Subfactors and Hadamard Matrices

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Subfactors $N \subset M$ $\longrightarrow$ $\mathcal{G}_{N,M}$

Inclusions of infinite dimensional factors with finite index: $dim_N M < \infty$

Group-like objects generalizing:
-f.g. groups
-Hopf algebras

Jones' Basic Construction

$$N \subset M \subset M_1 = < M, e_1 > \quad \subset B(L^2(M,\tau))$$

$$e_1 = proj^{.L^2(M,\tau)}_{L^2(N,\tau)}$$

$$N \subset M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} M_2 \overset{e_3}{\subset} M_3 \overset{e_4}{\subset} \ldots$$

The Standard Invariant $\mathcal{G}_{N,M}$

$$\begin{array}{ccccccc} N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & \ldots \\ \cup & & \cup & & \cup & & \\ M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & \ldots \end{array}$$

Roughly, $\mathcal{G}_{N,M}$ consists of two bipartite graphs and some 'rules' for fitting them.

Fact: $\mathcal{G}_{N,M}$ is a complete invariant for *amenable* subfactors (S.Popa '92).

Fact: The squares of inclusions

$$\begin{array}{ccc} A & \subset & B \\ \cup & & \cup \\ C & \subset & D \end{array}$$

that show up in $\mathcal{G}_{N,M}$ are extremely rigid!

$$A \ominus C \perp D \ominus C$$

Such a square is called a *commuting square*.

Conversely, one can construct subfactors from commuting squares:

$$\begin{array}{ccccccccc} A & \subset & B & \subset & B_1 = < B, e_1 > & \subset & \ldots & \nearrow & M \\ \cup & & \cup & & \cup & & & & \cup \\ C & \subset & D & \subset & D_1 = < D, e_1 > & \subset & \ldots & \nearrow & N \end{array}$$

The factors M, N obtained are isomorphic to R, the hyperfinite factor.

Examples of commuting squares.

• Commuting squares arising from finite groups G:

$$\begin{array}{ccc} l^\infty(G) & \subset & B(l^2(G)) \\ \cup & & \cup \\ \mathbb{C} & \subset & \mathbb{C}[G] \end{array}$$

• When $G = \mathbb{Z}_n$ we obtain:

$$\begin{array}{ccc} \mathcal{D}_n & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & F_n \mathcal{D}_n F_n^* \end{array}$$

where F_n is the *Fourier matrix*:

$$F_n = \frac{1}{\sqrt{n}} (e^{2i\pi kl})_{0 \le k,l \le n-1}$$

• Commuting squares from Hadamard matrices:

$$\begin{array}{ccc} \mathcal{D}_n & \subset & M_n(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} & \subset & U \mathcal{D}_n U^* \end{array}$$

Commuting square $\Rightarrow \sqrt{n}U$ is a *Hadamard matrix*

(i.e. orthogonal rows, entries on the unit circle)

Examples of complex Hadamard matrices.

• For every $n \geq 2$: the Fourier matrix F_n.

For example: $F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

• (U. Haagerup '96, P. Dita '04) For every $n = kl$ non-prime there exist parametric families of non-equivalent complex Hadamard matrices:

Let $A = (a_{i,j}) \in M_k(\mathbb{C})$ and $B_1, ..., B_k \in M_m(\mathbb{C})$ be complex Hadamard matrices. Then:

$$H = \begin{pmatrix} a_{1,1}B_1 & a_{1,2}B_2 & ... & a_{1,k}B_k \\ a_{2,1}B_1 & a_{2,2}B_2 & ... & a_{2,k}B_k \\ . & & & . \\ . & & & . \\ . & & & . \\ a_{k,1}B_1 & a_{k,2}B_2 & ... & a_{k,k}B_k \end{pmatrix}$$

is a Hadamard matrix.

• For example: $n = 4$, $A = F_2$, $B_1 = F_2$, $B_2 \simeq F_2$

$$F_4(\lambda) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \lambda & -\lambda \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\lambda & \lambda \end{pmatrix}, |\lambda| = 1$$

(S. Popa '81) For n prime is F_n the only complex Hadamard matrix of order n?

(De la Harpe-Jones + Munemasa-Watatani, '92) For $n \geq 7$ prime there exists at least one non-Fourier Hadamard matrix.

