W^*-superrigidity and uniqueness of Cartan subalgebras

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Von Neumann algebras

Murray and von Neumann, 1936 : an algebra of bounded operators on a Hilbert space, closed under adjoint, closed in the weak topology.

Main examples

- Countable group $\Gamma \rightsquigarrow$ group von Neumann algebra $L\Gamma$.

  Generated by unitary operators $(u_g)_{g\in\Gamma}$ on $\ell^2\Gamma$ given by $u_g\delta_h = \delta_{gh}$.

- Group action $\Gamma \curvearrowright (X,\mu)$ on a measure space by non-singular transformations $\rightsquigarrow$ crossed product $L^\infty(X) \rtimes \Gamma$.

  Generated by $L^\infty(X)$ and unitary operators $(u_g)_{g\in\Gamma}$ satisfying $u_g u_h = u_{gh}$ and $u_g^* F(\,\cdot\,) u_g = F(g\,\cdot\,)$.

Central problem:
Classify $L\Gamma$ and $L^\infty(X) \rtimes \Gamma$ in terms of the group (action) data.

II_1 factors

Simple von Neumann algebras: those that cannot be written as a direct sum of two. We call them factors.

Murray - von Neumann classification of factors: types I, II and III.

⇝ II_1 factors M are those factors that admit a trace $\tau : M \to \mathbb{C}$: $\tau(xy) = \tau(yx)$.

⇝ Arbitrary von Neumann algebras can be 'assembled' from II_1 factors (Connes, Connes & Takesaki).

Von Neumann algebras coming from groups and group actions

- $\mathrm{L}\Gamma$ is a II_1 factor if and only if Γ has infinite conjugacy classes (icc).
- $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a II_1 factor if $\Gamma \curvearrowright (X, \mu)$ is free ergodic and probability measure preserving (pmp).

Ergodicity: Γ-invariant subsets have measure 0 or 1.

Unexpected isomorphisms between II_1 factors

Connes, 1975 : all amenable II_1 factors are isomorphic.

$\rightsquigarrow$ All $L\Gamma$ for Γ amenable icc, are isomorphic.
All $L^\infty(X) \rtimes \Gamma$ for Γ amenable and $\Gamma \curvearrowright (X, \mu)$ free ergodic pmp, are isomorphic.

Reminder: von Neumann's amenability

- Given a unitary representation $\pi : \Gamma \to \mathcal{U}(H)$, we say that $\xi_n \in H$, $\|\xi_n\| = 1$, is a sequence of **almost invariant vectors** if $\|\pi(g)\xi_n - \xi_n\| \to 0$ for all $g \in \Gamma$.
- A group Γ is **amenable** if the regular representation on $\ell^2\Gamma$ given by $\pi(g)\delta_h = \delta_{gh}$ admits a sequence of almost invariant vectors.
- **Examples:** abelian groups, solvable groups.

Kazhdan's property (T) and rigidity

Rigidity in the title: (partially) recovering $\Gamma \curvearrowright X$ from $L^\infty(X) \rtimes \Gamma$.

Away from amenability: **Kazhdan's property (T).**

- A group Γ has **property (T)** if every unitary representation with almost invariant vectors, has a non-zero invariant vector.
  Ex. $SL(n, \mathbb{Z})$ for $n \geq 3$, lattices in higher rank simple Lie groups.
- A subgroup $\Lambda < \Gamma$ has **relative property (T)** if every unitary rep. of Γ with almost invariant vectors, has a non-zero Λ-invariant vector.
  Example: $\mathbb{Z}^2 < SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$.

Rigidity for crossed product II_1 factors

Free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$

⟶ **Orbit equivalence relation** given by $x \sim y$ iff $\Gamma \cdot x = \Gamma \cdot y$.

⟶ **II_1 factor** $\mathrm{L}^\infty(X) \rtimes \Gamma$.

Definition

Two actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ are called

- **conjugate,** if there exist $\Delta : X \to Y$ and $\delta : \Gamma \to \Lambda$ s.t. $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$.
- **orbit equivalent,** if there exist $\Delta : X \to Y$ s.t. $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$.
- **W*-equivalent,** if $\mathrm{L}^\infty(X) \rtimes \Gamma \cong \mathrm{L}^\infty(Y) \rtimes \Lambda$.

- Obviously, conjugacy implies orbit equivalence.
- **Singer (1955):** an orbit equivalence amounts to a W*-equivalence mapping $\mathrm{L}^\infty(X)$ onto $\mathrm{L}^\infty(Y)$.
- **Rigidity:** prove W*-equivalence ⇒ OE ⇒ conjugacy!

