Critical exponents in matrix models

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- Motivation for non commutative geometry
- Definition of the model and main features
- Phases
- Numerical results

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- The idea of uncertainty relations between coordinate and momentum plays crucial role in QM

$$[x, p_x] = i\hbar \quad <=> \quad \Delta x \Delta p \geq \frac{\hbar}{2}$$

- Dirac [1] was one of the first to realize that the above scheme could be applied in attempt to quantize the space itself by promoting the coordinates to some operators satisfying commutator relations. One particular realization is given by [2]. It replaces the coordinates on the sphere by the algebra M_n of $n \times n$ $su(2)$ generators

$$[X_a, X_b] = i\epsilon_{abc} X_c$$

A state in which the coordinates obey those commutation relations is called Fuzzy sphere

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Overview of the model

A particular system that has Fuzzy sphere state as ground state is defined by the action[3] :

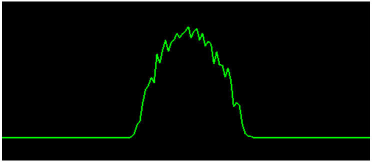
$$\mathcal{S}[X] = NTr\Big(-\frac{1}{4}[X_a, X_b]^2 + \frac{2i\alpha}{3}\epsilon_{abc}X_aX_bX_c\Big) \qquad (1)$$

Where:

- X_a are traceless hermitian $N \times N$ matrices
- α is parameter of the model. If we make the substitution $X_a \to \alpha D_a$, we see that α^4 could be interpreted as dimensionless inverse temperature. It would be convenient to use $\tilde{\alpha} = \frac{\alpha}{\sqrt{N}}$
- The system has two distinct phases separated by the critical value $\tilde{\alpha}^* = \left(\frac{8}{3}\right)^{\frac{3}{4}}$

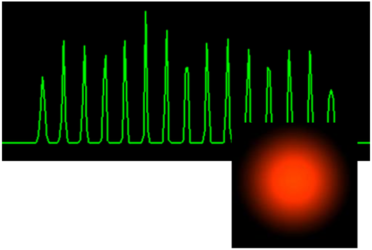
Between matrix phase and fuzzy sphere

Matrix phase



The ground state is determined by mutually commuting random matrices

Fuzzy sphere phase



The matrices have N distinct eigenvalues

Matrix phase

- Occurs for $\tilde{\alpha} < \tilde{\alpha}^*$. High temperature regime
- In this phase YM term is more important and X_a are commuting matrices
- Expectation value of the action is $< S >=< YM >= \frac{3(N^2-1)}{4} > 0$
- Eigenvalues of the matrices are even distributed around zero inside the parabola $f(x) = \frac{3}{4a^3}(a^2 - x^2)$
- Specific heat $C_v = \frac{<S^2>-<S>^2}{N^2} = \frac{<S>}{N^2} - \tilde{\alpha}^4 \frac{d}{d\tilde{\alpha}^4}\left(\frac{<S>}{N^2}\right) = \frac{3}{4}$

Fuzzy sphere phase

- Occurs for $\tilde{\alpha} > \tilde{\alpha}^*$. Low temperature phase
- In this phase the ground state is realized by N-dimensional $su(2)$ generators
$$X_a \sim \alpha L_a$$
- Expectation value of the action is $< S >= -\frac{\alpha^4 N c_2 c_2^{adj}}{12} < 0$
- Eigenvalues of the matrices are $\lambda = \{-\alpha\frac{N-1}{2}, -\alpha\frac{N-3}{2} \ldots \alpha\frac{N-1}{2}\}$
- Specific heat $C_v = 1$

- As the ground state in the fuzzy sphere regime is proportional to L_a. We can define a scaling field ϕ and make the ansatz

$$X_a = \phi\alpha L_a + A_a$$

- We can now compute the effective free energy of the system in terms of ϕ using the statistical mechanics relation $F = -T\ln(Z)$

$$\frac{F}{N^2} = \frac{3}{4}\ln\tilde{\alpha}^4 + \frac{\tilde{\alpha}^4}{2}\left(\frac{\phi^4}{4} - \frac{\phi^3}{3}\right) + \ln\tilde{\alpha}\phi$$

- The equilibrium state of a system is characterized by a minimum of the free energy. This could be used to compute explicitly the values of some observables while the system is in fuzzy sphere phase.

Phase transition

Naturally rises the question what is the behavior of the system in the border between those two completely different regimes. In large N limit some of the properties of the system become divergent in the point $\tilde{\alpha}^*$

- The theory of phase transitions suggests that near a phase transition the divergent observables as function of temperature behave like

$$A(T) \sim A_c + A_\pm \times |T - T^*|^{-\frac{\alpha}{4}} \sim A_c + A_\pm \times |\tilde{\alpha} - \tilde{\alpha}^*|^{-\alpha} \quad (2)$$

Where α is called critical exponent and has specific value for every phase transition

