An example of quantum group fusion rules

and a nonabelian weight lattice

joint work with T. Banica

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Matrix compact quantum groups

Definition

A matrix compact quantum group is a pair (A, u) where A is a unital C^*-algebra generated by the entries of $u \in M_n(A)$ such that

1. there exists $\Delta : A \to A \otimes A$ such that $\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}$,
2. u and $\bar{u} = (u_{ij}^*)$ are invertible in $M_n(A)$.

Examples :

1. $G \subset U_n$ compact : take $A = C(G)$, $u = i_{\mathrm{can}} : G \to M_n(C)$, $\Delta(f) = ((g, h) \mapsto f(gh)) \in C(G \times G) \simeq C(G) \otimes C(G)$.
2. $\Gamma = \langle g_1, \dots, g_n \rangle$ finitely generated group : take $A = C^*(\Gamma)$ or $C_r^*(\Gamma)$, $u = \mathrm{diag}(g_1, \dots, g_n)$, and $\Delta(g) = g \otimes g$.
3. $A_o(n) = \langle u_{ij} \mid u = \bar{u}$ unitary$\rangle$. Heuristically "$A_o(n) = C(O_n^+)$" where O_n^+ is the "free orthogonal quantum group".

Fusion rules

There is a notion of *corepresentation* for compact quantum groups.

Theorem (Peter-Weyl-Woronowicz)

Corepresentations of (A, u) are direct sums of irreducibles. Irreducible corepresentations are finite-dimensional.

Fusion rules : for u, $v \in \mathrm{Irrep}(A, u)$, write $u \otimes v = \bigoplus m_{uv}^w w$.

Examples :

1. $A = C(G)$: usual decomposition of tensor products of irreducible representations of G. Have $u \otimes v \simeq v \otimes u$.
2. $A = C^*(\Gamma)$: irreducibles correspond to elements of Γ and $u \otimes v = uv$. Thus (m_{uv}^w) is the multiplication table of Γ.
3. $A = A_o(n)$. Same fusion rules as $SU(2)$: irreducibles u_k, $k \in \mathbb{N}$ with $u_k \otimes u_l = u_{|k-l|} \oplus u_{|k-l|+2} \oplus \cdots \oplus u_{k+l}$ [Banica].

Half-liberated orthogonal groups

Observe that $C(O_n) = A_o(n)/\langle ab = ba \mid a, b \in \{u_{ij}\}\rangle$.
In other words we have $O_n \subset O_n^+$.

There is an intermediate "liberation" $O_n \subset O_n^* \subset O_n^+$ associated to an intermediate algebra $A_o^*(n)$:

Definition (Banica-Speicher)

$$A_o^*(n) = A_o(n)/\langle abc = cba \mid a, b, c \in \{u_{ij}\}\rangle$$

Non-trivial fact : $A_o(n) \neq A_o^*(n) \neq C(O_n)$ for $n \geq 3$.

Half-liberated orthogonal groups

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Brauer diagrams

For each k, l consider the following subspaces of $L(\mathbb{C}^{kn}, \mathbb{C}^{ln})$

$$\mathrm{Hom}_{O_n^+}(u^{\otimes k}, u^{\otimes l}) \subset \mathrm{Hom}_{O_n^*}(u^{\otimes k}, u^{\otimes l}) \subset \mathrm{Hom}_{O_n}(u^{\otimes k}, u^{\otimes l})$$

There is a standard procedure to produce linear maps $\mathbb{C}^{kn} \to \mathbb{C}^{ln}$ from partition diagrams with k upper points and l lower points. Then

1. $\mathrm{Hom}_{O_n}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Span}\{\text{all pair partitions}\}$
2. $\mathrm{Hom}_{O_n^*}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Span}\{\text{pair part. with even number of crossings}\}$
3. $\mathrm{Hom}_{O_n^+}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Span}\{\text{non-crossing pair partitions}\}$

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Projective version

Projective version of (A, u) : sub-C^*-algebra $PA = \langle u_{ij} u_{kl}^* \rangle \subset A$.