W*-superrigidity

Popa's strong rigidity theorem (2004)

Let Γ be a property (T) group and $\Gamma \curvearrowright (X, \mu)$ a free ergodic pmp action.
Let Λ be an icc group and $\Lambda \curvearrowright (Y, \eta) = (Y_0, \eta_0)^\Lambda$ its Bernoulli action.
If $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$, then the groups Γ and Λ are isomorphic and their actions conjugate.

First theorem ever deducing conjugacy out of isomorphism of II_1 factors.

Definition

A free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$ is called W*-superrigid if any W*-equivalent action must be conjugate.
In other words: $L^\infty(X) \rtimes \Gamma$ remembers the group action.

Compare: the assumptions in Popa's theorem are asymmetric.

First W^*-superrigidity theorem

Theorem (Popa-V, 2009)

For a large family of amalgamated free product groups $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ the Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ is W*-superrigid.

- Concrete examples: $\Gamma = SL(3,\mathbb{Z}) *_{T_3} (T_3 \times \Lambda)$ with T_3 the upper triangular matrices and $\Lambda \neq \{e\}$ arbitrary.
- Theorem covers more general families of group actions.

  Example. All mixing actions of $SL(3,\mathbb{Z}) *_{T_3} SL(3,\mathbb{Z})$ are W*-superrigid.

  Recent improvement (Houdayer - Popa - V, 2010).
  All free ergodic pmp actions of $SL(3,\mathbb{Z}) *_\Sigma SL(3,\mathbb{Z})$ are W*-superrigid, with $\Sigma < SL(3,\mathbb{Z})$ the subgroup of matrices x with $x_{31} = x_{32} = 0$.
- Peterson (2009) proved existence of virtually W*-superrigid actions.

How to establish W*-superrigidity for $\Gamma \curvearrowright (X, \mu)$

Assume that $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$.

To prove conjugacy of $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$, one needs two things.

Part 1 Prove that $L^\infty(X)$ and $L^\infty(Y)$ are unitarily conjugate.

- Conclusion of part 1: $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ follow orbit equivalent.
- Part 1 amounts to proving that $L^\infty(X) \rtimes \Gamma$ has a unique group measure space Cartan subalgebra, up to unitary conjugacy.

Part 2 Orbit equivalence superrigidity for $\Gamma \curvearrowright X$.

- Prove that every action that is OE with $\Gamma \curvearrowright X$ must be conjugate with $\Gamma \curvearrowright X$.
- Zimmer, Furman, Monod-Shalom, Popa, Ioana, Kida, ...
- For several Γ the **Bernoulli action** is OE superrigid (Popa).

$\rightsquigarrow$ Both parts are very hard. Even more difficult: both together for the same group action.

Cartan subalgebras

Definition

A Cartan subalgebra A of a II_1 factor M is a maximal abelian subalgebra such that $\{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M (i.e. its linear span is weakly dense in M).

Example : $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$, which we call a group measure space Cartan subalgebra.

Generic example : $\mathrm{L}^\infty(X) \subset \mathrm{L}_\Omega(\mathcal{R})$ where $\mathcal{R}$ is a type II_1 equivalence relation on (X, μ) and Ω is a 2-cocycle.

Remark : not all II_1 equivalence relations can be implemented by an essentially free group action !

Very difficult problem : uniqueness and non-uniqueness of Cartan subalgebras in II_1 factors.

Uniqueness of Cartan subalgebras

Theorem (Ozawa-Popa, 2007)

Let $\mathbb{F}_n \curvearrowright (X,\mu)$ be a free ergodic profinite action, $n \geq 2$. Then, $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$ has a unique Cartan subalgebra up to unitary conjugacy.

- The II_1 factor $M = \mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$ has the complete metric approx. property : there exist normal finite rank linear $\theta_k : M \to M$ such that $\|\theta_k(x) - x\|_2 \to 0$ for all $x \in M$ and $\limsup_k \|\theta_k\|_{\mathrm{cb}} = 1$.
- **Part 1.** If $\mathrm{L}^\infty(Y) \subset M$ and if $\mathcal{G} \subset \mathcal{U}(M)$ is a group of unitaries v satisfying $v\mathrm{L}^\infty(Y)v^* = \mathrm{L}^\infty(Y)$, then $\mathcal{G} \curvearrowright Y$ is weakly compact.
- **Part 2.** Ozawa-Popa prove next that if $\mathrm{L}^\infty(Y)$ cannot be conjugated into $\mathrm{L}^\infty(X)$, then $\mathcal{G}$ generates an amenable von Neumann algebra.

