A trace on the C*-algebra of geometric operators on self-similar graphs

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Sibiu, June 12th, 2007

Outline

1. Traces for geometric operators on graphs and CW-complexes
2. Applications:
   - Geometric invariants for self-similar CW-complexes (joint work with F. Cipriani and T. Isola, [math.OA/0607603]).
   - Ihara Zeta functions (joint work with T. Isola and M. Lapidus, [math.OA/0605753, to appear on Proceedings of Bedlewo - math.OA/0608060, to appear on Transactions AMS - math.OA/0608229]).
   - Bose-Einstein condensation for pure-hopping Hamiltonian on graphs (joint work with F. Fidaleo and T. Isola, work in progress).

Amenable graphs.

Graph: $X = (VX, EX)$, VX vertices, EX edges.
Simple: an edge is an unordered pair of distinct vertices.
Distance: Let $\partial : \ell^2(EX) \to \ell^2(VX)$ the boundary operator, and say that for $v_1 \neq v_2 \in VX$, $d(v_1, v_2) = 1$ if $(\partial^* v_1, \partial^* v_2) \neq 0$. Then endow VX with the corresponding path distance.
Amenable graph: An exhaustion $\mathcal{K} = \{K_n\}_{n \in \mathbb{N}}$ by finite subgraphs of X is amenable if

$$\lim_n \frac{|V\mathcal{F}K_n|}{|VK_n|} = 0,$$

where $v \in V\mathcal{F}K_n$ if $v \in VK_n$ and has distance 1 from a vertex in $(VK_n)^c$. A graph is amenable if it has an amenable exhaustion.

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A trace on finite propagation operators.

An operator $T \in \mathcal{B}(\ell^2(VX))$ has *finite propagation* ρ if $d(v_1, v_2) > \rho$ implies $(\delta_{v_1}, T\delta_{v_2}) = 0$.

Proposition

The norm closure $\mathcal{A}_X^{FP}$ of finite propagation operators is a C^*-algebra.

$(X, \mathcal{K})$ amenable graph, $\mathcal{A} \subset \mathcal{A}_X^{FP}$ C*-algebra.

Limit condition: $\forall T \in \mathcal{A} \quad \exists \lim_n \frac{\operatorname{tr} TP_n}{\operatorname{tr} P_n}$, where P_n denotes the orthogonal projection on $\ell^2(VK_n)$.

Theorem

In this case $\tau(T) = \lim_n \frac{\operatorname{tr} TP_n}{\operatorname{tr} P_n}$ is a trace state on $\mathcal{A}$.

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Geometric operators.

A *local isomorphism* of the graph X is a triple $\big(s(\gamma), r(\gamma), \gamma\big)$ where $s(\gamma), r(\gamma)$ are subgraphs of X and $\gamma : s(\gamma) \to r(\gamma)$ is a graph isomorphism.

Let $\mathcal{G}$ be a family of local isomorphisms. $T \in \mathcal{B}(\ell^2(VX))$ is *$\mathcal{G}$-geometric* if $\exists \rho > 0$:

- T has finite propagation ρ,
- if $\gamma \in \mathcal{G}$, $B(v, \rho) \subset s(\gamma)$, $B(\gamma v, \rho) \subset r(\gamma)$ then $(T\lambda_\gamma - \lambda_\gamma T)\delta_v = 0$ and $(T^*\lambda_\gamma - \lambda_\gamma T^*)\delta_v = 0$, where λ_γ is the partial isometry determined by
$$\begin{cases} \lambda_\gamma \delta_v = \delta_{\gamma v} & \text{if } v \in s(\gamma) \\ \lambda_\gamma \delta_v = 0 & \text{if } v \notin s(\gamma) \end{cases}$$

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The norm closure $\mathcal{A}_{X,\mathcal{G}}$ of the set of $\mathcal{G}$-geometric operators is a C^*-algebra.

Recall that $\Delta := \partial\partial^* = D - A$, where D is the diagonal degree matrix and A is the adjacency matrix. Then D and A are geometric operators.

