K-fibrations and non-commutative Torus bundles

joint with

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If $p : Y \to X$ is a fibration with X path connected, then all fibres $Y_x = p^{-1}(\{x\})$ are homotopy equivalent, and the fibration behaves as a “locally trivial fibre bundle” up to homotopy.

C*-Algebra bundles (or $C_0(X)$-algebras)

A C*-algebra bundle over X is a C*-algebra $A = A(X)$ together with a nondegenerate *-homomorphism

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If $I_x := \Phi\big(C_0(X \setminus \{x\})\big)A$, then

$$A_x := A/I_x$$

is the fibre of A at $x \in X$. If $a \in A$, then

$$x \mapsto \|a_x\|, \quad a_x := a + I_x \in A_x$$

is always upper semi continuous and vanishes at ∞.

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Examples of C*-algebra bundles

- If $f : Y \to X$ is a continuous map, then $C_0(Y)$ is a C*-algebra bundle over X with fibre $C_0(Y_x)$ via $\Phi : C_0(X) \to C_b(Y) = M(C_0(Y)); \Phi(g) = g \circ f$.

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- Continuous trace algebras (the case $B = \mathcal{K}$).
- Let $A([0,1])$ be given as

$$\{f : [0,1] \to M_2(\mathbb{C}); f \text{ continuous and } f(0) = \begin{pmatrix} f_{11}(0) & 0 \\ 0 & f_{22}(0) \end{pmatrix}\}$$

Then $A_t = M_2(\mathbb{C})$ for $t \neq 0$ and $A_0 = \mathbb{C}^2$.

Examples of C*-algebra bundles

- Heisenberg group algebra: $C^*(H_2) = C^*(U, V, W)$ where U, V, W are unitaries with relations

$$UV = WVU, \quad UW = WU, \quad VW = WV.$$

Functional calculus: $\Phi : C(\mathbb{T}) \xrightarrow{\cong} C^*(W) \subseteq C^*(U, V, W)$. We get $A_z = C^*(U_z, V_z)$ with relation $U_z V_z = z V_z U_z$. Thus $A_z = A_\theta$ if $z = e^{2\pi i \theta}$.

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- If $1 \to Z \to H \to G \to 1$ is a central group extension, then $C^*(H)$ is a C*-algebra bundle over $\widehat{Z}$ via

$$C_0(\widehat{Z}) \cong C^*(Z) \to ZM(C^*(H))$$

where $g \in C^*(Z)$ acts on $C^*(H)$ via convolution. The fibre $C^*(H)_\chi$ for $\chi \in \widehat{Z}$ is a twisted group algebra $C^*(G, \omega_\chi)$ of G.

Examples of C*-algebra bundles

- Crossed products $A(X) \rtimes G$ by fibre-wise actions. We then have $(A \rtimes G)_x = A_x \rtimes G$.

K-theory, KK-theory and $\mathcal{R}KK$-theory

KK-theory:

$$KK_0(A, B) = \{(E, \phi, T)\}/ \sim$$

with E a Hilbert B-module, $\phi \rightarrow \mathcal{L}_B(E)$ a *-homomorphism and T a “generalized” Fredholm operator.

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$$KK_1(A,B) = KK_0(C_0(\mathbb{R}) \otimes A, B) = KK_0(A, C_0(\mathbb{R}) \otimes B)$$

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Composition: $KK_i(A,B) \times KK_j(B,C) \to KK_{i+j}(A,C)$

A and B are KK-equivalent if $\exists\ x \in KK(A,B)$, $y \in KK(B,A)$

such that $x \otimes y = [\mathrm{id}_A]$ and $y \otimes x = [\mathrm{id}_B]$.