$\rightsquigarrow$ (Ozawa, December 2010). Part 1 also works if M only has the completely bounded approximation property.

$\rightsquigarrow$ (Chifan-Sinclair, this month). Part 2 works if we replace $\mathbb{F}_n$ by any hyperbolic group.

Uniqueness of Cartan subalgebras

Theorem (Chifan-Sinclair, 2011)

Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic profinite action of any hyperbolic group Γ. Then, $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra.

- The II_1 factor $M = L^\infty(X) \rtimes \mathbb{F}_n$ has the complete metric approx. property : there exist normal finite rank linear $\theta_k : M \to M$ such that $\|\theta_k(x) - x\|_2 \to 0$ for all $x \in M$ and $\limsup_k \|\theta_k\|_{\mathrm{cb}} = 1$.
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Back to W*-superrigidity

Theorem (Popa-V, 2009)

Let $\Gamma = \Gamma_1 * \Gamma_2$ be the free product of an infinite property (T) group Γ_1 and a non-trivial group Γ_2. Then $L^\infty(X) \rtimes \Gamma$ has **unique group measure space Cartan** subalgebra for arbitrary free ergodic pmp $\Gamma \curvearrowright (X, \mu)$.

- **Open problem:** uniqueness of arbitrary Cartan subalgebras?
- Theorem also holds for certain $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$, which is crucial to obtain examples of W*-superrigid actions.
- Crucial idea: given another group measure space decomposition $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$, some of the rigidity of $\Gamma_1 < \Gamma$ automatically transfers to some rigidity for Λ.

Why it only works for **group measure space Cartan :**
Writing $L^\infty(X) \rtimes \Gamma = M = L^\infty(Y) \rtimes \Lambda$ we obtain an embedding $\Delta : M \to M \bar{\otimes} M$ given by $\Delta(bv_s) = bv_s \otimes v_s$ for all $b \in L^\infty(Y)$ and $s \in \Lambda$.

Uniqueness of group measure space Cartan

Main question

Which groups Γ have the property that $L^\infty(X) \rtimes \Gamma$ has a unique group measure Cartan for all free ergodic pmp actions $\Gamma \curvearrowright X$?

- (Popa - V, 2009) All $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ where Σ is amenable and weakly malnormal and where Γ contains a non-amenable subgroup with the relative property (T).
- (Fima - V, 2010) All $\Gamma = \mathrm{HNN}(H, \Sigma, \theta)$ satisfying the same cond's.
- (Chifan - Peterson, 2010) All Γ that admit a non-amenable subgroup with the relative property (T) and that admit an unbounded 1-cocycle into a mixing representation. Moreover the same holds for a direct product of two such groups.

$\rightsquigarrow$ **Unified statement** (V, 2010). Same as Chifan - Peterson, but only requiring a representation that is mixing relative to amenable subgroups on which the cocycle is bounded.

Ioana's W*-superrigidity for Bernoulli actions

Theorem (Ioana, 2010)

If Γ is any icc property (T) group, then the Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ is W*-superrigid.

- A real tour de force!

  Ioana proves a very deep structural theorem about all possible embeddings $\Delta : M \to M \bar{\otimes} M$ when $M = L^\infty(X) \rtimes \Gamma$ and $\Gamma \curvearrowright (X, \mu)$ is the Bernoulli action of a property (T) group.

- This structural theorem is applied to the embedding $\Delta : M \to M \bar{\otimes} M$ that comes from another group measure space decomposition.

Other consequences : II_1 factors that cannot be written as (twisted) group von Neumann algebras.

Uniqueness of Cartan subalgebras: open problems

Open problem

Does $L^\infty(X) \rtimes \mathbb{F}_n$ have a unique Cartan subalgebra for arbitrary free ergodic pmp actions $\mathbb{F}_n \curvearrowright (X, \mu)$?

Answer is conjecturally yes. Maybe even for all groups having non-zero first L^2-Betti number.

Open problem

Let $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ be the Bernoulli action of a non-amenable group Γ. Does $L^\infty(X) \rtimes \Gamma$ have a unique Cartan subalgebra.

Answer is again conjecturally yes. There is even no counterexample when $\Gamma \curvearrowright (X, \mu)$ is a mixing action of any non-amenable group Γ.