CW-complexes

Let X be a (regular, bounded) CW-complex:

- $\mathcal{E}_j(X)$ j-cells, $\mathcal{E}(X) = \bigcup_{j=1}^{p} \mathcal{E}_j(X)$
  $\ell^2(\mathcal{E}(X)) = \oplus_{j=1}^{p} \ell^2(\mathcal{E}_j(M))$,
- Boundary: $\partial_j : \ell^2(\mathcal{E}_j(X)) \to \ell^2(\mathcal{E}_{j-1}(X))$
- Distance: $\sigma \in \mathcal{E}_{j-1}(M)$, $\tau \in \mathcal{E}_j(M)$, $d(\sigma, \tau) = 1$ if $(\sigma, \partial\tau) \neq 0$. d is extended to a distance on $\mathcal{E}(X)$ via path length.

Amenable exhaustions, local isomorphisms, finite propagation operators and $\mathcal{G}$-geometric operators can be defined analogously.

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Theorem

The norm closure $\mathcal{A}_{X,\mathcal{G}}$ of the set of $\mathcal{G}$-geometric operators is a C^*-algebra. If $\mathcal{A}_{X,\mathcal{G}}$ satisfies the limit condition, with P_n the orthogonal projection on $\oplus_j \ell^2(\mathcal{E}_j K_n)$,

$$\tau_{X,\mathcal{G},\mathcal{K}}(T) = \lim_n \frac{\operatorname{tr} TP_n}{\operatorname{tr} P_n}$$

is a trace state on $\mathcal{A}_{X,\mathcal{G}}$.

Periodic graphs (or CW-complexes)

X a simple graph with bounded degree, VX countably infinite, $\Gamma < Aut(X)$ discrete group, acting freely [i.e. Γ_v is trivial, $\forall v \in VX$], X/Γ finite graph.

Γ gives global isomorphisms for X, hence we may consider the Γ-geometric operators, and the corresponding C*-algebra $\mathcal{A}_{X,\Gamma}$. The weak closure of $\mathcal{A}_{X,\Gamma}$ is endowed with the trace state

$$\tau_{X,\Gamma}(T) = \frac{1}{|\mathcal{F}_0|} \sum_{v \in \mathcal{F}_0} (v, Tv),$$

where $\mathcal{F}_0 \subset X$ contains one representative for any point of X/Γ. If Γ is amenable, Følner condition gives an amenable exhaustion. Then, $\forall T \in \mathcal{A}_{X,\Gamma}$,

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Self-similar CW-complexes

An amenable CW-complex $(X, \mathcal{K})$ is *self-similar* if:

1. $\mathcal{G}(n, n+1)$ finite set of local isomorphisms such that:
   - $s(\gamma) = K_n$,
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2. Let $\mathcal{G}$ denote the set of all admissible products of γ's and γ^{-1}'s, $\mathcal{G}(n) = \{\gamma \in \mathcal{G} : s(\gamma) = K_n\}$. Then we ask that

$$\lim_n \frac{|\mathcal{F}_{\mathcal{G}}(\mathcal{E}_j K_n)|}{|\mathcal{E}_j K_n|} = 0, \quad \mathcal{F}_{\mathcal{G}}(\mathcal{E}_j K_n) = \bigcup_{\gamma \in \mathcal{G}(n)} \gamma_j^{-1} \mathcal{F}(\mathcal{E}_j \gamma(K_n)).$$

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$A_{X,\mathcal{G}}$ satisfies the limit condition, namely we get a trace $\tau_{X,\mathcal{G},\mathcal{K}}$.

