Linearization & Connes Embedding Property

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Overview

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This is joint work with Ken Dykema.

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1. Haagerup-Thorbjørnsen's C^*-algebra linearization 'trick'.
2. A von Neumann algebra linearization theorem.
3. About Connes embedding property (CEP) & applications of the theorem.
4. Horn theorem.
5. Bercovici Li's large N Horn theorem.
6. A new equivalent condition to CEP.

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Let A (resp. B) be a unital C^*-algebra generated by selfadjoints $X_1, \ldots, X_k$ (resp.$Y_1, \ldots, Y_k$) such that for all positive integers N and for all $a_0, \ldots, a_k \in \mathbb{M}_N(\mathbb{C})_{sa}$,

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$$a_0 \otimes 1 + a_1 \otimes X_1 + \ldots + a_k \otimes X_k$$

and

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have the same spectrum.

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have the same spectrum.
Then there exists an isomorphism ϕ between A and B such that $\phi(X_i) = Y_i$.

Question: How about a von Neumann (non-commutative) version ?

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Let τ be a faithful trace on M and χ be a faithful trace on N.
Let $c < d$ be positive real numbers and suppose that for all $n \in \mathbb{N}_*$ and all $a_1, \ldots a_k$ in $\mathbb{M}_n(\mathbb{C})_{s.a.}$ whose spectra are contained in the interval $[c, d]$,

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Then there exists an isomorphism $\phi : M \to N$ such that $\phi(X_i) = Y_i$ and $\chi \circ \phi = \tau$.

Notation

For $a_1, a_2 \in \mathbb{M}_N(\mathbb{C})_{sa}$, we call ev_{a_1,a_2} the algebra morphism

$$\mathbb{C}\langle x_1, x_2\rangle \to \mathbb{C}$$

given by

$$ev_{a_1,a_2}(P) = P(a_1, a_2).$$

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- Step I: For all monomial P in k non-commuting variables,

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is equivalent to proving that

$$\cap_{N\geq 1}\cap_{a_1,a_2\in \mathbb{M}_N(\mathbb{C})_{sa}} Ker(\mathrm{Tr}\circ ev_{a_1,a_2}) = \{[a,b], a, b \in \mathbb{C}\langle x_1, x_2\rangle\}$$

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- Observation: hypothesis $\mathrm{distr} \sum_i a_i \otimes X_i = \mathrm{distr} \sum_i a_i \otimes Y_i$ is equivalent to

$$Tr \circ \tau((\sum_i a_i \otimes X_i)^k) = Tr \circ \chi((\sum_i a_i \otimes Y_i)^k)$$

for all integers k.

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  for all integers k.
- Using the cyclicity of the trace and an appropriate description of

$$\{[a, b], a, b \in \mathbb{C}\langle x_1, x_2\rangle\},$$

  the moment condition can be seen to be equivalent to

$$\cap_{N\geq 1} \cap_{a_1,a_2 \in \mathbb{M}_N(\mathbb{C})_{sa}} Ker(\mathrm{Tr} \circ ev_{a_1,a_2}) = \{[a, b], a, b \in \mathbb{C}\langle x_1, x_2\rangle\}$$

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  In other words: don't be descriptive, use RMT instead.
  (more precisely, second order freeness theory by J. Mingo and R. Speicher)

Reminder of second order freeness

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For $x_1, \ldots, x_r$ complex valued random variables having moments of all orders, recall that the classical cumulant $C_r(x_1, \ldots, x_r)$ is defined by

$$C_r(x_1, \ldots, x_r) = \frac{\partial^r}{\partial t_1 \ldots \partial t_r} \log E(e^{\sum t_i x_i})|_{t_i=0}.$$

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A random matrice sequence $A = (A_N)_{N\in\mathbb{N}}$ has a second order limit distribution if for all $m, n \geq 1$ the limits

$$\alpha_n^A := \lim_{N\to\infty} C_1(\mathrm{tr}(A_N^n))$$

and

$$\gamma_{m,n}^A := \lim_{N\to\infty} C_2(\mathrm{Tr}(A_N^m), \mathrm{Tr}(A_N^n))$$

exists and if for all $r \geq 3$, and all $n(1), \ldots, n(r) \geq 1$,

$$\lim_{N\to\infty} C_r\big(\mathrm{Tr}(A_N^{n(1)}), \ldots, \mathrm{Tr}(A_N^{n(r)})\big) = 0.$$

