Strongly singular MASA's and mixing actions

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M denotes a finite von Neumann algebra (with separable predual), and τ denotes a normal, tracial state on M.

A will be an abelian von Neumann subalgebra of M and B will be an arbitrary von Neumann subalgebra of M.

A *singular MASA* in M is an abelian von Neumann subalgebra A whose normalizer in M

$$\mathcal{N}_M(A) := \{u \in U(M) \ : \ uAu^* = A\}$$

is equal to $U(A)$, i.e. is as small as possible (Dixmier, 1954).

S. Popa (1983) : Every separable type II_1 factor contains singular MASA's. However, it is hard in general to prove that a given MASA is singular.

Example. (F. Radulescu, 1991) Let $L(F_N)$ be the factor associated to the non abelian free group on N generators $X_1, \ \ldots \ X_N$ and let A be the abelian von Neumann subalgebra generated by $X_1 + \ \ldots \ + X_N + X_1^{-1} + \ \ldots \ + X_N^{-1}$. Then A is a singular MASA in $L(F_N)$. A is called the *radial* or *Laplacian* subalgebra.

T. Bildea (2007) : For every positive integer k, the corresponding Laplacian subalgebra is a singular MASA in $L(F_N)^{\bar{\otimes} k}$.

A. Sinclair and R. Smith (2002) : An abelian von Neumann subalgebra A of M is *strongly singular* if

$$\|E_A - E_{uAu^*}\|_{\infty,2} := \sup_{x \in M, \|x\| \leq 1} \|E_A(x) - E_{uAu^*}(x)\|_2 \geq \|u - E_A(u)\|_2$$

for every unitary $u \in M$. Obviously, strong singularity implies singularity.

In fact, it was proved by Sinclair, Smith, White and Wiggins in 2005 that all singular MASA's are strongly singular. Nevertheless, it is sometimes easier to prove directly strong singularity.

A sufficient condition (denoted henceforth by (SS)) :
Proposition. *(Robertson, Sinclair, Smith, 2003) Suppose that the pair $A \subset M$ satisfies the following condition : $\forall x, y \in M$ and $\forall \varepsilon > 0$, there exists $v \in U(A)$ such that*

$$\|E_A(xvy) - E_A(x)vE_A(y)\|_2 \leq \varepsilon.$$

Then A is a strongly singular MASA in M.

Proof. Fix $u \in U(M)$ and $\varepsilon > 0$, and take $x = u^*$, $y = u$. There exists $v \in U(A)$ such that

$$\|E_A(u^*vu) - E_A(u^*)vE_A(u)\|_2 = \|E_A(v^*u^*vu) - E_A(u^*)E_A(u)\|_2 \leq \varepsilon.$$

(Commutativity of A is essential here !) Hence, we get :

$$\begin{array}{rcl} \|E_A - E_{uAu^*}\|_{\infty,2}^2 & \geq & \|v - uE_A(u^*vu)u^*\|_2^2 \\ & = & \|u^*vu - E_A(u^*vu)\|_2^2 \\ & = & 1 - \|E_A(u^*vu)\|_2^2 \\ & \geq & 1 - (\|E_A(u^*)vE_A(u)\|_2 + \varepsilon)^2 \\ & \geq & 1 - (\|E_A(u)\|_2 + \varepsilon)^2 \\ & = & \|u - E_A(u)\|_2^2 - 2\varepsilon\|E_A(u)\|_2 - \varepsilon^2. \end{array}$$

As ε is arbitrary, we get the desired inequality. $\square$

Earlier, Sinclair and Smith (2002) used a stronger condition (AH) : Given $v \in U(A)$, the conditional expectation E_A is an *asymptotic homomorphism with respect to* v if

$$\lim_{|k|\to\infty} \|E_A(xv^k y) - E_A(x)v^k E_A(y)\|_2 = 0$$

for all $x, y \in M$.