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Selfsimilar fractal	**Selfsimilar prefractal**
$w_1, \dots w_q$ contracting similarities of $\mathbb{R}^p$	with the same scaling constant
$\mathcal{P}$ open, $w_i\mathcal{P} \subset \mathcal{P}$ $w_i\mathcal{P} \cap w_j\mathcal{P} = \emptyset$	convex polyhedron, $w_i\overline{\mathcal{P}} \cap w_j\overline{\mathcal{P}}$ =face of $\overline{\mathcal{P}}$
$W\mathcal{P} = \bigcup_{i=1}^q w_i\mathcal{P}$ $\{W^n\mathcal{P}\}_{n\in\mathbb{N}} \to F$ w.r.t. Hausdorff metrics	$w_{I\vert_n} := w_{I_n} \cdots w_{I_1}$ $\{K_n := w_{I\vert_n}^{-1} W^n\mathcal{P}\}_{n\in\mathbb{N}} \nearrow X$ amenable exhaustion

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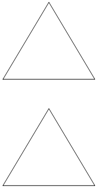

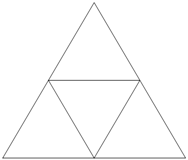

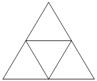

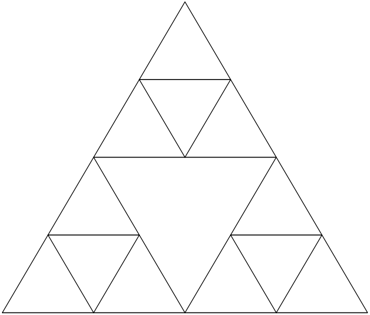

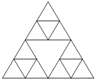

Small perturbations of graphs

Let $(X, \mathcal{K})$ be an amenable graph, $\mathcal{A}_X^{FP}$ the norm closure of the *-algebra of finite propagation operators.

Ess. zero: $T \sim 0$ if $\lim_n \dfrac{\text{tr } TP_n}{\text{tr } P_n} = 0$, $T \in \mathcal{A}_X^{FP}$.

Proposition

$\mathcal{I} := \{T \in \mathcal{A}_X^{FP} : T \sim 0\}$ is a closed two-sided ideal in $\mathcal{A}_X^{FP}$. If $\mathcal{A}_{X,\mathcal{G}}$ satisfies the limit condition for a given $\mathcal{G}$, $\tau_{X,\mathcal{G},\mathcal{K}}$ extends to a trace on $\mathcal{A}_{X,\mathcal{G}} + \mathcal{I}$.

Definition

Let $(X_1, \mathcal{K}_1)$, $(X_2, \mathcal{K}_2)$ be amenable graphs. $X_1 \sim X_2$ if there exists $(X, \mathcal{K})$ such that

- X_i *is a subgraph of* X;
- $K_{n,i} = K_n \cap X_i$;
- $I_{X_1} \sim I_{X_2}$;
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Theorem

Consider a graph $(X, \mathcal{G}, \mathcal{K})$ for which $\mathcal{A}_{X,\mathcal{G}}$ satisfies the limit condition and a graph $(X', \mathcal{K}')$ such that $X \sim X'$. We get a C^*-algebra $\mathcal{I}_{X'} \subset \mathcal{A}' \subset \mathcal{A}_{X'}^{FP}$ satisfying the limit condition, hence a trace τ' on it. If $T \in \mathcal{A}_{X,\mathcal{G}}$, $T' \in \mathcal{A}'$, and $T \sim T'$, then $\tau(T) = \tau'(T')$.

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Outline

L^2-invariants for self-semilar CW-complexes.

Let $(X, \mathcal{G}, \mathcal{K})$ a self-similar CW-complex, $T \in \mathcal{A}_{X,\mathcal{G}}$ selfadjoint, set $\mu_T : \int f \, d\mu_T = \tau(f(T))$, $f \in \mathcal{C}_0(\mathbb{R})$.
L^2-Betti numbers: $\beta_i = \mu_{\Delta_j}(\{0\})$, $j = 0, \dots, p$.
Novikov-Shubin numbers: α_j, $j = 1, \dots, p$ such that $\mu_{\partial_j^* \partial_j}([0, t)) - \beta_j \sim t^{\alpha_j/2}$.
Following Lott and Lück, we define also L^2-invariants for the relative complex $(X, \partial X)$.