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$\mathcal{R}KK$-Theory: Suppose $A(X)$ and $B(X)$ are C*-algebra bundles over X. Then

$$\mathcal{R}KK(X; A(X), B(X)) = \{(E, \phi, T)\}/ \sim$$

such that the left and right actions of $C_0(X)$ on E coincide.
Kasparov product over X:

$$\mathcal{R}KK(X; A(X), B(X)) \times \mathcal{R}KK(X; B(X), D(X)) \to \mathcal{R}KK(X; A(X), D(X)$$

K-fibrations

Let $A(X)$ be a C*-algebra bundle over X and let $f : Z \to X$ be a continuous map. Then we define the pull-back $f^*A(Z)$ of $A(X)$ along f as

$$f^*A(Z) = C_0(Z) \otimes_{C_0(X)} A(X)$$

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$A(X)$ is a K-fibration if for **every** compact contractible space Δ the evaluation map $\mathrm{ev}_v : f^*A(\Delta) \to A_{f(v)}$ induces an isomorphism $K_*(f^*A(\Delta)) \cong K_*(A_{f(v)})$.

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$A(X)$ is an $\mathcal{R}KK$-fibration, if $f^*A(\Delta) \sim_{\mathcal{R}KK} C(\Delta, A_{f(v)})$.

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We also have the following theorem, which follows from some deep result of Dadarlat:

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If $A(X)$ is a continuous and nuclear C*-algebra bundle. Then

$$A(X) \text{ is a } KK\text{-fibration} \Longleftrightarrow A(X) \text{ is an } \mathcal{R}KK\text{-fibration.}$$

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Idea: Let $X = \Delta$ and consider

$$\begin{aligned} KK(A_x, A(\Delta)) &\stackrel{\otimes C(\Delta)}{\to} \mathcal{R}KK(\Delta; C(\Delta, A_x), C(\Delta, A(\Delta)) \\ &\to \mathcal{R}KK(\Delta; C(\Delta, A_x), A(\Delta)), \end{aligned}$$

where the last map is given by restriction on the diagonal.

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- Theorem. (E-Oyono-Nest)
  Suppose G is an amenable group which acts fibre-wise on the C*-algebra bundle $A(X)$. Then:
  If $A(X) \rtimes K$ is a K-fibration (resp. KK-fibration) for all compact subgroups K of G, then $A(X) \rtimes G$ is a K-fibration (resp. KK-fibration).
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- **Corollary.** If $A(X)$ is a K-fibration (resp. KK-fibration) then the same is true for $A(X) \rtimes \mathbb{Z}^n$ or $A(X) \rtimes \mathbb{R}^n$ for every fibre-wise action $\alpha : \mathbb{Z}^n, \mathbb{R}^n \to \mathrm{Aut}(A(X))$.

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- Corollary. If $A(X)$ is a continuous-trace algebra over X and if G is an amenable group acting fibre-wise on $A(X)$, then $A(X) \rtimes G$ is an $\mathcal{R}KK$-fibration.

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- The Heisenberg group algebra $C^*(H_2)(\mathbb{T}) = C^*(U, V, W)(\mathbb{T})$ is an $\mathcal{R}KK$-fibration:
  There is a fibre-wise action of $\mathbb{T}^2$ on $C^*(H_2)$ given by

  $$\alpha_{(z,w)}(U) = zU, \;\; \alpha_{(z,w)}(V) = wV, \;\; \alpha_{(z,w)}(W) = W.$$

  With crossed-product $C^*(H_2) \rtimes \mathbb{T}^2 \cong C(\mathbb{T}, \mathcal{K})$. It follows then from Takesaki-Takai duality that

  $$C^*(H_2) \otimes \mathcal{K} \cong C(\mathbb{T}, \mathcal{K}) \rtimes \mathbb{Z}^2$$

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- If $1 \to Z \to H \to G \to 1$ is a central extension with G amenable, then $C^*(H)(\widehat{Z})$ is an $\mathcal{R}KK$-fibration.
  (Since $C^*(H) \otimes \mathcal{K} \cong C_0(\widehat{Z},\mathcal{K}) \rtimes G$ for some fibre-wise action of G)

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- The C*-algebra bundle

$$A([0,1]) = \{f : [0,1] \to M_2(\mathbb{C}) : f(0) = \begin{pmatrix} f_{11}(0) & 0 \\ 0 & f_{22}(0) \end{pmatrix}\}$$

is NOT a K-fibration.