Both conditions (SS) and (AH) remind mixing properties of group actions because of the following equality (A abelian is crucial) :

$$\|E_A(vxv^* y) - E_A(x)E_A(y)\|_2 = \|E_A(xv^* y) - E_A(x)v^* E_A(y)\|_2$$

$\forall x, y \in M,\ \forall v \in U(A)$.

Let Γ be a (countable) group and let α be a τ-preserving action of Γ on M. Recall that it is *weakly mixing* if, for every finite set $F \subset M$ and for every $\varepsilon > 0$, there exists $g \in \Gamma$ such that

$$|\tau(\alpha_g(a)b) - \tau(a)\tau(b)| < \varepsilon \quad \forall a, b \in F.$$

Relative version (S. Popa, 2005) :
If $1 \in B \subset M$ is a von Neumann subalgebra such that $\alpha_g(B) = B \; \forall g \in \Gamma$, the action α is called *weakly mixing relative to* B if, for every finite set $F \subset M \ominus N$, for every $\varepsilon > 0$, one can find $g \in \Gamma$ such that

$$\|E_B(x^*\alpha_g(y))\|_2 \leq \varepsilon \quad \forall x, y \in F.$$

Lemma. *(S. Popa) α is weakly mixing relative to B if and only if, every $\xi \in L^2(\langle M, e\rangle, \mathrm{Tr})$ which is Γ-invariant belongs to $L^2(e\langle M, e\rangle e) = L^2(Be)$.* (The corresponding action on $\langle M, e\rangle$ is given by $\alpha_g^B(xey) := \alpha_g(x)e\alpha_g(y)$ for all $x, y \in M$.)

In this talk from now on : G is a (countable) subgroup of the unitary group $U(A)$ and it acts on M by conjugation :

$$\sigma_v(x) = vxv^* \quad \forall v \in G,\ \forall x \in M.$$

Definition 1. The abelian von Neumann subalgebra A is *weakly mixing in M* if there exists a subgroup G of $U(A)$ such that the corresponding action by conjugation is weakly mixing relative to A in Popa's sense.

As already observed, it is equivalent to say that for every finite set $F \subset M$ and for every $\varepsilon > 0$, there exists $v \in G$ such that

$$\|E_A(xvy) - E_A(x)vE_A(y)\|_2 \leq \varepsilon \quad \forall x, y \in F.$$

Proposition 1. *If A is weakly mixing in M then it is a strongly singular MASA in M.*

Proposition 2. *Let Γ_0 be an abelian group which acts on a finite von Neumann algebra N and which preserves a trace τ, then the abelian von Neumann subalgebra $A = L(\Gamma_0)$ of the crossed product $M = N \rtimes \Gamma_0$ is weakly mixing in M iff the action of Γ_0 is.*

Examples.

Let Γ be a group and let Γ_0 be an abelian subgroup of Γ. Set $M = L(\Gamma)$ and $A = L(\Gamma_0)$. Here is a condition which ensures that A is a weakly mixing MASA in M :

Proposition 3. *(Robertson, Sinclair, Smith, 2003) If the pair* (Γ, Γ_0) *satisfies the following condition (SS) :*
for all finite sets $C, D \subset \Gamma \setminus \Gamma_0$*, one can find* $\gamma \in \Gamma_0$ *such that* $g\gamma h \notin \Gamma_0$ *for every* $g \in C$ *and every* $h \in D$
then $L(\Gamma_0)$ *is a weakly mixing MASA in* $L(\Gamma)$*.*

Geometric examples (RSS 2003) : Γ is a group of isometries of some metric space (X, d) and there exists a Γ_0-invariant subset Y of X such that

(C1) there exists a compact set $K \subset Y$ such that $\Gamma_0 K = Y$;

(C2) if $Y \subset_\delta g_1 Y \cup g_2 Y \cup \ \ldots \ \cup g_n Y$ for some g_j's in Γ, and some $\delta > 0$, then there exists j such that $g_j \in \Gamma_0$.