- Covering case: homotopy invariance of α_j and β_j (Dodziuk 1977, Gromov-Shubin 1991).
- Self-similar case: rough isometry invariance of α for graphs (when the NS-numbers are defined). (Follows by Hambly-Kumagai, 2004)

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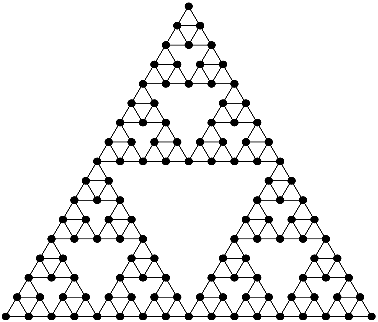
Examples of self-similar graphs



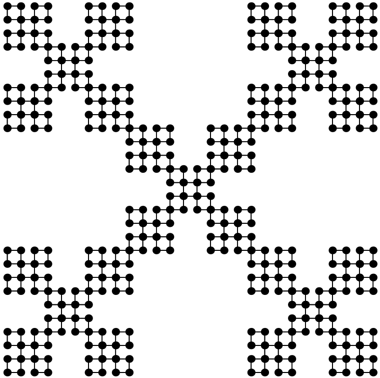
Sierpinski graph

$\alpha = 2\frac{\log 3}{\log 5}$

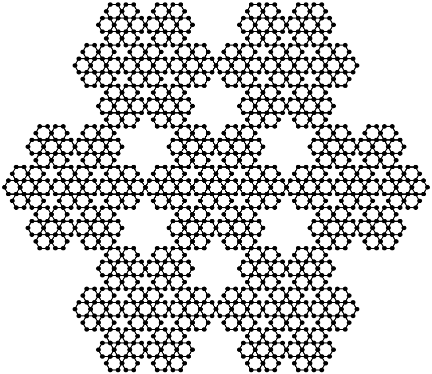
$\beta_0 = 0$

$\beta_1 = \frac{1}{3}$

M.T.Barlow, 2003



Vicsek graph
$\alpha = 2\frac{\log 5}{\log 15}$
$\beta_0 = 0$
$\beta_1 = \frac{1}{7}$
J.Kigami, M.L.Lapidus, 2001



Lindstrom graph

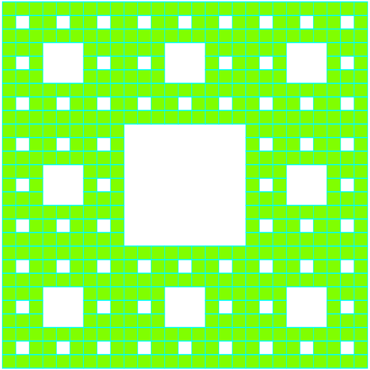
$\alpha = 2\frac{\log 7}{\log 12.89027}$

$\beta_0 = 0$

$\beta_1 = \frac{1}{5}$

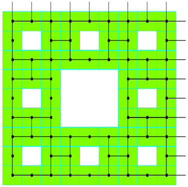
T.Kumagai, 1993

A 2-dimensional example



A 2-dimensional prefractal complex X: the Sierpinski carpet CW-complex.

Computation of $\alpha_2(X)$



Is the 2-Laplacian of X the Laplacian of a graph G, where the 2-cells are the vertices, and 2 vertices are connected by an edge if the corresponding cells have a 1-cell in common?

