(Supersymmetric) Quantum Electrodynamics on Moyal Space

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Outline

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   - Calculus on the noncommutative Minkowski space
   - Gauge theory and covariant coordinates
   - The Yang-Feldman formalism

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Motivation

We study quantum electrodynamics on Moyal space, which is generated by coordinates subject to

$$[q^\mu, q^\nu] = i\sigma^{\mu\nu}.$$

This is motivated by

- A semiclassical analysis leading to uncertainty relations between coordinates [Doplicher, Fredenhagen & Roberts 94]
- The appearance of such commutation relations in a particular limit of string theory ($\sigma^{0i} = 0$) [Schomerus 99, Seiberg & Witten 99]
- Typically, QFTs on this space exhibit a distortion of the dispersion relation. Quantum electrodynamics seems to be a very interesting testbed.

Formal vs Strict

- The noncommutativity can either be implemented in a formal or a strict sense,

$$(f \star h)(x) = e^{\frac{i}{2}\partial^y_\mu \sigma^{\mu\nu} \partial^z_\nu} f(y)h(z)|_{x=y=z} \quad \text{vs} \quad f(q)h(q).$$

- The Seiberg-Witten map uses the formal expansion to relate gauge theories on commutative and noncommutative spaces.
- The formal expansion is also the basis for the twist approach of Wess et al.
- In the formal approach the fact that noncommutative spaces are intrinsically nonlocal is hidden in the appearance of derivatives of arbitrary order.
- It is in general not clear whether the expansion converges.
- Here we consider strict noncommutativity.

Euclidean vs Lorentzian

- In QFT on ordinary flat spacetime, it is often convenient to work in Euclidean signature. The Osterwalder-Schrader theorem then relates the results obtained on Euclidean space to the ones on the physical Lorentzian space.
- It is straightforward to derive modified Feynman rules in the noncommutative case from a Euclidean path integral. [Filk 96]
- Due to the absence of Osterwalder-Schrader reflection positivity, it is not clear what this tells us about the Lorentzian case.
- A naive application of these rules in the Lorentzian setting leads to a violation of unitarity for $\sigma^{0i} \neq 0$. [Gomis & Mehen 00]
- The reason for this is an inappropriate definition of time-ordering. [Bahns, Doplicher, Fredenhagen & Piacitelli 02]
- In the Lorentzian case, one can use the Hamiltonian or the Yang-Feldman approach.

Hamiltonian vs Yang-Feldman

- In the Hamiltonian approach, one postulates a Hamiltonian $H(t)$ and expands the time evolution in the coupling constant.
- In the Yang-Feldman approach, one directly uses the equation of motion.
- In the commutative case, the Hamiltonian approach yields the Feynman rules. The Yang-Feldman rules are more complicated than the Feynman rules, but are believed to be equivalent.
- In the NC case, the two approaches differ. The combinatorics of the Hamiltonian approach is in general more complicated.
- In the Hamiltonian approach, the interacting field does, at tree level, not fulfill the equation of motion. [Bahns 04]
  For NCQED, this leads to a violation of transversality at tree level. [Ohl, Rückl & Zeiner 03]

Outline

Calculus on the noncommutative Minkowski space I

- We assume that the commutation relations can be integrated to Weyl form
$$e^{ikq}e^{ipq} = e^{i(k+p)q}e^{-\frac{i}{2}k_\mu\sigma^{\mu\nu}p_\nu} = e^{i(k+p)q}e^{-\frac{i}{2}k\sigma p}.$$
The factor $e^{-\frac{i}{2}k\sigma p}$ is called twisting factor.
- Functions of the noncommuting coordinates are defined by
$$f(q) = \frac{1}{(2\pi)^2}\int \mathrm{d}^4k\ e^{-ikq}\hat{f}(k); \quad \hat{f}(k) = \frac{1}{(2\pi)^2}\int \mathrm{d}^4k\ e^{ikx}f(x).$$
- The product of two such functions is given by
$$f(q)h(q) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\ e^{-ikq}\int \mathrm{d}^4l\ \hat{f}(l)\hat{h}(k-l)e^{\frac{i}{2}k\sigma l}.$$
- For $f(x) \in \mathcal{S}(\mathbb{R}^4)$, one obtains a topological $*$-algebra $\mathcal{S}$. A subalgebra $\mathcal{M}$ of its multiplicator algebra is convenient (it contains the q's and the e^{ikq}'s). [Gracia-Bondia & Varilly 88]