Then the pair (Γ, Γ_0) satisfies condition (SS). In a first class of examples, Γ is the fundamental group of a compact locally symmetric space, hence $X = SL_n(\mathbb{R})/SO_n(\mathbb{R})$ and $\Gamma_0 = \langle \gamma_0 \rangle$ where γ_0 is the class of a geodesic of minimal length.
In a second class, X is a locally finite Euclidean building and $\Gamma = Aut(X)$ satisfying analoguous suitable conditions.

Let F be the Thompson's group ; it admits the following presentation :

$$F = \langle x_0,\ x_1,\ \ldots \mid x_i^{-1} x_n x_i = x_{n+1},\ 0 \leq i < n \rangle.$$

Let Γ_0 be the subgroup generated by x_0. Then (J 2005) the pair (F, Γ_0) satisfies a (strictly) stronger condition than (SS) :

Definition 2. Let Γ be a group and let Γ_0 be an abelian subgroup of Γ. Then the pair (Γ, Γ_0) is said to satisfy *condition (ST)* if, for all finite sets $C, D \subset \Gamma \setminus \Gamma_0$, there exists a finite set $E \subset \Gamma_0$ such that $g\gamma h \notin \Gamma_0$ for all $\gamma \in \Gamma_0 \setminus E$.

It turns out that condition (ST) is completely characterized by the pair of von Neumann algebras $L(\Gamma_0) \subset L(\Gamma)$.

To see that, say that a subset S of the unitary group $U(M)$ is *almost orthonormal* if, for every $\varphi \in M_\star$ and for every $\varepsilon > 0$, there exists a finite subset $E \subset S$ such that $|\varphi(u)| \leq \varepsilon$ for all $u \in S \setminus E$.

For example, if $(u_k)_{k\geq 1}$ is a sequence of unitaries that tend to 0 weakly, then $S = \{u_k \mid k \geq 1\}$ is almost orthonormal ; the image in $L(\Gamma)$ of any infinite subgroup Γ_1 of Γ is almost orthonormal.

Let Γ be a (countable) group and let α be a τ-preserving action of Γ on M. Recall that it is *strongly mixing* if, for every finite set $F \subset M$ and for every $\varepsilon > 0$, there exists a finite set $E \subset \Gamma$ such that

$$|\tau(\alpha_g(a)b) - \tau(a)\tau(b)| < \varepsilon$$

for all $a, b \in F$ and all $g \notin E$.

Definition 3. Let M and τ be as in Section 1, let A be an abelian, unital von Neumann subalgebra of M and let G be a subgroup of $U(A)$. We say that the action of G is *strongly mixing relative to* A if, for all $x, y \in M$, one has :

$$\lim_{u \to \infty, u \in G} \|E_A(uxu^{-1}y) - E_A(x)E_A(y)\|_2 = 0.$$

Definition 4. Let M and A be as above. We say that A is *strongly mixing* in M if, for every almost orthonormal infinite subgroup G of $U(A)$, the action of G by inner automorphisms on M is strongly mixing relative to A.

Theorem. *Let Γ be an infinite group and let Γ_0 be an infinite abelian subgroup of Γ. Let $M = L(\Gamma)$ and $A = L(\Gamma_0)$ be as above. Then the following conditions are equivalent :*

(1) *the action of Γ_0 by inner automorphisms on M is strongly mixing relative to A;*

(2) *the pair (Γ, Γ_0) satisfies condition (ST), i.e. for all finite subsets $C, D \subset \Gamma \setminus \Gamma_0$ there exists a finite subset $E \subset \Gamma_0$ such that $gg_0h \notin \Gamma_0$ for all $g_0 \in \Gamma_0 \setminus E$, all $g \in C$ and all $h \in D$;*