$\rightsquigarrow$ Ioana : uniqueness of group measure space Cartan when $\Gamma \curvearrowright (X, \mu)$ is the Bernoulli action of an icc property (T) group.

Groups and algebras

Countable group G $\rightsquigarrow$ a variety of algebras, like $\mathbb{C}G, \mathrm{C}^*_\mathrm{r}G, \mathrm{L}G$,

$\rightsquigarrow$ how does the isomorphism class depend on G ?

The group algebra $\mathbb{C}G$ acts on $\ell^2 G$ by left convolution operators.

- The C*-algebra $\mathrm{C}^*_\mathrm{r}G$ is the norm closure of $\mathbb{C}G$.
- The von Neumann algebra $\mathrm{L}G$ is the weak closure of $\mathbb{C}G$.

General principle

In the passage from $\mathbb{C}G$ to $\mathrm{L}G$ the memory of G tends to fade away.

Illustration for torsion free abelian groups

- $\mathrm{C}^*_\mathrm{r}G$ remembers G as the group of connected components of $\mathcal{U}(\mathrm{C}^*_\mathrm{r}G)$. Indeed, $G = \mathbb{Z}^{n_1} \hookrightarrow \mathbb{Z}^{n_2} \hookrightarrow \cdots$.
- All $\mathrm{L}G$ are the same diffuse abelian von Neumann algebra.

C^* versus von Neumann

Both $\mathbb{C}G$ and $C^*_r G$ tend to remember G:

- **Higman's conjecture:** if G is torsion free, the only invertible elements in $\mathbb{C}G$ are the multiples of elements of G.

  (Proven for orderable groups. Implies Kaplansky's conjecture.)

- There are **no examples** of torsion free $G \ncong \Lambda$ with $C^*_r G \cong C^*_r \Lambda$.

Group von Neumann algebras LG are very flexible:

- (Connes, 1976) All LG for G icc and amenable, are isomorphic.
- (Dykema, 1993) If $n \geq 2$ and $\Gamma_1, \ldots, \Gamma_n$ infinite amenable, then $L(\Gamma_1 * \cdots * \Gamma_n) \cong L\mathbb{F}_n$.
- (Ioana, 2006) The $L(\mathbb{F}_n \wr \mathbb{Z})$, $n \geq 2$, are isomorphic.
  Recall: $H \wr \Gamma = H^{(\Gamma)} \rtimes \Gamma$.
- (Bowen, 2009) The $L(H \wr \mathbb{F}_2)$, H non-trivial abelian, are isomorphic.

The big open problems

- Are the free group factors $L\mathbb{F}_n$, $n \geq 2$, isomorphic?
- Connes' rigidity conjecture: if G and Λ are icc property (T) groups and $LG \cong L\Lambda$, then $G \cong \Lambda$.
- Are the $L(SL(n, \mathbb{Z}))$, $n \geq 3$, isomorphic?

Note: Connes' rigidity conjecture would imply that the LG for G icc property (T), **remember the group G.**

Indeed, whenever $LG \cong L\Lambda$, the group Λ must be icc property (T).

W*-superrigidity of group von Neumann algebras (Ioana-Popa-V, '10)

We prove the first W*-superrigidity theorem for certain group von Neumann algebras LG : whenever Λ is a group and $LG \cong L\Lambda$, one must have $G \cong \Lambda$.

W^*-superrigidity theorem

Ioana-Popa-V, 2010

Let Γ_0 be **any** non-amenable group. Consider

$$G = \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{(I)} \rtimes (\Gamma_0 \wr \mathbb{Z}) \quad \text{where} \quad I = (\Gamma_0 \wr \mathbb{Z})/\mathbb{Z} = \Gamma_0^{(\mathbb{Z})}.$$

If Λ is any group such that $L\Lambda \cong LG$, then $\Lambda \cong G$.

Moreover, the isomorphism $L\Lambda \cong LG$ must be group-like.

- We can actually treat a wider class of generalized wreath product groups $(\mathbb{Z}/n\mathbb{Z})^{(I)} \rtimes \Gamma$.

Plain wreath products never work though, because... (IPV 2010)

Let Γ be any torsion-free group and H_0 any non-trivial finite abelian group. There exists a torsion-free group Λ such that $L\Lambda \cong L(H_0 \wr \Gamma)$. In particular, $\Lambda \not\cong H_0 \wr \Gamma$.

Let $n \geq 2$ and H_0 any non-trivial finite abelian group. There are infinitely many non-isomorphic groups Λ for which $L\Lambda \cong L(H_0 \wr PSL(n, \mathbb{Z}))$.