Inductive limits of projective C^*-algebras.

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Introduction I

Shape theory:

- a tool to study global properties of spaces
- better than homotopy theory if a space has singularities

Idea:

- approximate a space by nicer spaces (building blocks)
- study approximating system instead of original space

	commutative world	noncommutative world
object:	metric space X	separable C^*-algebra A
building blocks:	absolute neighborhood retracts X_k	semiprojective C^*-algebras A_k
approx-imation:	limit (= inverse limit) $X \cong \varprojlim(\ldots \to X_2 \to X_1\}$	colimit (=inductive limit) $A \cong \varinjlim(A_1 \to A_2 \to \ldots\}$

Introduction II

- problem: Are there enough building blocks in order to approximate every space?
- commutative world: Yes.
  (every metric spaces is an inverse limit of ANRs)
- noncommutative world: We don't know.

Question 1.1 (Blackadar)

Which C^*-algebras are inductive limits of semiprojectives?

Theorem 1.2 (Sørensen, T)

$C(X)$ is semiprojective $\Leftrightarrow$ X is an ANR with $\dim(X) \leq 1$.

Theorem 1.3 (Loring, Shulman)

For every C^*-algebra A, the cone $CA = C_0((0,1]) \otimes A$ is an inductive limit of projective C^*-algebras.

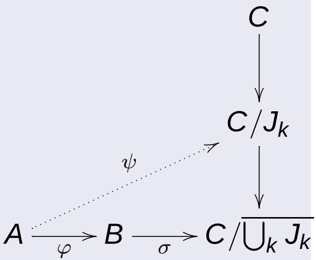
Noncommutative shape theory I

- Blackadar developed noncommutative shape theory for all separable C^*-algebras
- to avoid possible problems with too few building blocks, change notion of approximation:

Definition 2.1

*A morphism $\varphi\colon A \to B$ is called **(weakly) semiprojective**, abbreviated by **(W)SP**, if:*

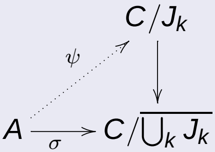
- *$\forall$ C with increasing sequence of ideals $J_1 \lhd J_2 \lhd \ldots \lhd C$, $\sigma\colon B \to C/\overline{\bigcup_k J_k}$ (and $\varepsilon > 0$ and finite subset $F \subset A$)*
- *$\exists$ k and $\psi\colon A \to C/J_k$ such that the diagram commutes (up to ε on F):*



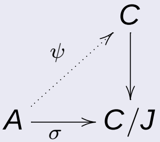
Definition 2.2

*If in the above definition, there is always a lift $\sigma : A \to C$, then the morphism is called **(weakly) projective**.*
A C^*-algebra A is called (weakly) (semi-)projective, if the morphisms $\mathrm{id}_A : A \to A$ is.

A semiprojective:



A projective:



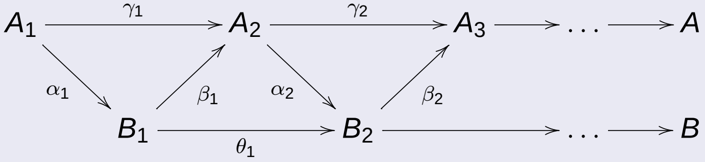
Theorem 2.3 (Blackadar)

*Every C^*-algebras is the inductive limit of an inductive system with semiprojective connecting maps. Such a system is called **shape system**.*

Noncommutative shape theory III

Definition 2.4

A and B are ***shape equivalent****, denoted $A \sim_{Sh} B$, if they have shape systems with intertwinings that make the following diagram commute up to homotopy:*



If only upper triangles commute, say A is ***homotopy dominated*** *by B, denoted $A \precsim_{Sh} B$.*

Remark 2.5

Shape theory extends homotopy theory:

$A \simeq B \Rightarrow A \sim_{Sh} B; \quad A \preceq B \Rightarrow A \precsim_{Sh} B$

converses hold if A, B are SP

Noncommutative shape theory IV

For X a compact, connected, metric space, and $x \in X$, set:

$$C_0(X_0) := C_0(X \setminus \{x\})$$

Example 2.6 (Dadarlat)

If X, Y are compact, connected, metric spaces, then:

$$C_0(X_0) \sim_{Sh} C_0(Y_0) \quad \Leftrightarrow \quad (X, x) \sim_{Sh} (Y, y)$$

This means: noncommutative shape theory = classical shape theory for commutative C^*-algebras. However:

$$C_0(X_0) \otimes \mathbb{K} \sim_{Sh} C_0(Y_0) \otimes \mathbb{K} \quad \Leftrightarrow \quad K^*(X, x) \cong K^*(Y, y)$$

Inductive limits of projective C^*-algebras I

Need criterion to decompose a C^*-algebra as inductive limit.
For example: Given $A = \varinjlim A_k$ and $A_k = \varinjlim_l A_k^l$. When is A an inductive limit of some algebras A_k^l ?

Theorem (Dadarlat, Eilers: $AAH \neq AH$)

There exists $A = \varinjlim A_k$ such that each A_k is AH (an inductive limit of homogeneous algebras), but A is not AH.

Proposition 3.1 (T)

$A = \varinjlim A_k$, each $A_k = \varinjlim_l A_k^l$ inductive limit of f.g. WSP algebras $A_k^l \Rightarrow A$ is inductive limit of some algebras A_k^l.