Indeed any boundary 1-cell of X allows the random walk to "fall in the hole", implying the corresponding NS-invariant to be trivial. To avoid this, one should excise the boundary, and consider the relative complex $(X, \partial X)$. Then

$$\alpha_2(X, \partial X) = \alpha(G).$$

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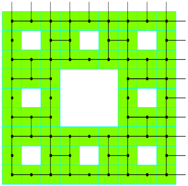

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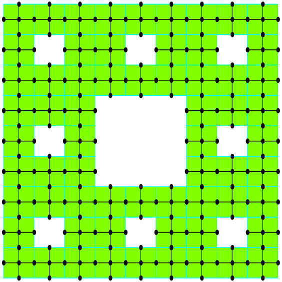

Let us consider now the graph G', obtained by X via a "cross-square" transformation, which was studied in [M.T.Barlow, R.F.Bass, (1999)].
G and G' are roughly isometric, hence

$$\alpha_2(X, \partial X) = \alpha(G) = \alpha(G') \in [1.67, 1.87].$$

Outline

1. Traces for geometric operators on graphs and CW-complexes

2. Applications
   - Geometric invariants for self-similar CW-complexes
   - Ihara Zeta functions
   - BE condensation for pure-hopping Hamiltonian on graphs

Ihara Zeta function for finite simple graphs

[Ihara (1966), Hashimoto (1989), Bass (1992)]

Let X be a finite graph.

- Path of length m: $C = (v_0, ..., v_m)$, $(v_i, v_{i+1}) \in EX$ (Closed: $v_0 = v_m$).
- Connected: any two vertices are connected by a path.

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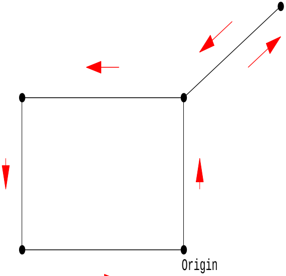
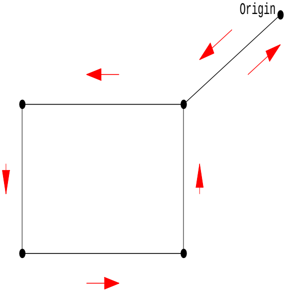
- No backtracking / no tail: $v_i \neq v_{i+2}$ [mod m].



Backtracking



Tail

A closed path with no backtracking and no tail is primitive if it is not obtained by going around some other path $r \geq 2$ times.
A cycle is a closed path modulo the starting point.
$\mathcal{P}$ denotes the class of primitive cycles (infinitely many!).
Def. [Zeta function]

$$Z_X(u) := \prod_{C \in \mathcal{P}} (1 - u^{|C|})^{-1}, \qquad u \in \mathbb{C}.$$

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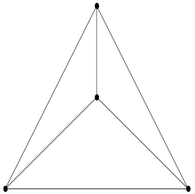

Example: $|VX| = 4, |EX| = 6,$
$\chi(X) = -2, r = 3, \kappa_X = 16,$
$\frac{1}{Z_X(u)} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$

Determinant formula
[Ihara (1966), Hashimoto (1989), Bass (1992)]

Theorem (Determinant formula)

$$\frac{1}{Z_X(u)} = (1-u^2)^{-\chi(X)} \det(\Delta(u)),$$

where $\Delta(u) = I - Au + Qu^2$,
$A =$ *adjacency matrix,* $Q = \mathrm{diag}(deg(v_1) - 1, \ldots, deg(v_n) - 1)$,
$\chi(X) = |VX| - |EX| =$ *Euler char. of* X.

Obs. $\Delta(1) = (Q + I) - A =$ graph Laplacian.

Properties of Ihara Z_X.

- Hashimoto (1989), Northshield (1998)
  $r :=$ rank of fundamental group $\pi_1(X) \equiv |EX| - |VX| + 1$ is the order of the pole of $Z_X(u)$ at $u = 1$. If $r > 1$

  $$\lim_{u \to 1^-} Z_X(u)(1-u)^r = -\frac{1}{2^r(r-1)\kappa_X},$$

  $\kappa_X =$ number of spanning trees in X.
- Hashimoto (1992), Horton, Stark, Terras (2006)
  R_X radius of convergence of Z_X
  $g.c.d.\{|C| : C \in \mathcal{P}\} = 1$. Then
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Regular graphs.

