Free Talagrand Inequality, a Simple Proof

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A Joke

If $F : [0, 1] \to \mathbb{R}$ is a smooth convex function such that $F(0) = F'(0) = 0$, then

$$F(t) \geq 0 \text{ for any } t \in [0, 1].$$

Proof. F is convex $\Longrightarrow$ F' is nondecreasing $\square$

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The Wasserstein Distance on $\mathbb{R}^n$

$$W(\mu,\nu) := \sqrt{\inf_{\pi\in\Pi(\mu,\nu)} \iint |x-y|^2 \pi(dx,dy)},$$

$\Pi(\mu,\nu)$: probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals μ and ν.

- This is a distance for the topology of weak convergence on the probability measures with second moment.

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- For given μ and ν, there is a “unique” transport map T such that $\pi = (T, Id)_* \nu$ is the minimizer for $W(\mu,\nu)$ and

$$W(\mu,\nu)^2 = \int |x - T(x)|^2 \nu(dx).$$

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$$W(\mu,\nu)^2 = \int |x - T(x)|^2 \nu(dx).$$
- $\mu_t = ((1-t)x + tT(x))_*\nu$ is the geodesic path for W with
$$W(\mu_t,\nu) = tW(\mu,\nu).$$

Classical Talagrand Inequality

Relative entropy: $H(\mu|\nu) = \begin{cases} \int f \log(f) d\nu & \mu \ll \nu, f = \frac{d\mu}{d\nu} \\ \infty & \mu \not\ll \nu. \end{cases}$

Theorem (Talagrand). *Assume* $\nu(dx) = e^{-\xi(x)}dx$ *is a probability measure on* $\mathbb{R}^n$ *such that* $\xi(x) - \rho|x|^2$ *is a convex function. Then*

$$\rho W(\mu,\nu)^2 \leq H(\mu|\nu). \tag{$T(\rho)$}$$

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- Notice here the fact that n does not appear in $T(\rho)$.
- $T(\rho)$ extends to infinite dimensional case with the Wiener measure in place of ν.

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Proof. Take $n = 1$ and the (smooth) map T such that $\mu = T_*\nu = f\nu$. Then

$$T'(x)f(T(x))e^{-\xi(T(x))} = e^{-\xi(x)}$$

$$E(\mu|\nu) - \rho W(\mu,\nu)^2 = \int (\xi(T(x)) - \xi(x) - \log(T'(x)) - \rho(x - T(x))^2)\nu(dx).$$

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Take $T_t(x) = (1 - t)x + tT(x)$, $\mu_t = (T_t)_* \nu$ and

$$\begin{aligned} F(t) &= H[\mu_t|\nu] - \rho W(\mu_t, \nu) \\ &= \int (\xi(T_t) - \xi(x) - \log(T_t'(x)) - \rho(x - T_t(x))^2) \nu(dx). \end{aligned}$$

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- $F(0) = 0$ and

$$F'(0) = \int \left(\xi'(x)e^{-\xi(x)}(T(x) - x) - (T'(x) - 1)e^{-\xi(x)}\right) dx$$
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- Now, joke to get that $F(t) \geq 0$. In particular

$$
H(\mu|\nu) - \rho W(\mu,\nu)^2 = F(1) \geq 0.
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Talagrand, Log Sobolev and HWI Inequalities

A measure ν is said to satisfy $T(\rho)$ if for any probability μ

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ν satisfies Log Sobolev with constant ρ ($LSI(\rho)$) if for any μ,

$$H(\mu|\nu) \leq \frac{1}{4\rho} I(\mu|\nu)$$

where $I(\mu|\nu) = \int \left|\nabla \log\left(\frac{d\mu}{d\nu}\right)\right|^2 d\mu = 4 \int \left|\nabla \sqrt{\frac{d\mu}{d\nu}}\right|^2 d\nu.$

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ν satisfies $HWI(\rho)$ inequality if for any μ:

$$H(\mu|\nu) \leq W(\mu|\nu) \sqrt{I(\mu|\nu)} - \rho W(\mu|\nu)^2.$$

Theorem (Otto and Villani). *Assume* $\nu = e^{-\xi(x)}dx$ *is a probability measure on* $\mathbb{R}^n$.

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4. *In particular for* $\xi(x) - \rho|x|^2$ *is convex for* $\rho > 0$, *then* $T(\rho)$ *is equivalent with* $LSI(\rho)$.

The main idea

The proof is based on the geometric/PDE interpretation of the gradient of the entropy functional $H(\mu|\nu)$.