(U. Haagerup '94) For $n = 5$ the Fourier matrix is the only complex Hadamard matrix.

(S. Popa) For n prime, is the number of complex Hadamard matrices of size n finite?

(M. Petrescu '94) For $n = 7, 13, 19, 31$ and 79 there exist parametric families of complex Hadamard matrices.

$$U(t) = \frac{1}{\sqrt{7}} \begin{pmatrix} tw & tw^4 & w^5 & w^3 & w^3 & w & 1 \\ tw^4 & tw & w^3 & w^5 & w^3 & w & 1 \\ w^5 & w^3 & \bar{t}w & \bar{t}w^4 & w & w^3 & 1 \\ w^3 & w^5 & \bar{t}w^4 & \bar{t}w & w & w^3 & 1 \\ w^3 & w^3 & w & w & w^4 & w^5 & 1 \\ w & w & w^3 & w^3 & w^5 & w^4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$|t| = 1, w$ root of order 6 of unity

Classification results for Hadamard matrices.

(U. Haagerup '94). Complete classification for $n = 3, 4, 5$.

(K. Beauchamp + R.N. '06). Classification of *self-adjoint* Hadamard matrices of order 6.

$$H(\theta) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \bar{x} & -y & -\bar{x} & y \\ 1 & x & -1 & t & -t & -x \\ 1 & -\bar{y} & \bar{t} & -1 & \bar{y} & -\bar{t} \\ 1 & -x & -\bar{t} & y & 1 & \bar{z} \\ 1 & \bar{y} & -\bar{x} & -t & z & 1 \end{pmatrix}$$

where: $\theta \in [-\pi, -arcos(\frac{-1+\sqrt{3}}{2})] \cup [arcos(\frac{-1+\sqrt{3}}{2}), \pi]$

$$y = exp(i\theta), \ \ z = \frac{1 + 2y - y^2}{y(-1 + 2y + y^2)}$$

$$x = \frac{1 + 2y + y^2 - \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{1 + 2y - y^2}$$

$$t = \frac{1 + 2y + y^2 - \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{-1 + 2y + y^2}$$

Classification of 6×6 self-adjoint Hadamard matrices

• If the diagonal contains a -1:

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & b & c & d \end{matrix}$$

Orthogonality of rows 1,2 implies $a+b+c+d=0$, hence $a=-b$ or $a=-c$ or $a=-d$.

-If the third diagonal element is 1:

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a & b & -b \\ 1 & \bar{a} & 1 & x & y & z \end{matrix}$$

Orthogonality of rows 2,3 implies:

$$1-a\bar{x}+b\bar{y}-b\bar{z}=0,\ \text{etc...}$$

-If the third diagonal element is -1:

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & a & -a & b & -b \\ 1 & \bar{a} & -1 & x & y & z \end{matrix}$$

$$\bar{a}+x+y+z=0,\ \text{etc...}$$

• If the diagonal contains just 1's:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & x & \bar{y} & . & . \\ 1 & \bar{x} & 1 & z & . & . \\ 1 & y & \bar{z} & 1 & . & . \\ 1 & . & . & . & . & . \\ 1 & . & . & . & . & . \end{pmatrix}$$

Using the orthogonality of rows 1,2,3:

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & x & \bar{y} & u & v \\ 1 & \bar{x} & 1 & z & s & t \end{matrix}$$

$$2 + x + \bar{y} = -(u + v)$$

$$2 + x + \bar{z} = -(\bar{s} + \bar{t})$$

$$1 + 2\bar{x} + yz = -(\bar{u}s + \bar{v}t)$$

Since

$$\begin{aligned} (u+v)(\bar{s}+\bar{t})(\bar{u}s+\bar{v}t) =& 2 + (u\bar{v} + \bar{u}v) + (s\bar{t} + \bar{s}t) \\ &+ (\bar{u}\bar{t}vs + ut\bar{v}\bar{s}) \end{aligned}$$

is a real number, we have:

$$(2 + x + \bar{y})(2 + x + \bar{z})(1 + 2\bar{x} + yz) \in \mathbb{R}$$

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & x & \bar{y} & . & . \\ 1 & \bar{x} & 1 & z & . & . \\ 1 & y & \bar{z} & 1 & . & . \\ 1 & . & . & . & . & . \\ 1 & . & . & . & . & . \end{pmatrix}$$

$$(2 + x + \bar{y})(2 + x + \bar{z})(1 + 2\bar{x} + yz) \in \mathbb{R}$$

$$(2 + y + \bar{z})(2 + y + \bar{x})(1 + 2\bar{y} + zx) \in \mathbb{R}$$

$$(2 + z + \bar{x})(2 + z + \bar{y})(1 + 2\bar{z} + xy) \in \mathbb{R}$$

If x, y, z are distinct, after some work we obtain:

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 7$$

Which implies $x = y = z = 1$, contradiction!