A graph is regular if any vertex has the same degree. If X is $(q+1)-$regular,

$$Z_X(q^{-s}) = \prod_{C\in\mathcal{P}} (1-(q^{|C|})^{-s})^{-1}.$$

Cf. the Euler product formula for the Riemann zeta function:

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X satisfies the Riemann Hypothesis if

$$\begin{cases} Z_X^{-1}(q^{-s}) = 0 \\ \Re s \in (0,1) \end{cases} \implies \Re s = \frac{1}{2}$$

Theorem (Ihara (1966), Lubotzky (1994))

X $(q+1)-$regular satisfies the Riemann Hypothesis iff X is a Ramanujan graph, namely

$$\lambda \in \sigma(A), |\lambda| < q+1 \implies |\lambda| \le 2\sqrt{q}.$$

Ramanujan graphs are important in communication networks.

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Ihara Zeta function for covering graphs

Let X be a connected simple graph with bounded degree, VX countably infinite, $\Gamma < Aut(X)$ discrete group, acting freely [i.e. Γ_v is trivial, $\forall v \in VX$], X/Γ finite graph.

Definition (Zeta function)

$$Z_{X,\Gamma}(u) := \prod_{[C]\in\mathcal{P}/\Gamma} (1 - u^{|C|})^{-1/|\Gamma_C|},$$

where $\mathcal{P}/\Gamma$ denotes the set of primitive cycles modulo Γ-translations, and Γ_C = the stabilizer of a cycle C.

It gives a holomorphic function in a suitable disc.

Ihara Zeta function for self-similar graphs

[G.I.L. (2006)]

Definition (Zeta function)

$$Z_{X,\mathcal{G}}(u) := \prod_{[C]\in\mathcal{P}/\mathcal{G}} (1 - u^{|C|})^{-\mu(C)},$$

where classes of cycles are modulo local isomorphisms in $\mathcal{G}$*, and* $\mu(C) =$ *average multiplicity of* C *is defined as*

$$\mu(C) = \lim_n \frac{|\{C' \subset K_n : C' \sim_{\mathcal{G}} C\}|}{|K_n|}.$$

N.B. Self-similarity implies the limit exists and is finite.

Proposition

μ_C depends only on the size of C, namely the least $m \in \mathbb{N}$ s.t. $C \subset \gamma(K_m)$, for some $\gamma \in \mathcal{G}$.

- For the Gasket graph, $\mu(C) = \frac{2}{3^{p+1}}$,
- For the Vicsek graph, $\mu(C) = \frac{1}{3 \cdot 5^p}$,
- For the Lindstrom graph, $\mu(C) = \frac{1}{5 \cdot 7^p}$,

where p is the size of C

An analytic determinant on $(\mathcal{A}, \tau)$

Proposition

Let $\mathcal{A}_0 = \{A \in \mathcal{A} : co(\sigma(A)) \not\ni 0\}$*, and set*

$$\det A = \exp \circ \tau \circ \log A, \qquad A \in \mathcal{A}_0,$$

where $\log A = \frac{1}{2\pi i} \int_{\mathcal{C}} \log \lambda \, (\lambda - A)^{-1} \, d\lambda$*, and* $\mathcal{C}$ *is a simple curve surrounding* $co(\sigma(A))$*. Then* det *is well defined and analytic on* $\mathcal{A}_0$*.*

- $\det(zA) = z^{\tau(I)} \det(A)$, for any $z \in \mathbb{C} \setminus \{0\}$,
- if A is normal, $\det(A) = \det(U)\det(H)$, where $A = UH$ is the polar decomposition,
- if A is positive, $\det(A) = Det(A)$, where the latter is the Fuglede–Kadison determinant.

$\det AB \neq \det A \det B$:

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- even if $A, B, AB \in \mathcal{A}_0$ and A and B commute, the product property may be violated.
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The determinant formula

Theorem

$$\frac{1}{Z_{X,\Gamma}(u)} = (1-u^2)^{-\chi^{(2)}(X)} \det_{\mathcal{G}}(\Delta(u)),$$ *for u in a suitable disc.*

- Periodic case: $\chi^{(2)}(X) = \chi(X/\Gamma)$.
- Self-similar case: $\chi^{(2)}(X) = \lim_{n\to\infty} \frac{\chi(K_n)}{|K_n|}$.

