The Baum-Connes Conjecture

The Baum-Connes Conjecture

Conjecture

$$\mu : K_*^G(\underline{E}G) \xrightarrow{\cong} K_*(C_r^* G)$$

$$(\textit{Topology}) \qquad\qquad (\textit{Analysis})$$

The Baum-Connes Conjecture

Conjecture

$$\mu : K_*^G(\underline{E}G) \xrightarrow{\cong} K_*(C_r^* G)$$

$$(\mathit{Topology}) \qquad\qquad (\mathit{Analysis})$$

Kasparov's approach:

The Baum-Connes Conjecture

Conjecture

$$\mu : K_*^G(\underline{E}G) \xrightarrow{\cong} K_*(C_r^* G)$$

$$(Topology) \qquad\qquad (Analysis)$$

Kasparov's approach:

- $\exists$ canonical idempotent $\gamma \in R(G) := KK^G(\mathbb{C}, \mathbb{C})$.

The Baum-Connes Conjecture

Conjecture

$$\mu : K_*^G(\underline{E}G) \xrightarrow{\cong} K_*(C_r^* G)$$

$$(Topology) \qquad\qquad (Analysis)$$

Kasparov's approach:

- $\exists$ canonical idempotent $\gamma \in R(G) := KK^G(\mathbb{C}, \mathbb{C})$.
- If $\gamma = 1$ then BC holds for all discrete subgroups of G.

Method of proof:

Method of proof:

1. Construct explicit model of γ using "elliptic" differential complex over the flag variety.

The Baum-Connes Conjecture

Method of proof:

1. Construct explicit model of γ using "elliptic" differential complex over the flag variety.
2. Homotopy: $\gamma \sim 1$.

Method of proof:

1. Construct explicit model of γ using "elliptic" differential complex over the flag variety.
2. Homotopy: $\gamma \sim 1$.

This has been achieved for all **rank one** semisimple Lie groups.

The Baum-Connes Conjecture

Method of proof:

1. Construct explicit model of γ using "elliptic" differential complex over the flag variety.
2. Homotopy: $\gamma \sim 1$.

This has been achieved for all **rank one** semisimple Lie groups.

Open case: $\mathrm{SL}(3, \mathbb{Z})$.

The Baum-Connes Conjecture

Method of proof:

1. Construct explicit model of γ using "elliptic" differential complex over the flag variety.
2. Homotopy: $\gamma \sim 1$.

This has been achieved for all **rank one** semisimple Lie groups.

Open case: $\mathrm{SL}(3, \mathbb{Z})$.

Remark. $\gamma \neq 1$ if G has property T.

History

Constructions of γ in this way:

History

Constructions of γ in this way:

Group	Differential Complex	Elliptic for...

History

Constructions of γ in this way:

Group	Differential Complex	Elliptic for...
$\mathrm{SO}(n,1)$	de Rham	standard ΨD calculus [Kasparov '86]

History

Constructions of γ in this way:

Group	Differential Complex	Elliptic for...
$\mathrm{SO}(n,1)$ (incl. $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$)	de Rham	standard ΨD calculus [Kasparov '86]

History

Constructions of γ in this way:

Group	Differential Complex	Elliptic for...
$\mathrm{SO}(n,1)$ (incl. $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$)	de Rham	standard ΨD calculus [Kasparov '86]
$\mathrm{SU}(n,1)$	Rumin	Heisenberg calculus [Julg-Kasparov '95]

History

Constructions of γ in this way:

Group	Differential Complex	Elliptic for...
$\mathrm{SO}(n,1)$ (incl. $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$)	de Rham	standard ΨD calculus [Kasparov '86]
$\mathrm{SU}(n,1)$	Rumin	Heisenberg calculus [Julg-Kasparov '95]
$\mathrm{Sp}(n,1)$	quaternionic Rumin	quaternionic Heis. calculus [Julg '02]

History

Constructions of γ in this way:

Group	Differential Complex	Elliptic for...
$\mathrm{SO}(n,1)$ (incl. $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})$)	de Rham	standard ΨD calculus [Kasparov '86]
$\mathrm{SU}(n,1)$	Rumin	Heisenberg calculus [Julg-Kasparov '95]
$\mathrm{Sp}(n,1)$	quaternionic Rumin	quaternionic Heis. calculus [Julg '02]
$\mathrm{SL}(3,\mathbb{C})$	Bernstein-Gelfand-Gelfand	??? [Y. '11]

Geometry of flag varieties

Geometry of flag varieties

G	—	complex semisimple Lie group.
B	—	minimal parabolic subgroup
$\mathcal{X} = G/B$	—	flag variety.

Geometry of flag varieties

G	—	complex semisimple Lie group.
B	—	minimal parabolic subgroup
$\mathcal{X} = G/B$	—	flag variety.

$\mathcal{X}$ admits many fibrations over *partial* flag varieties.

Geometry of flag varieties

G	—	complex semisimple Lie group.
B	—	minimal parabolic subgroup
$\mathcal{X} = G/B$	—	flag variety.

$\mathcal{X}$ admits many fibrations over *partial* flag varieties.

P — non-minimal parabolic

Geometry of flag varieties

G — complex semisimple Lie group.
B — minimal parabolic subgroup
$\mathcal{X} = G/B$ — flag variety.

$\mathcal{X}$ admits many fibrations over *partial* flag varieties.

P — non-minimal parabolic

$$eg,\ P = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & \circledast & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \text{ in } G = \mathrm{SL}(n, \mathbb{C})$$

Geometry of flag varieties

G	—	complex semisimple Lie group.
B	—	minimal parabolic subgroup
$\mathcal{X} = G/B$	—	flag variety.

$\mathcal{X}$ admits many fibrations over *partial* flag varieties.

P — non-minimal parabolic

$$eg,\ P = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & \circledast & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \text{ in } G = \mathrm{SL}(n, \mathbb{C})$$

$G/B \twoheadrightarrow G/P$ — smooth fibration.

Geometry of flag varieties

$$\begin{array}{lcl} G & — & \text{complex semisimple Lie group.} \\ B & — & \text{minimal parabolic subgroup} \\ \mathcal{X} = G/B & — & \text{flag variety.} \end{array}$$

$\mathcal{X}$ admits many fibrations over *partial* flag varieties.

$$P \quad — \quad \text{non-minimal parabolic}$$

$$eg,\ P = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & \circledast & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \text{ in } G = \mathrm{SL}(n, \mathbb{C})$$

$$G/B \twoheadrightarrow G/P \qquad \text{— smooth fibration.}$$

Problem: Index theory on **multiply foliated** manifolds.