Calculus on the noncommutative Minkowski space II

- Derivations on $\mathcal{M}$ can be defined as

$$\partial_\mu f(q) = (\partial_\mu f)(q) = -i\sigma^{-1}_{\mu\nu}[q^\nu, f(q)].$$

- A trace on $\mathcal{S}$ is given by

$$\int \mathrm{d}^4 q \ f(q) = \int \mathrm{d}^4 x \ f(x) = (2\pi)^2 \hat{f}(0).$$

- A metric is introduced in an ad hoc way by using, e.g.,

$$L = \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \partial_\mu\phi\partial^\mu\phi$$

as the kinetic term in the Lagrangean.

Gauge theory I

- Inspired by the Serre-Swan theorem, we describe gauge theories by finitely generated projective $\mathcal{M}$-modules E. The case of electrodynamics is obtained by choosing $E = \mathcal{M}$ and the metric $E \times E \ni (\psi, \phi) \mapsto \psi^* \phi \in \mathcal{M}$.
- We also have to choose a differential calculus over $\mathcal{M}$. We choose the one generated by $\mathrm{d}q^\mu$ subject to $[q^\mu, \mathrm{d}q^\nu] = 0$.
- There is a natural pairing between m-forms and symmetric tensor products of derivations ∂_μ given by

$$\langle f \mathrm{d}q^{\mu_1} \dots \mathrm{d}q^{\mu_m}, \partial_{\nu_1} \otimes \dots \otimes \partial_{\nu_n} \rangle = \delta^m_n f \sum_\pi (-1)^{\mathrm{sign}(\pi)} \delta^{\mu_1}_{\nu_{\pi(1)}} \dots \delta^{\mu_m}_{\nu_{\pi(m)}}.$$

- Given a connection D on E and choosing a normalized basis section $s \in E$, we define the vector potential A_μ as

$$\langle Ds, \partial_\mu \rangle = -iesA_\mu.$$

If D is metric, then the A_μ are self-adjoint.

Gauge theory II

- The field strength corresponding to the above connection is

$$F_{\mu\nu} = \tfrac{i}{e}\langle D^2 s, \partial_\mu \otimes \partial_\nu \rangle = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu].$$

- Under an infinitesimal gauge transformation $\delta_\lambda s = -ies\lambda$, A_μ and $F_{\mu\nu}$ transform as

$$\delta_\lambda A_\mu = \partial_\mu \lambda - ie[A_\mu, \lambda], \qquad \delta_\lambda F = ie[\lambda, F].$$

- With the action

$$S = \tfrac{1}{4} \int \mathrm{d}^4 q \; F_{\mu\nu} F^{\mu\nu}$$

one obtains the equation of motion

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ie[A_\mu, F^{\mu\nu}] = 0.$$

Covariant coordinates

Problem: The naive local observable

$$\int \mathrm{d}^4q \; F_{\mu\nu} f^{\mu\nu}(q)$$

is not gauge invariant, as F transforms covariantly, and $f(q)$ does not.

Solution: The covariant coordinates

$$X^\mu = q^\mu + e\sigma^{\mu\nu} A_\nu$$

transform covariantly. [Madore, Schraml, Schupp & Wess 00]

Proof: $\delta_\lambda X^\mu = e\sigma^{\mu\nu}\partial_\nu\lambda - ie^2\sigma^{\mu\nu}[A_\mu, \lambda] = -ie[X^\mu, \lambda]$.

When the universal differential calculus is employed, such a construction is possible for arbitrary $a \in \mathcal{M}$. [Bahns, Doplicher, Fredenhagen & Piacitelli 10]

Local observables

- We may now define the local, gauge invariant observable

$$\int \mathrm{d}^4 q \; F_{\mu\nu} f^{\mu\nu}(X).$$

- Elements $f(X)$ can be defined analogously to $f(q)$:

$$f(X) = (2\pi)^{-2} \int \mathrm{d}^4 k \; e^{-ikX} \hat{f}(k).$$

- We can write

$$e^{ikX} = e^{ikq} \sum_{N=0}^{\infty} (ie)^N (2\pi)^{-2N} \int \prod_{i=1}^{N} \mathrm{d}^4 k_i \; e^{-ik_1 q} \dots e^{-ik_N q}$$
$$\times \, k\sigma\hat{A}(k_1) \dots k\sigma\hat{A}(k_N) P_N(-ik\sigma k_1, \dots, -ik\sigma k_N)$$

with a certain polynomial P_N.