The determinant formula in the periodic case was fist proved by Clair and Mokhtari-Sharghi, but the determinant was defined in a completely different way.
Recall that $\Delta(u) = I - Au + Qu^2$, hence takes values in a commutative C^*-algebra if and only if Q is constant, namely the graph is regular.

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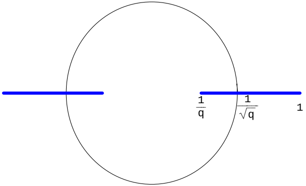
$\chi^{(2)}(X)$ for some self-similar graphs

- Gasket graph $\chi^{(2)}(X) = -1$.
- Vicsek graph $\chi^{(2)}(X) = -\frac{1}{3}$.
- Lindstrom graph $\chi^{(2)}(X) = -\frac{1}{2}$.
- Carpet graph $\chi^{(2)}(X) = -\frac{10}{11}$.

The Functional Equation



We now assume the graph X to be $(q+1)$-regular, possibly up to a finite number of vertices. Then $Z_X(u)$ extends to the complement of the curve Ω.

- The Riemann zeta admits a so called completion $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$, where Γ is the usual Gamma function, satisfying the functional equation $\xi(s) = \xi(1-s)$.
- Setting $u = q^{-s}$, the reflection $s \to 1-s$ becomes the reflection $u \to \frac{1}{qu}$.

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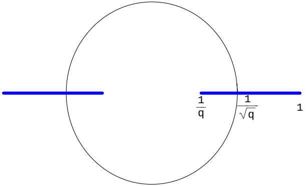

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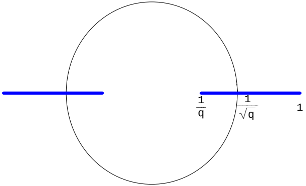

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Theorem (Functional equation)

Ihara zeta function admits a completion

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Such completion satisfies $\xi_X(u) = \xi_X\left(\frac{1}{qu}\right)$.

N.B.: The curve Ω is invariant under the reflection $u \to \frac{1}{qu}$. While in the finite graph case the singularities are only polar, hence the functional equation is a relation between meromorphic functions on $\mathbb{C}$, for infinite graphs there are branching points. The question whether the extension of the domain of $Z_{X,\Gamma}$ by means of the determinant formula is compatible with an analytic extension from the defining domain is a non-trivial issue, see [Clair, Zeta functions of graphs with $\mathbb{Z}$ actions, math.NT/0607689].

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Approximation by finite graphs

Theorem (Clair, Mokhtari-Sharghi)

X covering graph with constant degree, and assume Γ is residually finite, i.e. $\Gamma_n \searrow \{e\}$ normal subgroups of Γ, $[\Gamma : \Gamma_n] < \infty$. Then

$$Z_{X,\Gamma}(u) = \lim_{n\to\infty} Z_{B_n}(u)^{\frac{1}{[\Gamma:\Gamma_n]}}, \qquad u \in \Omega,$$

where $B_n := X/\Gamma_n$ be the tower of finite coverings of $B := X/\Gamma$.

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Outline

1. Traces for geometric operators on graphs and CW-complexes

2. Applications
   - Geometric invariants for self-similar CW-complexes
   - Ihara Zeta functions
   - BE condensation for pure-hopping Hamiltonian on graphs

Bose-Einstein condensation

$(X, \mathcal{K})$ amenable graph, H Hamiltonian, H_n the restriction to K_n. The grand-canonical ensemble on K_n is described by:

- the C*-algebra $\mathfrak{A}(K_n) := CCR(L^2(K_n))$,
- the time-evolution, implemented by the second quantization of e^{itH_n},
- the KMS state $\omega_{\beta,\mu}$ such that $\omega_{\beta,\mu}(a^*(x)a(y)) = (x, (e^{\beta H_n - \mu} - I)^{-1}y)$, (2p. funct.) $\rho_n(\beta, \mu) = \frac{\mathrm{tr}((e^{\beta H_n - \mu} - I)^{-1})}{|K_n|}$, (density).