If ν is a fixed measure and μ is another measure, then

$$I(\mu|\nu) = \|\mathrm{grad}_\mu H(\cdot|\nu)\|^2.$$

$LSI(\rho)$ translates in this context

$$H(\mu|\nu) \leq \frac{1}{4\rho}\|\mathrm{grad}_\mu H(\cdot|\nu)\|^2.$$

The "gradient flow" μ_t:

$$\dot{\mu}_t = -\mathrm{grad}_{\mu_t} H(\cdot|\nu) \quad \text{with} \quad \mu_0 = \mu.$$

This combined with $LSI(\rho)$ implies that

$$\frac{d}{dt} H(\mu_t|\nu) = \langle \mathrm{grad}_{\mu_t} H(\cdot|\nu), \dot{\mu}_t \rangle = -\|\mathrm{grad}_{\mu_t} H(\cdot|\nu)\|^2 \leq -4\rho H(\mu_t|\nu)$$

which yields that $H(\mu_t|\nu)$ converges exponentially fast to 0,
$\lim_{t\to\infty} \mu_t = \nu$ and

$$g(t) = W(\mu, \mu_t) + \sqrt{H(\mu_t|\nu)/\rho}$$

is nonincreasing in t. Therefore, $0 = g(\infty) \leq g(0)$ results with $T(\rho)$:

$$W(\mu, \nu) \leq \sqrt{H(\mu_t|\nu)/\rho}.$$

The Free Counterpart

Free entropy with potential Q for probability measures on $\mathbb{R}$:

$$E(\mu) = \int Q(x)\mu(x) - \iint \log|x-y|\mu(dx)\mu(dy).$$

There is a unique probability measure μ_Q such that

$$E(\mu_Q) = \inf_{\mu} E(\mu).$$

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The variational characterization of μ_Q: for a.e. $x \in \operatorname{supp}\mu$

$$Q(x) = 2\int \log|x-y|\mu_Q(dx) \quad \Rightarrow Q'(x) = \int \frac{2}{x-y}\mu_Q(dx).$$

If $Q(x) = x^2/2$ the minimizer for the free entropy is

$$\mu_Q(dx) = s(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}dx.$$

The analog of the relative entropy is played by

$$E(\mu|\mu_Q) = E(\mu) - E(\mu_Q).$$

The relative free information:

$$I(\mu|\mu_Q) = \int (H\mu(x) - Q'(x))^2 \mu(dx)$$

where $H\mu(x) = \int \frac{2}{x-y}\mu(dx)$.

Free Talagrand:

Theorem (Biane and Voiculescu). *For* $Q(x) = x^2/2$*,*

$$\frac{1}{2}W(\mu, s)^2 \leq E(\mu|s).$$

The technique of the proof is similar to the one of Otto and Villani for the classical case.

The Idea: If S is a semicircular free with X, then

$$X(t) = e^{-t/2}X + (1 - e^{-t})^{1/2}S, \quad \frac{d}{dt}H(X(t)|S) = I(X(t)|S)$$

$t \to W(X(t), S) - \sqrt{2H(X(t)|S)}$ is nondecreasing

$$\lim_{t\to\infty}(W(X(t), S) - \sqrt{2H(X(t)|S)}) = 0.$$

Random Matrices and Free Entropy

On $\mathcal{M}_n^{sa}(\mathbb{C})$ consider the probability measure

$$\mathcal{P}_Q^n(dA) = \frac{1}{Z_n(Q)} e^{-n\mathrm{Tr_n Q(A)}} dA.$$

The distribution of the eigenvalues is given by

$$\Lambda_n(dx) = \frac{1}{\overline{Z}_n(Q)} e^{-n\sum_{i=1}^n Q(x_i)} \prod_{1\le i<j\le n} (x_i - x_j)^2 \prod_{i=1}^n dx_i$$

$$= \frac{1}{\overline{Z}_n} \exp\left(-n^2\left(\frac{1}{n}\sum_{i=1}^n Q(x_i) - \frac{1}{n^2}\sum_{1\le i\ne j\le n} \log|x_i - x_j|\right)\right)\prod_{i=1}^n dx_i.$$

B. Arous and A. Guionnet:

Take λ_n to be the distribution on of $\eta_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}$ under Λ_n. Then λ_n satisfies a "Large Deviation Principle" with rate function given by $H(\mu|\mu_Q)$ i.e. for any measurable set A of probability measures on $\mathbb{R}$,

$$-\inf_{\mu\in\mathring{A}} E(\mu|\mu_Q) \leq \liminf_{n\to\infty} \frac{1}{n^2}\log(\lambda_n(A))$$
$$\leq \limsup_{n\to\infty} \frac{1}{n^2}\log(\lambda_n(A)) \leq -\inf_{\mu\in\overline{A}} E(\mu|\mu_Q)$$

In particular $\eta_n \to \mu_Q$.

Hiai, Petz and Ueda:

$$\mathcal{P}^n_Q(dA) = \frac{1}{Z_n(Q)} e^{-n\mathrm{Tr_n Q(A)}} dA.$$

For a given μ, take $Q_\mu(x) = \int \log|x-y|\mu(dy)$.