The K-theory group bundle

Suppose $A(X)$ is a K-fibration. Then the K-theory group bundle consists of the collection

$$\mathcal{K}_*(A(X)) := \{K_*(A_x) : x \in X\}$$

together with isomorphisms $c_\gamma : K_*(A_x) \to K_*(A_y)$ for every continuous path $\gamma : [0,1] \to X$ from x to y given by the composition

$$c_\gamma : K_*(A_x) \overset{ev_{0,*}^{-1}}{\to} K_*(\gamma^* A[0,1]) \overset{\mathrm{ev}_{1,*}}{\to} K_*(A_y).$$

We then have $c_{\gamma \circ \gamma'} = c_\gamma \circ c_{\gamma'}$ and c_γ only depends on the homotopy class of γ.

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Proof. If $\Gamma : [0,1]^2 \to X$ is a homotopy for γ and γ', then show that both maps coincide with

$$K_*(A_x) \overset{ev_{0,0,*}^{-1}}{\longrightarrow} K_*(\Gamma^* A([0,1]^2)) \overset{\mathrm{ev}_{1,1*}}{\longrightarrow} K_*(A_y).$$

The K-theory group bundle

Observations.

- If X is simply connected and path connected, and if $A(X)$ is a K-fibration, then $\mathcal{K}_*(A(X))$ is the trivial bundle $X \times K_*(A_x)$. The trivialization map is given by

$$(y, K_*(A_y)) \to (y, K_*(A_x)); (y, \mu) \mapsto (y, c_{y,x}(\mu))$$

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  where $c_{y,x} = c_\gamma$ for any chosen path γ from x to y.
- In general, if X is path connected, there is an action of $\pi_1(X)$ on $K_*(A_x)$, and $\mathcal{K}_*(A(X))$ is the trivial bundle if and only if this action is trivial.

The Leray-Serre Spectral sequence

Let $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$ be the sceleton of a finite simplicial complex X. Put $A_p := A(X_p)$, $A_{p,p-1} = A(X_p \setminus X_{p-1})$. We then have short exact sequences

$$0 \to A_{p,p_1} \to A_p \to A_{p-1} \to 0$$

which gives the long exact sequences

$$K_q(A_{p,p-1}) \xrightarrow{\iota} K_q(A_p) \xrightarrow{j} K_q(A_{p-1}) \xrightarrow{\partial} K_{q+1}(A_{p,p-1}) \to$$

Now put $\mathcal{A}^{p,q} = K_q(A_p)$ and $E_1^{p,q} = K_q(A_{p,p-1})$. Then we get the exact couple

$$\begin{tikzcd} \bigoplus_{p,q}\mathcal{A}^{p,q} \arrow[rr, "j"] & & \bigoplus \mathcal{A}^{p,q} \arrow[dl, "\partial"] \\ & \bigoplus E_1^{p,q} \arrow[ul, "\iota"] & \end{tikzcd}$$

The Leray-Serre Spectral sequence

Let $\{E_r^{p,q}, dr : E_r^{p,q} \to E_r^{p+r,q+1}\}$ be the spectral sequence derived from the above exact couple. We have

$$d_1 : E_1^{pq} = K_q(A_{p,p-1}) \xrightarrow{\iota} K_q(A_p) \xrightarrow{\partial} K_{q+1}(A_{p+1,p}) = E_1^{p+1,q+1}$$

The higher terms are derived from this iterative by

$$E_{r+1}^{p,q} = (\mathrm{kernel}\, dr / \mathrm{image}\, dr)_{p,q}.$$

This process stabilizes eventually with

$$E_\infty^{p,p-q} := F_p^q / F_{p+1}^q, \quad \text{for } F_p^q := \mathrm{kernel}\big(K_q(A(X)) \to K_q(A_p)\big)$$