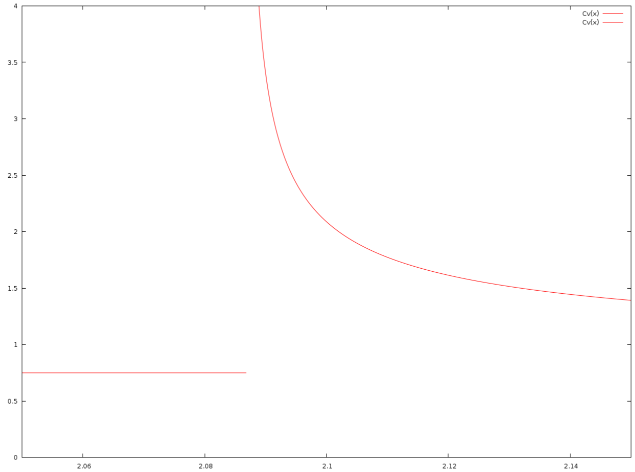
By substitution of extremal values of ϕ near the critical point. For the specific heat we get

$$C_v(\tilde{\alpha}) = \begin{cases} \frac{29}{36} + \frac{1}{2^{\frac{11}{8}} 3^{\frac{7}{8}}} \frac{1}{\sqrt{\tilde{\alpha} - \tilde{\alpha}^*}} + \mathcal{O}((\tilde{\alpha} - \tilde{\alpha}^*)^{\frac{1}{2}}) & \text{for } \tilde{\alpha} > \tilde{\alpha}^* \\ \frac{3}{4} & \text{for } \tilde{\alpha} < \tilde{\alpha}^* \end{cases}$$

The above expressions tell us that the phase transition is rather unusual as the critical exponents for the specific heat are different when the temperature approaches the critical point from above or from below

$$\alpha = \begin{cases} 0 \text{ or } A_- = 0 & \text{for } \tilde{\alpha} - \tilde{\alpha}^* \to 0_- \\ \frac{1}{2} & \text{for } \tilde{\alpha} - \tilde{\alpha}^* \to 0_+ \end{cases}$$

Phase transition and finite systems



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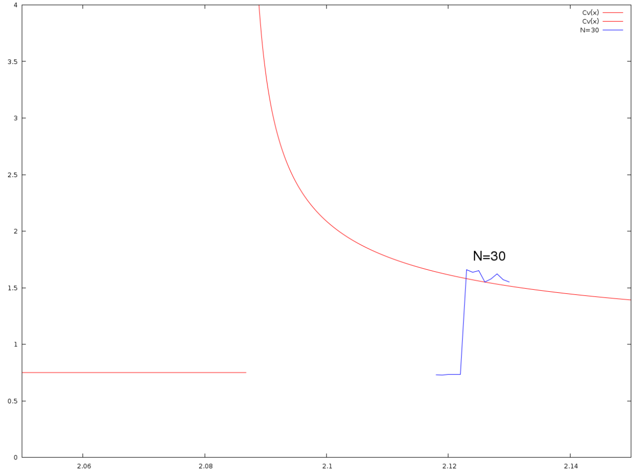

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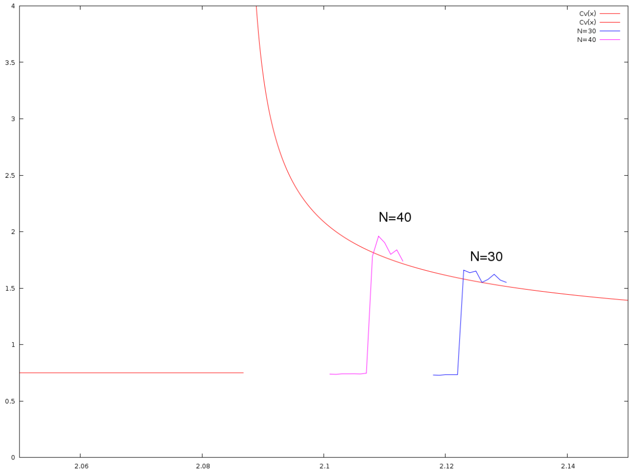

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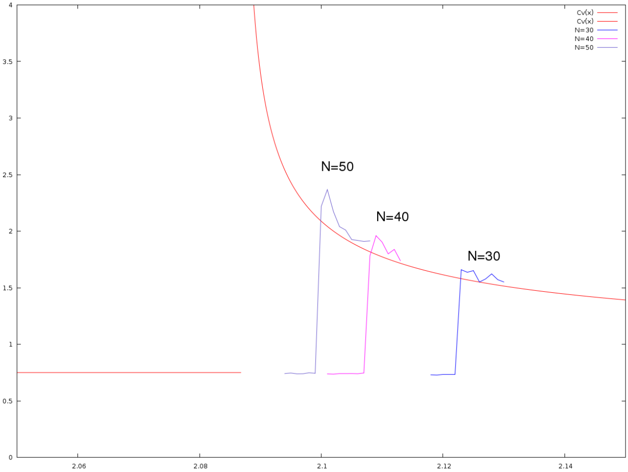