If $Q(x) - \rho x^2$ is a convex function, then $A \to nTr_n Q(A) - n\rho|A|^2$ is convex, and from classical Talagrand:

$$n\rho W(\mathcal{P}^n_{Q_\mu}, \mathcal{P}^n_Q)^2 \leq H(\mathcal{P}^n_{Q_\mu}|\mathcal{P}^n_Q)$$

$$W(\mu, \mu_Q)^2 \leq \liminf_{n\to\infty} W(\mathcal{P}^n_{Q_\mu}, \mathcal{P}^n_Q)^2/n$$

$$\lim_{n\to\infty} H(\mathcal{P}^n_{Q_\mu}|\mathcal{P}^n_Q)/n^2 = E(\mu|\mu_Q)$$

$$\rho W(\mu, \mu_Q)^2 \leq E(\mu|\mu_Q).$$

Free Talagrand, Log Sobolev and HWI Inequalities

For a given potential Q, the free Talagrand $T(\rho)$:

$$\rho W(\mu, \mu_Q)^2 \leq E(\mu|\mu_Q), \quad \text{for any } \mu.$$

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Log Sobolev $LSI(\rho)$ if for any μ,

$$E(\mu|\mu_Q) \leq \frac{1}{4\rho} I(\mu|\mu_Q)$$

where $I(\mu|\mu_Q) = \int (H\mu(x) - Q'(x))^2 \mu(dx)$, $H\mu(x) = \int \frac{2}{x-y} \mu(dx)$.

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$HWI(\rho)$ inequality if for any μ:

$$E(\mu|\mu_Q) \leq W(\mu, \mu_Q) \sqrt{I(\mu|\mu_Q)} - \rho W(\mu, \mu_Q)^2.$$

The Simpler Proof of $T(\rho)$

If $Q(x) - \rho x^2$ is convex for a certain $\rho > 0$, then $T(\rho)$ holds true.

Let T be the transport map of μ_Q into μ, i.e. $\mu = T_*\mu_Q$. T is nondecreasing. Set $T_t(x) = (1-t)x + tT(x)$, $\mu_t = (T_t)_*\mu_Q$ and

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\begin{aligned}
F(t) &= E(\mu_t|\mu_Q) - \rho W(\mu_t, \mu_Q)^2 \\
&= -E(\mu_Q) + \int \left(Q(T_t(x)) - \rho(x - T_t(x))^2\right) \mu_Q(dx) \\
&\quad -2 \iint_{x>y} \log(T_t(x) - T_t(y))\mu_Q(dx)\mu_Q(dy)
\end{aligned}
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- F is convex;
- $F(0) = 0$;
- From $Q'(x) = 2 \int \frac{1}{x-y} \mu_Q(dx)$, it follows that

$$F'(0) = \int (T(x) - x) Q'(x) \mu_Q(dx) - 2 \iint \frac{T(x) - x}{x - y} \mu_Q(dx) \mu_Q(dy)$$
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- Therefore, $F(t) \geq 0$ and then

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Using these ideas and some from Erausquin, M. Ledoux reproved $LSI(\rho)$ and, for the first time, proved $HWI(\rho)$.

The circle case

$Q : \mathbf{T} \approx (-\pi, \pi] \to \mathbb{R}$ is the potential and call $V(x) = Q(e^{ix})$.

$$E(\mu) = \int V(x)\mu(dx) - \iint \log|e^{ix} - e^{iy}|\mu(dx)\mu(dy)$$

There is a measure μ_Q such that $E_Q(\mu)$ is minimized. We define then

$$E(\mu|\mu_Q) = E(\mu) - E(\mu_Q).$$

$$I_Q(\mu) = \int [H\mu(x) - V'(x)]^2 \mu(dx) - \left(\int V'(x)\mu(dx)\right)^2$$

where

$$H\mu(x) = \int \cot(x-y)\mu(dy).$$

If $Q(e^{ix}) - \rho x^2$ is convex on $\mathbb{R}$, then $T(\rho)$ is

$$(\rho + 1/4)W(\mu, \mu_Q)^2 \le E[\mu|\mu_Q],$$

the $LSI(\rho)$ is

$$E[\mu|\mu_Q] \le \frac{1}{1+4\rho} I(\mu|\mu_Q)$$

and $HWI(\rho)$ is

$$E^Q(\mu|\mu_Q) \le W(\mu, \mu_Q) I(\mu)^{1/2} - (\rho + 1/4)W(\mu, \mu_Q)^2$$

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- Is there a version of Talagrand's inequality for a tuple of non-commutative random variable with the logarithmic entropy replaced by Voiculescu's free entropy? Partial result was proved using random matrix approximation by Hiai and Ueda for the case

$$E(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} \tau(Q_i(a_i)) - \chi(a_1, a_2, \ldots, a_n).$$