V(a)^{oo} \cap C'$ such that

$a_0(1-z_0) = c(1-z_0) \in C^{oo}(1-z_0)$. Hence by Theorem 3, $a_0 = P(a)$ and so by Theorem 2, Kadison-Singer problem is positive for all $t \in \beta\mathbb{N} \setminus \mathbb{N}$.

Conversely if $V(a)^{oo}(1-z_0) \cap C'(1-z_0) \cap B(\mathcal{H})(1-z_0) = \emptyset$ for some $a \in B(\mathcal{H})$, $P(a) \notin V(a)^{oo} \cap C'$; in fact, if $P(a) \in V(a)^{oo} \cap C'$, then $P(a)(1-z_0) \in V(a)^{oo}(1-z_0) \cap C'(1-z_0) \cap B(\mathcal{H})(1-z_0)$. $\square$

Therefore the following problem would be interesting in the aspect of axiomatic set theory in operator algebras.

Problem 1 *Can we extend Markov-Kakutani fixed point theorem to the Calkin algebra in the set theory related to operator algebras?*

Finally we shall state one more problem. Let D be a commutative W^*-subalgebra of $B(\mathcal{H})^{**}$ such that $C^{oo} \subsetneqq D \subset C'$ in $B(\mathcal{H})^{**}$. Then there exists a norm-one projection Q of $B(\mathcal{H})^{**}$ onto D. If one can take the Q which is $\sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^*)$-continuous, then $D = Q(B(\mathcal{H})^{**}) = \sigma(B(\mathcal{H})^{**}, B(\mathcal{H})^*)$-closure of $Q(B(\mathcal{H}))$; hence $Q(B(\mathcal{H})) \not\subset C^{oo}$. Therefore Kadison-Singer problem is negative for some $t \in \beta\mathbb{N} \setminus \mathbb{N}$. The following problem is interesting in the theory of operator algebras within ZFC.

Problem 2 *Is there a commutative W^*-subalgebra D of C' in $B(\mathcal{H})^{**}$ satisfying the above condition?*

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