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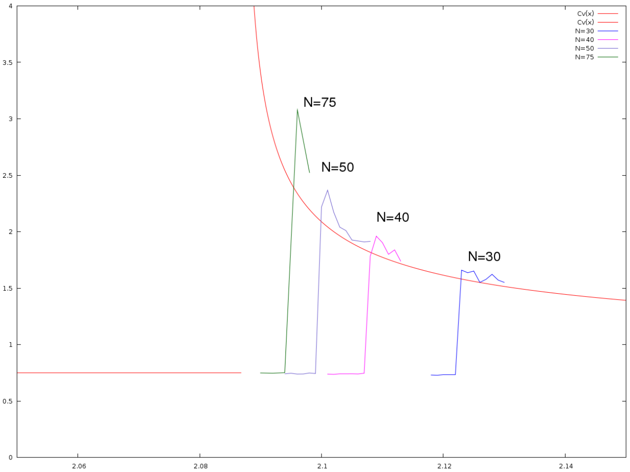

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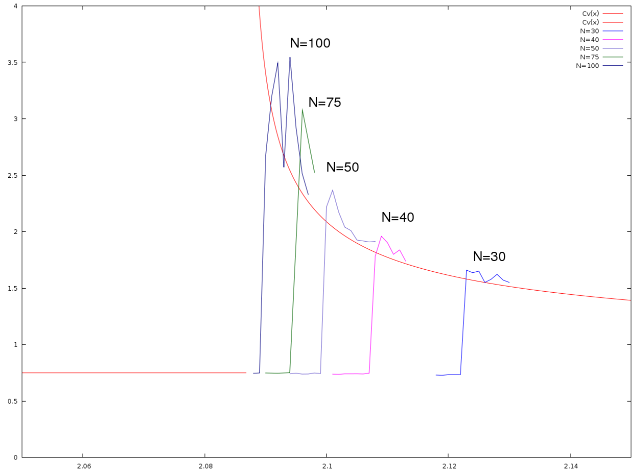


The numerical results indicate some effects which are due to the finite size of the system under consideration

- The critical point is actually a function of the size of the matrices $\tilde{\alpha}^* \to \tilde{\alpha}_m(N)$
- Specific heat is not divergent and rather has a maximum in the critical point $C_v^* \to C_v^*(\tilde{\alpha}_m(N))$

In order to evaluate those effects, following C.Domb and Lebowitz [4] we introduce new critical exponents.

- $$|\tilde{\alpha}_m(N) - \tilde{\alpha}^*| \sim Const \times N^{-\lambda}$$
- $$C_v^*(\tilde{\alpha}_m(N)) \sim Const \times N^{\omega}$$

All three exponents α, λ and ω characterize the phase transition of the specific heat. Under the assumptions (which are known to be true for lattice theories)

- The size of the system scales with the size of the matrices

$$\xi(\alpha_m(N)) \sim L \sim Const \times N \tag{3}$$

  The maximum correlation length for a system with finite size is in the same order as the size of the system L.

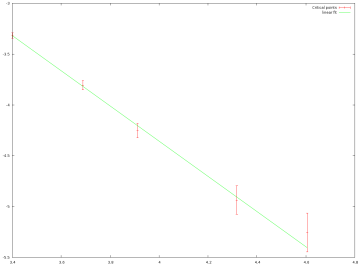
- The shift exponent λ could be related to the correlation length exponent ν

$$\lambda = \frac{1}{\nu} \tag{4}$$

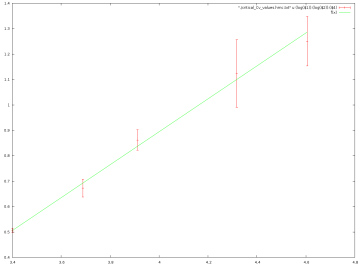
The theory of finite size scaling predicts the relation

$$\omega = \alpha \frac{1}{\nu} = \alpha \lambda \tag{5}$$

We have numerically obtained values of λ and ω.



$(\tilde{\alpha}_m(N) - \tilde{\alpha}^*)$ as function of N in double-logarithmic scale.



$C_v^*(\tilde{\alpha_m})$ as function of N in double-logarithmic scale.

The coefficients in the linear functions represent respectively λ and ω.

The numerical calculations for the exponents are summarized in the following table

Exponent	value	uncertainty	method
λ	1.73	0.06	numerical
ω	0.65	0.03	numerical
α	0.5	Na	theory

$$\lambda\alpha = 0.86 \pm 0.03$$

$$\omega = 0.65 \pm 0.03$$

The numerically computed values for ω and λ don't fulfill (5) within the estimated error

Possible explanations

- The assumption that the size of the system (and the maximal correlation length) is proportional to the matrix size (3) does not hold. In order this to be verified one can study the correlation matrix

$$< (X_a)_{ij}(X_b)_{kl} >$$

  However the above quantity is relatively hard to compute using numerical techniques as it is a matrix with $3N^4$ entries
- Alternativelly one can try to modify relation (3) to

$$L \sim Const \times N \to L \sim Const \times N^{\theta} \tag{6}$$

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- The assumption that the finite scaling depends only of the correlation length (4) does not hold. Which is unlikely since it is shown to be true for wide range of systems. Also RG techniques support it.
- α has a different value, which is due to higher order corrections
- Underestimating the errors in numerically computed quantities ω and λ and/or undetected systematic error in the numerical algorithm

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Thank you for your attention!

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