One tries to find, on a suitable CCR algebra $\mathfrak{A}(X)$ on which the second-quantization of e^{itH} implements automorphisms, a KMS state $\omega_{\beta,\mu}$ given by a weak* limit of states ω_{β,μ_n} on $\mathfrak{A}(K_n)$, with $\mu_n \longrightarrow \mu$.

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The case $X = \mathbb{Z}^d$, $H = \Delta$

When $\mu < 0$, $\lim_n \frac{\mathrm{tr}((e^{\beta H_n - \mu_n} - I)^{-1})}{|K_n|} = \tau((e^{\beta H - \mu} - I)^{-1}) < \infty$.

Critical density: $\rho_c(\beta) = \tau((e^{\beta H} - I)^{-1}) < \infty$ if $d > 2$.

Given β, ρ one find $\mu_n : \rho_n(\beta, \mu_n) = \rho$. Then:

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Pure hopping Hamiltonian for small perturbations of periodic graphs

The pure hopping Hamiltonian is the non-diagonal part of the Laplacian, i.e. $-A$. In order to set to 0 the lower bound of the spectrum, we consider $H = \|A\| I - A$. It turns out that this Hamiltonian is sensitive to small perturbations.

Classically, e.g. for $X = \mathbb{Z}^d$, $H = \Delta$,
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In our case, transience is still necessary for BEC, and possibly sufficient, but it is not related with $\rho_c < \infty$ or $d > 2$.
We expect that, when A is transient, we can choose sequences $\mu_n \to 0$ giving rise to KMS states with two-point functions

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We expect that, when A is transient, we can choose sequences $\mu_n \to 0$ giving rise to KMS states with two-point functions

$$(x, (e^{\beta H} - I)^{-1} y) + k(x, v)(v, y),$$

where v is a generalised Perron-Frobenius eigenvector for A, i.e. the 0-energy eigenvector.

The comb graphs

The comb graph $\mathbb{Z}^d \dashv (\mathbb{Z}, 0)$ consists of a copy of the graph $\mathbb{Z}^d$ (the basis) where at each vertex is attached a copy of the graph $\mathbb{Z}$ (the fiber) at the vertex 0.

The exaustion K_n is given by boxes with a given center on the basis and side $2n$.

We consider here the restriction of H to K_n with periodic conditions, or, equivalently, we approximate $\mathbb{Z}^d \dashv (\mathbb{Z}, 0)$ with $(\mathbb{Z}^d_{2n} \dashv ([-n, n], 0)$.

The comb graph $\mathbb{Z}^d \dashv (\mathbb{Z}, 0)$ may be considered as a small perturbation of infinitely many copies of $\mathbb{Z}$, the perturbation consisting in removing the basis. We easily get that A is transient *iff* $d > 2$. In this case the critical density is finite.

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Theorem

Let $d > 2, \beta > 0$. Then, $\forall k \geq 0$, there exists a sequence $\mu_n \to 0$ such that:

- *KMS states ω_{β,μ_n} converge to a state on $CCR(\cup_n L^2(K_n))$ with two-point function*

  $$(x, (e^{\beta H} - I)^{-1} y) + k(x, v)(v, y),$$

  where v is the generalised Perron-Frobenius eigenvector for A obtained as pointwise limit of the Perron-Frobenius eigenvectors v_n for A_n taking value 1 on the basis.
- *Such limit states extend to KMS states on $CCR(\{v_{\mathbf{i}j} \in \ell^2(\mathbb{Z}^d \dashv (\mathbb{Z}, 0)) : \sum_j v_{\mathbf{i}j}^2 \in S(\mathbb{Z}^d)\})$.*
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