Since $X_n = X$ we obtain a filtration

$$\{0\} = F_n^q \subseteq F_{n-1}^q \subseteq \cdots \subseteq F_{-1}^q = K_q(A(X)).$$

The Leray-Serre Spectral sequence

Theorem (E-Nest-Oyono) Suppose $A(X)$ is a K-fibration over the finite simplicial complex X. Then the E_2-term of the above described spectral sequence is given by $E_2^{p,q} \cong H^p(X, \mathcal{K}_q(A))$, the cohomology of X with coefficients in the K-theory group bundle $\mathcal{K}_*(A(X))$.

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- The case $A(X) = C(X)$ is the classical Atiyah-Hirzebruch spectral sequence for the K-theory of X.
- If $A(X)$ is a KK-fibration, then a similar result holds for the K-homology of $A(X)$.
- If $A(X) \sim_{\mathcal{R}KK} B(X)$, then the spectral sequences of $A(X)$ and $B(X)$ coincide, i.e., the spectral sequence is an invariant for $\mathcal{R}KK$-equivalence.

Non-commutative principle torus bundles

Let $p : Y \to X$ be a principal $\mathbb{T}^n$-bundle. Then by Phil Green:

$$C_0(Y) \rtimes \mathbb{T}^n \cong C_0(X, \mathcal{K}) \qquad \mathcal{K} := \mathcal{K}(L^2(\mathbb{T}^n))$$

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$$C_0(Y) \rtimes \mathbb{T}^n \cong C_0(X, \mathcal{K}) \qquad \mathcal{K} := \mathcal{K}(L^2(\mathbb{T}^n))$$

Definition. A C*-algebra bundle $A(X)$ is a non-commutative principal $\mathbb{T}^n$-bundle (or NCP $\mathbb{T}^n$-bundle), if it is equipped with a fibre-wise action $\alpha : \mathbb{T}^n \to \mathrm{Aut}(A(X))$ such that

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By Takesaki-Takai duality we get $A(X) \sim_M C_0(X, \mathcal{K}) \rtimes_{\widehat{\alpha}} \mathbb{Z}^n$ and vice versa, so that NCP-bundles are up to $C_0(X)$-linear Morita equivalence precisely the crossed products $C_0(X, \mathcal{K}) \rtimes \mathbb{Z}^n$ for fibre-wise actions of $\mathbb{Z}^n$.

Classification of NCP-bundles

Let H_n be the group generated by $\{f_1, \ldots, f_n, g_{ij}, 1 \leq i < j \leq n\}$ with relations $f_i f_j = g_{ij} f_j f_i$ and g_{ij} is central for all ij.

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Notice that $C^*(H_2)$ is the Heisenberg group algebra.

Classification of NCP-bundles

Theorem (E-Williams, 1996) Every NCP $\mathbb{T}^n$-bundle over a given space X is stably isomorphic to one of the form

$$Y * (f^* C^*(H_n))(X)$$

where $f : X \to \mathbb{T}^{\frac{n(n-1)}{2}}$ is a continuous map and $p : Y \to X$ is a (commutative) principal bundle over X.

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If $A(X)$ is any NCP $\mathbb{T}^n$-bundle, we can twist it by a commutative bundle $p : Y \to X$ by defining

$$Y * A(X) = (C_0(Y) \otimes_{C_0(X)} A(X))^{\mathbb{T}^n}$$

where $\mathbb{T}^n$ acts diagonally on the balanced tensor product.

Problems

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The K-theory bundles of NCP-bundles

We can explicitly compute the action of $\pi_1(X)$ on the fibre

$$K_*(A_x) \cong K_*(C(\mathbb{T}^n)) \cong \Lambda^*(\mathbb{Z}^n).$$

The key-result is the computation for the Heisenberg-bundle over $\mathbb{T}$. The fibre at $1 \in \mathbb{T}$ is $C(\mathbb{T}^2)$ and we get

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Lemma. (E-Nest-Oyono) Let γ be the positive generator of $\pi_1(\mathbb{T})$. Then $c_\gamma : K_1(C(\mathbb{T}^2)) \to K_1(C(\mathbb{T}^2))$ is trivial and $c_\gamma : K_0(C(\mathbb{T}^2)) \to K_0(C(\mathbb{T}^2))$ is given by the matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

with respect to the generators $\{[1], \beta\}$ of $K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$.