(3) *for every almost orthonormal infinite subset $S \subset U(A)$, for all $x, y \in M$ and for every $\varepsilon > 0$, there exists a finite subset $F \subset S$ such that*

$$\|E_A(uxu^*y) - E_A(x)E_A(y)\|_2 < \varepsilon \quad \forall u \in S \setminus F;$$

(4) *A is strongly mixing in M.*

As in the case of weak mixing, we also have :

Proposition 4. *Let Γ_0 be an abelian group which acts on a finite von Neumann algebra N and which preserves a trace τ, then the abelian von Neumann subalgebra $A = L(\Gamma_0)$ of the crossed product $M = N \rtimes \Gamma_0$ is strongly mixing in M iff the action of Γ_0 is.*

Proposition 4 and a theorem of K. Schmidt (1984) prove that strongly mixing MASA's are not weakly mixing in general :

Let Γ_0 be an infinite abelian group and let α be a measure-preserving, free, weakly mixing but not strongly mixing action on some standard probability space $(X, \mathcal{B}, \mu)$. Set $N = L^\infty(X, \mathcal{B}, \mu)$ and let M be the corresponding crossed product II_1-factor. Then the abelian subalgebra $A = L(\Gamma_0)$ is a weakly mixing MASA in M, but it is not strongly mixing.

Typical examples of strongly mixing actions : Consider a finite von Neumann algebra $B \neq \mathbb{C}$ gifted with some trace τ_B, let Γ_0 be an infinite abelian group that acts *properly* on a countable set X : for every finite set $Y \subset X$, the set $\{g \in \Gamma_0 \; ; \; g(Y) \cap Y \neq \emptyset\}$ is finite. Let $(N, \tau) = \bigotimes_{x \in X}(B, \tau_B)$ be the associated infinite tensor product. Then the corresponding Bernoulli shift action is the action σ of Γ_0 on N given by

$$\sigma_g(\otimes_{x \in X} b_x) = \otimes_{x \in X} b_{gx}$$

for every $\otimes_x b_x \in N$ such that $b_x = 1$ for all but finitely many x's. Then it is easy to see that properness of the action implies that σ is a strongly mixing action. The classical case corresponds to the simply transitive action by left translations on Γ_0.

Examples

From now on, we consider pairs (Γ, Γ_0) where Γ_0 is an abelian subgroup of Γ.

Y. Stalder (2006) : Let Γ be an HNN-extention $HNN(\Lambda, H, K, \phi)$ where H, K are subgroups of Λ and $\phi : H \to K$ is an isomorphism. Denote by t the stable letter such that $t^{-1}ht = \phi(h)$ for all $h \in H$ and by Γ_0 the subgroup generated by t. (Γ is generated by Λ and by t and it just has to satisfy relations of Λ and $t^{-1}ht = \phi(h)$.) For every positive integer j, the domain of ϕ^j, denoted by $\mathrm{Dom}(\phi^j)$, is defined by $\mathrm{Dom}(\phi) = H$ for $j = 1$ and, by induction, $\mathrm{Dom}(\phi^j) = \phi^{-1}(\mathrm{Dom}(\phi^{j-1}) \cap K) \subset H$ for $j \geq 2$.

Then

Proposition 5. *(Y. S.) Suppose that for each $\lambda \in \Lambda \setminus \{1\}$, there exists $j > 0$ such that $\lambda \notin \mathrm{Dom}(\phi^j)$. Then the pair (Γ, Γ_0) satisfies condition (ST), hence $L(\Gamma_0)$ is a strongly mixing MASA in $L(\Gamma)$.*

The pair $(F, \langle x_0 \rangle)$ is a special case of Proposition 5 :

Denote by F_k the subgroup of F generated by $(x_n)_{n \geq k}$ and let σ denote the "shift map" defined by $\sigma(x_n) = x_{n+1}$, for $n \geq 0$. Its restriction to F_k is an isomorphism onto F_{k+1}, and in particular, the inverse map $\phi : F_2 \to F_1$ is an isomorphism which satisfies $\phi(x) = x_0 x x_0^{-1}$ for every $x \in F_2$. It is evident that F is the HNN extension $HNN(F_2, F_1, \phi)$ with $t = x_0^{-1}$ as stable letter.