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$$Y_N\big(n(1), m(1), \ldots, n(p), m(p)\big) =$$

$$\mathrm{Tr}\big((A_N^{n(1)} - \alpha^A_{n(1)}1)(B_N^{m(1)} - \alpha^B_{m(1)}1)\cdots(A_N^{n(p)} - \alpha^A_{n(p)}1)(B_N^{m(p)} - \alpha^B_{m(p)}1)\big).$$

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The random matrix ensembles $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$ are asymptotically free of second order if every monomial in A_N, B_N has a second order limit distribution, and if for all $n, m \geq 1$

$$\lim_{N\to\infty} C_2\big(\mathrm{Tr}(A_N^n - \alpha^A_n 1), \mathrm{Tr}(B_N^m - \alpha^B_m 1)\big) = 0$$

...

...and for all $p, q \geq 1$ and $n(1), \ldots, n(p), m(1), \ldots, m(p), \tilde{n}(1), \ldots, \tilde{n}(q),\ \tilde{m}(1), \ldots, \tilde{m}(q) \geq 1$ we have

$$\lim_{N\to\infty} C_2\Big(Y_N\big(n(1), m(1), \ldots, n(p), m(p)\big),$$

$$Y_N\big(\tilde{n}(1), \tilde{m}(2), \ldots, \tilde{n}(q), \tilde{m}(q)\big)\Big) = 0$$

if $p \neq q$,

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if $p \neq q$, and otherwise

$$\lim_{N\to\infty} C_2\Big(Y_N\big(n(1), m(1), \ldots, n(p), m(p)\big),$$

$$Y_N\big(\tilde{n}(p), \tilde{m}(p), \ldots, \tilde{n}(1), \tilde{m}(1)\big)\Big) = \sum_{k=1}^{p} \prod_{i=1}^{p} \gamma^A_{n(i+k),\tilde{n}(i)} \gamma^B_{m(i+k),\tilde{m}(i)}$$

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Theorem (Johansson)

Let A_N be the GUE of dimension N and T_i the Chebyshev polynomial of second kind, then the real infinite dimensional random vector

$$\left(\frac{Tr(T_i(A_N)) - E(\mathrm{Tr}(T_i(A_N)))}{\sqrt{i}}\right)_{i\in\mathbb{N}}$$

tends in distribution towards independent standard real gaussian variables.

End of the proof

Let x be any element of $\mathbb{C}\langle x_1, x_2\rangle - [\mathbb{C}\langle x_1, x_2\rangle, \mathbb{C}\langle x_1, x_2\rangle]$.

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By the results of Johansson and Mingo-Speicher, it is possible to prove that the random element

$$\mathrm{Tr}(\mathrm{ev}_{X_1,X_2}(x)) - E(\mathrm{Tr}(\mathrm{ev}_{X_1,X_2}(x)))$$

converges towards a nontrivial gaussian variable as $N \to \infty$.

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Therefore there exists N, $a_1, a_2 \in \mathbb{M}_N(\mathbb{C})_{s.a.}$ (resp. b_1, b_2) such that

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- It is (probably) the first application of second order freeness beyond RMT.
- However, it could be important and instructive to look for a constructive proof.

Reminders about CEP

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Then R^ω is the quotient $\ell^\infty(\mathbb{N}, R)/I_\omega$, which is actually a von Neumann algebra.

Definition

The pair (M, τ) is said to have Connes' embedding property (CEP) if M can be embedded into an ultra power R^ω of the hyperfinite von Neumann algebra R in a trace–preserving way.

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Whether any finite vN algebra with separable predual has CEP is a BIG OPEN QUESTION.

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Theorem

Connes Embedding Property is equivalent to the existence of microstates.