The Yang-Feldman formalism

Ingredient: Eom with a well-posed Cauchy problem.

Example: ϕ^3 model, i.e., $(\Box + m^2)\phi = \lambda\phi^2$.

Ansatz: $\phi = \sum_{n=0}^{\infty} \lambda^n \phi_n$.

$\Rightarrow (\Box + m^2)\phi_n = \sum_{k=0}^{n-1} \phi_k \phi_{n-1-k}$.

- ϕ_0 is the free field. We identify it with the incoming field.
- $\phi_1(q) = \int \mathrm{d}^4x \; \Delta_{\mathrm{ret}}(x)\phi_0(q-x)\phi_0(q-x) = \Delta_{\mathrm{ret}} \times (\phi_0\phi_0)(q)$.
- $\phi_2(q) = \Delta_{\mathrm{ret}} \times (\phi_1\phi_0 + \phi_0\phi_1)(q)$.

Dispersion relations from two-point function

$$\langle \Omega | \phi(f)\phi(h) | \Omega \rangle = \int \mathrm{d}^4k \; \hat{f}(-k)\hat{h}(k)\Sigma(k).$$

$$\Sigma(k) = \hat{\Delta}_+(k) \left(1 + \lambda^2 \int \mathrm{d}^4l \; \hat{\Delta}_+(l)\hat{\Delta}_{\mathrm{ret}}(k+l)\tfrac{1}{2}(1 + \cos k\sigma l) \right).$$

Thus, $\Sigma(k) = \Sigma(k^2, (\sigma k)^2)$, hence distorted dispersion relations.

Outline

The goal

- We want to compute the two-point correlation function
$$\langle\Omega| \left(\int \mathrm{d}^4 q \; f^{\mu\nu}(X) F_{\mu\nu} \right) \left(\int \mathrm{d}^4 q \; h^{\lambda\rho}(X) F_{\lambda\rho} \right) |\Omega\rangle \qquad (1)$$
of the interacting field (defined by the Yang-Feldman formalism) to second order in e.
- Because of the presence of the higher order terms in the observables, the two-point function (1) contains, at order e^2, also three- and four-point functions of the photon field.
- Previously, the (time-ordered) two-point function
$$\langle\Omega|\mathrm{T} A_\mu(x) A_\nu(y)|\Omega\rangle$$
was calculated with the modified Feynman rules, and a severe distortion of the dispersion relation was found.
- Does the same happen in the Yang-Feldman formalism?
- Do the covariant coordinates help?

The Lagrangean

- In order to set up the Yang-Feldman series, we need a well-posed Cauchy problem. Thus, we have to break gauge invariance:

$$\Rightarrow \mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \partial_\mu B A^\mu + \tfrac{\alpha}{2} B^2 - \partial_\mu \bar{c} D^\mu c.$$

- Nonlinear eom $\Rightarrow$ Already pure NCQED is self-interacting.
- We do not add fermions. At the one-loop level, their contribution is as in the commutative case.
- This transforms covariantly under the BRST transformation

$$\begin{aligned} \delta_\xi A_\mu &= \xi D_\mu c, \\ \delta_\xi c &= \xi \tfrac{i}{2} e \{c, c\}, \\ \delta_\xi \bar{c} &= \xi B, \\ \delta_\xi B &= 0. \end{aligned}$$

- As usual, δ_ξ is nilpotent.

The two-point function I

The two-point function contains a lot of terms. We focus on those that contribute to the discrete spectrum. We write

$$\langle\Omega| \left(\int \mathrm{d}^4 q \; f^{\mu\nu}(X) F_{\mu\nu} \right) \left(\int \mathrm{d}^4 q \; h^{\lambda\rho}(X) F_{\lambda\rho} \right) |\Omega\rangle = \int \mathrm{d}^4 k \; \hat{f}^{\mu\nu}(-k) \hat{h}^{\lambda\rho}(k) \Sigma_{\mu\nu\lambda\rho}(k).$$

- At zeroth order, we obtain the usual

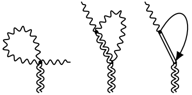
$$\Sigma_{\mu\nu\lambda\rho}(k) = -4(2\pi)^2 g_{\nu\rho} k_\mu k_\lambda \hat{\Delta}_+(k).$$

- The second order contribution from

$$4\langle\Omega| \left(\int \mathrm{d}^4 q \; f^{\mu\nu}(q) \partial_\mu A_\nu \right) \left(\int \mathrm{d}^4 q \; h^{\lambda\rho}(q) \partial_\lambda A_\rho \right) |\Omega\rangle$$

  corresponds to the two-point function that was calculated previously with the modified Feynman rules.