Subfactors associated to Hadamard Matrices.

$$\begin{array}{ccccccccc}
\mathcal{D}_n & \subset & M_n(\mathbb{C}) & \overset{e_3}{\subset} & \mathcal{P}_1 & \overset{e_4}{\subset} & \mathcal{P}_2 & \overset{e_5}{\subset} & \ldots \nearrow P_H \\
\cup & & \cup & & \cup & & \cup & & \\
\mathbb{C} & \subset & U\mathcal{D}_nU^* & \overset{e_3}{\subset} & \mathcal{Q}_1 & \overset{e_4}{\subset} & \mathcal{Q}_2 & \overset{e_5}{\subset} & \ldots \nearrow Q_H
\end{array}$$

Fact: (Ocneanu Compactness). Computing $\mathcal{G}_{Q_H \subset P_H}$ is equivalent to computing the row of inclusions:

$$\mathcal{D}'_n \cap \mathcal{Q}_0 \subset \mathcal{D}'_n \cap \mathcal{Q}_1 \subset \mathcal{D}'_n \cap \mathcal{Q}_2 \subset \mathcal{D}'_n \cap \mathcal{Q}_3 \subset \ldots$$

Formula for $\mathcal{D}'_n \cap \mathcal{Q}_1$:

$$\mathcal{P}_1 = M_n(\mathbb{C}) \otimes \mathcal{D}_n, \mathcal{Q}_1 = U_1 M_n(\mathbb{C}) U_1^*$$

$$\begin{aligned}
\mathcal{D}'_n \cap \mathcal{Q}_1 &= \mathcal{D}'_n \cap \mathrm{Ad}U_1(M_n(\mathbb{C}))) \\
&= \mathcal{D}'_n \cap \mathrm{Ad}U_1(e'_4 \cap \mathcal{P}_1) \\
&= \mathcal{D}'_n \cap \mathrm{Ad}U_1(\mathcal{P}_1) \cap \mathrm{Ad}U_1(e_4)' \\
&= (\mathcal{D}_n \otimes \mathcal{D}_n) \cap \mathrm{Ad}U_1(e_4)'
\end{aligned}$$

The second relative commutant is thus determined by the connected components of the graph of the *profile matrix*:

$$\mathrm{Ad}U_1(e_4) = (\sum_i u_{a,i}\bar{u}_{b,i}\bar{u}_{c,i}u_{d,i})_{a,b}^{c,d}$$

Fact: The five real Hadamard matrices of order 16 are classified by $\dim(\mathcal{D}_n' \cap \mathcal{Q}_1)$

Open questions: (V. Jones '99)

- Calculation of $\mathcal{G}_{\mathcal{Q}_H \subset \mathcal{P}_H}$?

- example of H yielding a subfactor with no extra structure in its standard invariant?

(W. Camp + R.N.'06). The known non-Dita, non-Fourier matrices of dimension $n \leq 10$ have no extra structure in the first three relative commutants.

(R.N.'06). Hadamard matrices of Dita type yield subfactors with intermediate subfactors:

$$\begin{array}{ccccc}
\mathcal{D}_m \otimes D_k & \subset & M_m(\mathbb{C}) \otimes M_k(\mathbb{C}) & \subset \ldots \nearrow & \mathcal{P}_H \\
\cup & & \cup & & \cup \\
\mathcal{D}_m \otimes I_k & \subset & U(M_m(\mathbb{C}) \otimes \mathcal{D}_k)U^* & \subset \ldots \nearrow & \mathcal{R}_H \\
\cup & & \cup & & \cup \\
\mathbb{C} & \subset & U(\mathcal{D}_m \otimes D_k)U^* & \subset \ldots \nearrow & \mathcal{Q}_H
\end{array}$$

In particular, the second commutant has some extra structure.