As a second example, consider the *Baumslag-Solitar group* $BS(m,n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$. Set $\Gamma_0 = \langle a \rangle$. Then $BS(m,n)$ is an HNN-extention $HNN(\mathbb{Z}, n\mathbb{Z}, m\mathbb{Z}, \phi)$ where $\phi(nk) = mk$ for every integer k. Then the pair $(BS(m,n), \langle a \rangle)$ satisfies hypothesis of Proposition 5 when $|n| \neq |m|$. Observe that these pairs (m,n) are precisely the values for which $BS(m,n)$ is an ICC group (Y. S.).

Let us look at examples where Γ_0 is not cyclic (inspired by Sinclair and Smith, 2005) : let $\mathbb{Q}$ be the additive group of rational numbers and denote by $\mathbb{Q}^\times$ the multiplicative group of nonzero rational numbers.

For each positive integer n, set

$$F_n = \{\frac{p}{q} \cdot 2^{kn} \ ; \ p, q \in \mathbb{Z}_{\mathrm{odd}}, \ k \in \mathbb{Z}\} \subset \mathbb{Q}^\times$$

and

$$F_\infty = \{\frac{p}{q} \ ; \ p, q \in \mathbb{Z}_{\mathrm{odd}}\} \subset \mathbb{Q}^\times.$$

Next, for $n \in \mathbb{N} \cup \{\infty\}$, set

$$\Gamma(n) = \left\{ \left(\begin{array}{cc} f & x \\ 0 & 1 \end{array} \right) \ ; \ f \in F_n, \ x \in \mathbb{Q} \right\}$$

and let $\Gamma_0(n)$ be the subgroup of diagonal elements of $\Gamma(n)$. $\Gamma(n)$ is an ICC, amenable group. Then the pair $(\Gamma(n), \Gamma(n)_0)$ satisfies condition (ST) for every n.

However, if we consider larger matrices, the corresponding pairs of groups do not satisfy condition (ST). Let us fix two positive integers m and n, and set

$$\Gamma(m,n) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & f_1 & 0 \\ 0 & 0 & f_2 \end{pmatrix} \; ; \; f_1 \in F_m,\ f_2 \in F_n,\ x, y \in \mathbb{Q} \right\}$$

and let $\Gamma_0(m,n)$ be the corresponding diagonal subgroup. Then we have :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which belongs to Γ_0 for all $f \in F_m$. Thus the pair $(\Gamma(m,n), \Gamma(m,n)_0)$ does not satisfy condition (ST), though it satisfies condition (SS).

S. Popa (1983) used so-called *malnormal* subgroups to obtain orthogonal pairs of von Neumann subalgebras ; they were also used by Sinclair and Smith (2002) to get asymptotic homomorphisms. Recall that a subgroup Γ_0 of Γ is *malnormal* in Γ if it satisfies :

$(\star)$ *For every* $g \in \Gamma \setminus \Gamma_0$, *one has* $g\Gamma_0 g^{-1} \cap \Gamma_0 = \{1\}$.

Then condition $(\star)$ implies condition (ST), and, if Γ_0 is torsion-free, the converse holds. However, condition $(\star)$ is strictly stronger than condition (ST) in general.

Let $\Gamma = \Gamma_0 \star_Z \Gamma_1$ be an amalgamated product with Γ_0 abelian, Z being finite such that $Z \neq \{1\}$ and that there exists $g \in \Gamma_1 \setminus Z$ such that $zg = gz$ for every $z \in Z$. Then $g\Gamma_0 g^{-1} \cap \Gamma_0 \supset Z$, and (Γ, Γ_0) does not satisfy $(\star)$, but it can be proved as for HNN-extentions that Γ satisfies condition (ST).