The K-theory group bundles of NCP-bundles

Scetch of proof. We have $\gamma : [0,1] \to \mathbb{T}; \gamma(t) = e^{2\pi i t}$. Recall that $\gamma^*(C^*(H_2)) = C[0,1] \otimes_\gamma C^*(H_2)$.

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$$U' = 1 \otimes_\gamma U \text{ and } V' = 1 \otimes_\gamma V \in C[0,1] \otimes_\gamma C^*(H_2)$$

Then $[U'], [V']$ are elements of $K_1(\gamma^*(C^*(H_2)))$ which restrict to the standard generators $[u], [v]$ of $K^1(C(\mathbb{T}^2))$ at 0 and 1.

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This implies that $c_\gamma([u]) = [u]$ and $c_\gamma([v]) = [v]$.

The K-theory group bundles of NCP-bundles

Action on $K_0(C(\mathbb{T}^2))$

For each $\theta \in [0,1]$ the generators of $K_0(A_\theta)$ are given by $[1]$ and the projective module E_θ which is a closure of $C_c(\mathbb{R})$ with respect to a certain A_θ-valued inner product and with right action of A_θ given by

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Rieffel computes $\tau([E_\theta]) = \theta + 1$, from which we conclude that $[E_{\theta+1}] = [E_\theta] + [1]$ for all irrational θ, and hence for all θ. Thus

$$c_\gamma([E_0]) = [E_1] = [E_0] + [1]$$

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One can check that $[E_0] = -[\beta] + [1]$ and the result then follows from the obvious fact $c_\gamma([1]) = [1]$.

The K-theory group bundles of NCP-bundles

Lemma (E-Nest-Oyon) Let $A(X) = Y * f^*(C^*(H_2))(X)$ for some function $f : X \to \mathbb{T}$ and some principal $\mathbb{T}^2$-bundle $p : Y \to X$. Assume that $x \in X$ with $f(x) = 1$. Then the action of $\gamma \in \pi_1(X)$ on $K_1(C(\mathbb{T}^2))$ is trivial and the action on $K_0(C(\mathbb{T}^2))$ is given on the generators $[1], [\beta]$ by the matrix

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where $< f, \gamma >$ is the winding number of $f \circ \gamma : \mathbb{T} \to \mathbb{T}$.

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Corollary. The K-theory group bundle of $A(X)$ is trivial if and only if f is homotopic to a constant map.

$\mathcal{R}KK$-triviality for NCP torus bundles

Theorem (E-Nest-Oyono) Let $A(X)$ be any NCP $\mathbb{T}^n$-bundle. Then $A(X)$ is $\mathcal{R}KK$-equivalent to a commutative bundle $p : Y \to X$ (or rather $C_0(Y)(X)$) if and only if the K-theory bundle of $A(X)$ is trivial.

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Proof. If two maps $f_1, f_2 : X \to \mathbb{T}^{\frac{n(n-1)}{2}}$ are homotopic, then one can show directly that $f_1^*(C^*(H_n))(X) \sim_{\mathcal{R}KK} f_2^*(C^*(H_n))$. The result then follows from the above and the classification of NCP-bundles.

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Theorem (E-Nest-Oyono) The NCP-bundle $A(X)$ is $\mathcal{R}KK$-equivalent to the trivial bundle $X \times \mathbb{T}^n$ if and only if the K-theory group bundle is trivial and the map

$$d_2 : H^0(X, K_1(A_x)) \to H^2(X, K_0(A_x))$$

in the Larey-Serre spectral sequence is the trivial map.