Arithmetic subalgebras for Bost-Connes systems

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The Bost-Connes problem ('95)

For a number field K construct a C^*-dynamical system (QSM)

$$\mathcal{A}_K = (A, (\sigma_t)_{t \in \mathbb{R}})$$

with the following six properties:

Analytic Properties

1) The partition function is given by the Dedekind zeta function $\zeta_K(s)$

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A system satisfying these properties is called **analytic BC system** for K.

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Arithmetic Properties

5) $\exists$ K-rational subalgebra $A^{arith} \subset A$ such that $\forall$ $f \in A^{arith}$ and $\varphi \in \sigma\text{-}KMS_\infty$ we have

$$\varphi(f) \in K^{ab}$$

Moreover, K^{ab} is generated in this way.

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A system $\mathcal{A}_K$ satisfying these properties is called **(full) BC system** for K and the corresponding subagebra A^{arith} an arithmetic subalgebra.

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Remark

The difficulty of constructing arithmetic subalgebras does come from its connection to Hilbert's 12th problem $\longleftarrow$ widely open for K not $\mathbb{Q}$ or imaginary quadratic (or CM)!

Reminder: The analytic BC-system $\mathcal{A}_K$

The monoid of (non-zero) integral ideals $I(\mathcal{O}_K)$ of $\mathcal{O}_K$ is acting naturally on the topological space

$$Y_K = \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} Gal(K^{ab}/K)$$

The C^*-dynamical system $\mathcal{A}_K$ is defined by the crossed product

$$\mathcal{A}_K = (C(Y_K) \rtimes I(\mathcal{O}_K), \sigma_t)$$

with time evolution

$$\sigma_t(fu_s) = \mathcal{N}(s)^{it} fu_s$$

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Theorem 2 - Functoriality (Y. '11)

There is a natural functor

$$K \mapsto \mathcal{E}_K$$

which recovers the functor $K \mapsto \mathcal{A}_K$ recently constructed by Laca, Neshveyev and Trifkovic ('10).

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1. The theory of endomotives by Connes, Consani and Marcolli ('05)

2. the theory of Λ-rings due to Borger and de Smit ('08)

1. On endomotives

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$$\begin{array}{ccccc} \text{algebraic EM} & \rightarrow & \text{analytic EM} & \text{"}\rightarrow\text{"} & \text{measured analytic EM} \\ \cap & & \cap & & \cap \\ K\text{-algebras} & \rightarrow & C^*\text{-algebras} & \text{"}\rightarrow\text{"} & C^*\text{-dynamical systems} \end{array}$$

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Motivation

The classical BC system can be seen to come from an algebraic endomotive and we want to prove that in fact every BC-type system does come from an algebraic endomotive.

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Definition

The algebraic endomotive $\mathcal{E}$ associated with the above data is defined as the K-algebra

$$\mathcal{E} = A \rtimes S$$

Generators and Relations

$\mathcal{E} = A \rtimes S$ has a presentation by adjoining to elements $a \in A$ new generators U_ρ and U_ρ^*, for $\rho \in S$, satisfying the relations

$$\begin{array}{cc} U_\rho^* U_\rho = 1, & U_\rho U_\rho^* = \rho(1), \\ U_{\rho_1\rho_2} = U_{\rho_1} U_{\rho_2}, & U_{\rho_1\rho_2}^* = U_{\rho_1}^* U_{\rho_2}^*, \\ U_\rho a = \rho(a) U_\rho, & a U_\rho^* = U_\rho^* \rho(a) \end{array}$$

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In particular

$$\mathcal{E} \subset \mathcal{E}^{an}$$

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Definition

The measured analytic endomotive $\mathcal{E}^{meas}$ (if it exists) is the C^*-dynamical system

$$\mathcal{E}^{meas} = (\mathcal{E}^{an}, (\sigma_t)_{t\in\mathbb{R}})$$

Résumé

In good situations an algebraic endomotive $\mathcal{E}^{alg} = A \rtimes S$ gives naturally rise to a C^*-dynamical system

$$(\mathcal{E}^{an}, (\sigma_t)_{t\in\mathbb{R}})$$

with

$$\mathcal{E}^{alg} \subset \mathcal{E}^{an}$$

(plus other nice properties, e.g. a natural $Gal(\overline{K}/K)$-action!)

Example: The classical Bost-Connes system as endomotive

See blackboard. If time permits!

What's next?

In order to construct interesting (algebraic) endomotives which are related to our Bost-Connes systems of HPLLN, we have to find the **right** class of finite, étale K-algebras!

2. On Λ-rings

Definition

An $\mathcal{O}_K$-algebra $\widetilde{E}$ is a Λ-ring if there exists for each (non-zero) prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ an endomorphism $f_{\mathfrak{p}}$ such that

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with

$$f_p : X \mapsto X^p$$

$\rightsquigarrow$ classical BC system

The Deligne-Ribet monoid

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where for each $\mathfrak{f} \in I(\mathcal{O}_K)$ we set $DR_{\mathfrak{f}} = I(\mathcal{O}_K)/_{\sim_{\mathfrak{f}}}$.

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Proposition

There is a natural equivariant isomorphism of (topological) monoids

$$Y_K = \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} Gal(K^{ab}/K) \cong DR_K$$

w.r.t. the natural action of $I(\mathcal{O}_K)$.

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Recall the analytic BC system

$$\mathcal{A}_K = (C(Y_K) \rtimes I(\mathcal{O}_K), \sigma_t)$$

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Theorem (Borger, de Smit)

The functor $E \mapsto Hom(E, \overline{K})$ induces an equivalence of categories

$$\left\{ \begin{array}{c} \text{finite, étale } K\text{-algebras} \\ \text{with integral } \Lambda\text{-structure} \end{array} \right\} \overset{1:1}{\Longleftrightarrow} \left\{ \begin{array}{c} \text{finite sets } + \\ \circlearrowleft_{cont.} DR_K \end{array} \right\}$$

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In particular

$$DR_{\mathfrak{f}} \cong Hom(E_{\mathfrak{f}}, \overline{K})$$

with finite, étale K-algebra $E_{\mathfrak{f}} \cong \prod_{\mathfrak{d}|\mathfrak{f}} K_{\mathfrak{d}}$ with $K_{\mathfrak{d}}$ a strict ray class field of K, i.e. abelian over K.

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Key observation II (due to $Y_K \cong DR_K$)

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In particular

$$DR_{\mathfrak{f}} \cong Hom(E_{\mathfrak{f}}, \overline{K})$$

with finite, étale K-algebra $E_{\mathfrak{f}} \cong \prod_{\mathfrak{d}|\mathfrak{f}} K_{\mathfrak{d}}$ with $K_{\mathfrak{d}}$ a strict ray class field of K, i.e. abelian over K.

Key observation II (due to $Y_K \cong DR_K$)

$$Y_K \cong Hom(\varinjlim E_{\mathfrak{f}}, \overline{K})$$

Finally: Our algebraic endomotive $\mathcal{E}_K$

Now we can define our desired algebraic endomotive

$$\mathcal{E}_K = ((E_{\mathfrak{f}}), I(\mathcal{O}_K))$$

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And, most importantly, one can identify

$$\mathcal{E}_K \subset C(Y_K) \rtimes I(\mathcal{O}_K)$$

as an arithmetic subalgebra of $\mathcal{A}_K$.

Mulțumesc!

Probably I have already overstepped my time! Therefore I will stop here and thank you very much for your attention!