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Suppose that a von Neumann algebra M with trace τ is generated by self–adjoint elements x_1 and x_2.
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Then M has Connes' embedding property if and only if there exists $y_1, y_2 \in (R^\omega)_{sa}$ such that for all $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})_{sa}$ whose spectra are contained in $[c, d]$,

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$$distr(a_1 \otimes x_1 + a_2 \otimes x_2) = distr(a_1 \otimes y_1 + a_2 \otimes y_2)$$

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Then M has Connes' embedding property if and only if for all $n \in \mathbb{N}_*$ and all $a_1, a_2 \in \mathbb{M}_n(\mathbb{C})_+$ there exists $y_1, y_2 \in R^\omega$ such that

$$\mathrm{distr}(x_1) = \mathrm{distr}(y_1) \tag{2}$$

$$\mathrm{distr}(x_2) = \mathrm{distr}(y_2) \tag{3}$$

$$\mathrm{distr}(a_1 \otimes x_1 + a_2 \otimes x_2) = \mathrm{distr}(a_1 \otimes y_1 + a_2 \otimes y_2) \tag{4}$$

hold.

Horn conjecture: setting

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Let (I, J, K) be a triple of subsets of $\{1, \ldots, n\}$. The eigenvalues $(\alpha_i), (\beta_i), (\gamma_i)$ are said to satisfy inequalities $(*IJK)$ iff

$$\sum_{i \in K} \gamma_i \leq \sum_{i \in I} \alpha_i + \sum_{i \in J} \beta_i.$$

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$$U_r^n = \{(I, J, K), \sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + r(r+1)/2\}.$$

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$$U_r^n = \{(I, J, K), \sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + r(r+1)/2\}.$$

- When $r = 1$, set $T_1^n = U_1^n$. In general,

$$T_r^n = \{(I, J, K) \in U_r^n, \ \textit{for all } p < r \textit{ and all } (F, G, H) \in T_p^r,$$

$$\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq \sum_{h \in H} k_h + p(p+1)/2$$

Horn conjecture: theorem

Theorem

A triple (α, β, γ) *occurs as eigenvalues of Hermitian* n *by* n *matrices* A, B, C *with* $C = A + B$ *if and only if*

$$\sum \gamma_i = \sum \alpha_i + \sum \beta_i$$

and the inequalities $(*IJK)$ *hold for every* (I, J, K) *in* T_r^n*, for all* $r < n$.

Large N scaling limit of Horn problem

Observation & theorem due to Bercovici-Li: for probability measures μ, ν with compact support, it is possible to well-define a convex body $K_{\nu,\mu}$ of measures by approximating μ, ν by the spectral distribution of matrices A, B of dimension N.

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Theorem (Bercovici-Li)

This convex body exactly characterizes what spectral measures occur for $A + B$, A and B belonging to a factor with CEP.

Matrix valued version

Observation: with N-dimensional spectral measures μ, ν and $a, b \in \mathbb{M}_n(\mathbb{C})_{sa}$, define a set $K_{\nu,\mu}^{a,b}$ of possible measures of

$$a \otimes A_N + b \otimes B_N$$

for $A_N \in \mathbb{M}_N(\mathbb{C})$ of spectral measure μ and $B_N \in \mathbb{M}_N(\mathbb{C})$ of spectral measure ν.

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- *Unlike $K_{\nu,\mu}^{1,1}$, $K_{\nu,\mu}^{a,b}$ may be non-convex.*
- *Let M be a II_1 factor.*

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One can show that this body exactly characterizes what spectral measures occur for $a \otimes A + b \otimes B$, A and B belonging to a factor with CEP.

Theorem

- *Unlike $K^{1,1}_{\nu,\mu}$, $K^{a,b}_{\nu,\mu}$ may be non-convex.*
- *Let M be a II_1 factor. This factor has CEP iff* $\mathrm{distr}(a \otimes A + b \otimes B) \in K^{a,b}_{\nu,\mu}$ *for all a, b and $A, B \in M$ of distribution (μ, nu)*

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Thanks !