Case $G \subset U_n$: $PC(G) = C(PG)$ where PG is the image of G in PU_n.

Proposition

The compact quantum group PO_n^* is isomorphic to PU_n.

Proof : pair partitions with even number of crossings correspond to pair partitions compatible with labelling $ababa \cdots$ of points.

Diagonal groups

Definition

Diagonal quotient of (A, u) : $C^*_u(L) = A/\langle u_{ij} = 0 \mid i \neq j\rangle$.
Diagonal group : $L = \langle u_{ii}\rangle$.

Examples :

1. $A = C(G)$, $G \subset U_n$ connected → $L = \hat{T}$ where $T = G \cap \mathbb{T}^n$
2. $A = A_o(n)$ → $L = (\mathbb{Z}/2\mathbb{Z})^{*n}$
3. $A = A_o^*(n)$ → $L = \langle e_i \cdot \tau\rangle \subset \mathbb{Z}^n \rtimes (\mathbb{Z}/2\mathbb{Z})$, $L \simeq \mathbb{Z}^{n-1} \rtimes (\mathbb{Z}/2\mathbb{Z})$

Example 1 : up to global conjugacy, T is a maximal torus and L is the weight lattice → "nonabelian weight lattice" in general ?

Proposition

If the u_{ii}'s are distinct in L, $C^*(L)$ is maximal as a cocommutative quotient of (A, u). This is the case for $A_o^*(n)$.

Weights for representations

(A, u) matrix compact quantum group with diagonal group L.

Quotient + decomposition into irreducibles yields :

$$r \in \mathrm{Rep}(A) \to r' \in \mathrm{Rep}(C^*(L)) \to \Sigma(r) \subset L \text{ (with repetitions).}$$

If $G \cap \mathbb{T}^n$ is a maximal torus of $G \subset U_n$, the sets of weights $\Sigma(r)$ classify irreducibles representations r. This also works for O_n^+, U_n^+ — but not S_n^+.

Proposition

If r, $s \in \mathrm{Irrep}(O_n^*)$ are distinct then $\Sigma(r) \neq \Sigma(s)$.

Proof : by comparison with the case $G = U_n$.

One can go on :

- dominant and positive weights $L_{++} \subset L_+ \subset L$
- highest weight $\lambda_r \in L_{++}$ for $r \in \mathrm{Irrep}(O_n^*)$

Fusion rules

There is an injective map $\psi : \mathrm{Irrep}(O_n^*) \to \mathrm{Irrep}(U_n)$ defined at the level of highest weights :

$$\psi(\lambda \cdot x) = \lambda \ \text{ for } \lambda \in \mathbb{Z}^n, \ x \in \mathbb{Z}/2\mathbb{Z}.$$

For $r \in \mathrm{Irrep}\, O_n^*$ and $t \in \mathrm{Irrep}\, U_n$, put $t^r = t$ if r is even, $t^r = \bar{t}$ else.

Proposition

Let $r, s \in \mathrm{Irrep}(O_n^*)$. Then $\psi(r \otimes s) = \psi(r) \otimes \psi(s)^r$.

Hence the fusion rules of O_n^* can easily be computed from the ones of U_n. They are noncommutative, although the ones of O_n and O_n^+ are!

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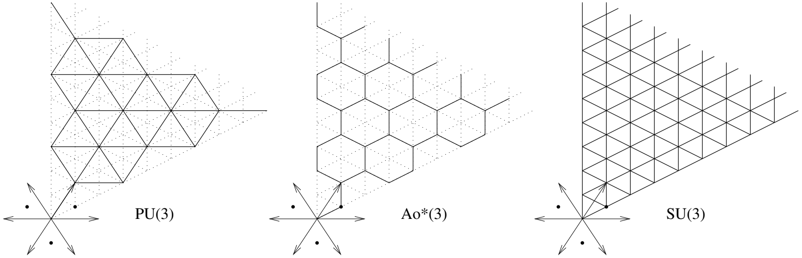
Cayley graphs :



Fusion rules

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Questions :

1. When is there a “good” nonabelian weight lattice ?
2. Analogues of O_n^* for other groups than U_n ?