The self-energy



From these graphs, we obtain

$$\Sigma^1_{\mu\nu\lambda\rho}(k) = -\tfrac{20}{3} e^2 g_{\nu\rho} k_\mu k_\lambda \hat{\Delta}_+(k) \ln \mu \sqrt{-(\sigma k)^2},$$
$$\Sigma^2_{\mu\nu\lambda\rho}(k) = -4e^2 k_\mu k_\lambda \frac{(\sigma k)_\nu (\sigma k)_\rho}{(\sigma k)^4} \left(8 \frac{\partial}{\partial m^2} \hat{\Delta}_+(k) - \frac{(\sigma k)^2}{3} \hat{\Delta}_+(k) \right).$$

This coincides with the results obtained with the modified Feynman rules. [Hayakawa 99]

- The first term corresponds to a momentum dependent field strength renormalization.
- The second term was interpreted as a severe distortion of the dispersion relation. [Matusis, Susskind and Toumbas 00]
- But: Not well-defined ⇒ Nonlocal renormalization ambiguity

The two-point function II

There are many other terms. Most of them do not contribute to the discrete spectrum or are not relevant for the present discussion. Using the $\mathcal{O}(e)$ contribution of the covariant coordinates in one observable, we obtain the contribution

$$\Sigma_{\mu\nu\lambda\rho}(k) = -8(2\pi)^2 k_\mu k_\lambda \hat{\Delta}_+(k) \\ \times \int \mathrm{d}^4 l \; [\hat{\Delta}_+(l) + \hat{\Delta}_+(-l)] \hat{\Delta}_R(k-l) \frac{\sin^2 \frac{k\sigma l}{2}}{\frac{k\sigma l}{2}} \left\{ (k\sigma)_\rho l_\nu + (k\sigma)_\nu l_\rho \right\} .$$

The loop integral is not well-defined. Formally, it is of the form

$$\frac{(\sigma k)_\nu (\sigma k)_\rho}{(\sigma k)^2} \times \text{ log. div. } + \text{ fin.}$$

Thus, we found another nonlocal divergence, stemming from the covariant coordinates.

Supersymmetric NCQED

- Upon introducing supersymmetry, the term proportional to $\frac{(\sigma k)_\mu (\sigma k)_\nu}{(\sigma k)^4}$ in the self-energy vanishes.
- The nonlocal divergence from the covariant coordinates is removed when the supersymmetric covariant coordinates

$$
\begin{aligned}
X^\mu &= q^\mu + e\sigma^{\mu\nu} \left(\frac{1}{4e} \bar{\sigma}_\nu^{\dot{\alpha}\alpha} \bar{D}_{\dot{\alpha}} \left(e^{-2eV} D_\alpha e^{2eV} \right) \right) \\
&= q^\mu + e\sigma^{\mu\nu} \left(A_\nu - i\theta\sigma_\nu \bar{\lambda} + i\lambda\sigma_\nu \bar{\theta} + \text{ higher orders in } \theta, \bar{\theta} \right)
\end{aligned}
$$

  are employed.
- The only modification to the one-particle spectrum is the nonlocal wave function renormalization

$$
\Sigma_{\mu\nu\lambda\rho}(k) = -4e^2 g_{\nu\rho} k_\mu k_\lambda \hat{\Delta}_+(k) \ln \mu \sqrt{-(\sigma k)^2}
$$

Outline

Summary & Outlook

- Rigorous and complete calculation of the photon self-energy at the one-loop level (for $k^2 > 0$).
- Severe distortion of the dispersion relation or interpretation as a nonlocal renormalization ambiguity.
- The covariant coordinates were fully taken into account.
- They also contribute nonlocal divergences.
- These vanish upon introducing supersymmetry and using observables appropriate for the supersymmetric case.
- Unfortunately, this only holds for unbroken supersymmetry.
- Use nonlocal counterterms and usual dispersion relation as renormalization condition?