Notation

$A\mathcal{P}$:= class of inductive limits of *f.g.* projective algebras

Theorem 3.2 (Loring, Shulman)

A is f.g. $\Rightarrow$ the cone $CA = C_0((0,1]) \otimes A$ lies in $A\mathcal{P}$

Inductive limits of projective C^*-algebras II

Theorem 3.3 (T)

Let A be a C^*-algebra. Then the following are equivalent:

1. *A lies in $\mathcal{AP}$*
2. *$A \sim_{Sh} 0$ (A has **trivial shape**)*
3. *A is inductive limit of (f.g.) cones*
4. *A is inductive limit of (f.g.) contractible C^*-algebras*

Remark 3.4

This generalizes Loring, Shulman, since $C_0((0,1]) \otimes A \simeq 0$

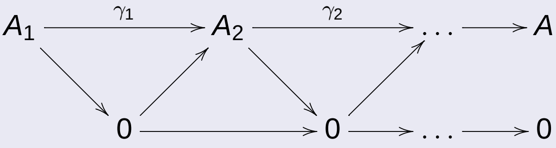
Corollary 3.5 (Closure properties of $\mathcal{AP}$)

$\mathcal{AP}$ is closed under countable direct sums, inductive limits, approximation by sub-C^*-algebras and maximal tensor products with any other C^*-algebra,
i.e., $A \otimes_{max} B \in \mathcal{AP}$ as soon as $A \in \mathcal{AP}$

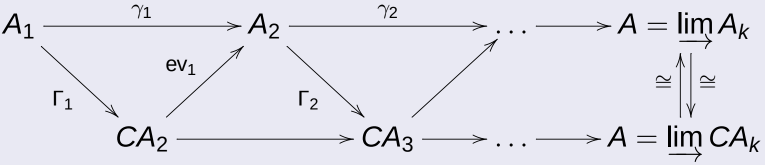
Inductive limits of projective C^*-algebras III

sketch of proof.

"(2)⇒(1)": $A \sim_{Sh} 0$ means:



$\Rightarrow \gamma_k \simeq 0$, which corresponds naturally to a morphism
$\Gamma_k : A_k \to CA_{k+1}$ such that $\gamma_k = \mathrm{ev}_1 \circ \Gamma_k$



each $CA_k \in \mathcal{AP} \Rightarrow A \in \mathcal{AP}$ [by criterion for inductive limit] □

Inductive limits of projective C^*-algebras IV

Corollary 3.6

Every contractible C^*-algebra is an inductive limit of projective C^*-algebras.

Remark 3.7

This is the non-commutative analogue of the following classical result: Every contractible space is an inverse limit of ARs.

Example 3.8

$$X := \{0\} \times [-1,1] \cup \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x \leq 1/\pi\}$$
$$X_0 := X \setminus \{(1/\pi, 0)\}$$

Then $C_0(X_0) \sim_{Sh} 0$, while $C_0(X_0) \not\simeq 0$.
For every algebra A, $C_0(X_0, A)$ is inductive limit of projectives.

Inductive limits of projective C^*-algebras V

Example 3.9 (Dadarlat)

There exists a commutative C^*-algebra $A = C_0(X, x_0)$ such that $A \otimes \mathbb{K} \simeq 0$ (in particular $A \otimes \mathbb{K} \sim_{Sh} 0$), while $A \not\sim_{Sh} 0$.

Corollary 3.10

Trivial shape does not pass to full hereditary sub-C^*-algebras.

Proposition 3.11 (T)

Let (A_k, γ_k) be an inductive system. Then there exists an inductive system (B_k, δ_K) with surjective connecting morphisms and such that $\varinjlim A_k \cong \varinjlim B_k$.
Moreover, we may assume $B_k = A_k * \mathcal{F}_\infty$, where $\mathcal{F}_\infty := C^*(x_1, x_2, \dots \mid \|x_i\| \leq 1)$ is the universal C^*-algebra generated by a countable number of contractive generators.
If A_k is (semi-)projective, then so is $A_k * \mathcal{F}_\infty$.

Inductive limits of projective C^*-algebras VI

Corollary 3.12

$A \sim_{Sh} 0 \Rightarrow A$ is inductive limit of projective C^*-algebra with surjective connecting morphisms.

Corollary 3.13

Projectivity does not pass to full hereditary sub-C^*-algebras.

Proof.

Use example of Dadarlat: $A \otimes \mathbb{K} \simeq 0$ but $A \not\sim_{Sh} 0$
$A \otimes \mathbb{K} \cong \varinjlim P_k$ with P_k projective and surjective connecting morphisms $\gamma_k : P_k \to P_{k+1}$
Consider $Q_k := \gamma_{\infty,k}^{-1}(A) \subset P_k$. Then $A \cong \varinjlim Q_k$.
$A \subset A \otimes \mathbb{K}$ full hereditary $\Rightarrow Q_k \subset P_k$ full hereditary.
If all Q_k were projective, then A would have trivial shape, a contradiction. Thus, some algebras Q_k are not projective. □

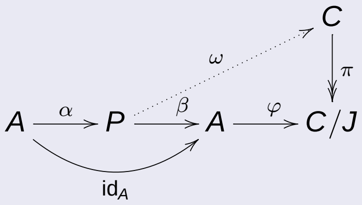
Relations between the different classes I

Lemma 4.1

Given $\alpha\colon A \to P$, $\beta\colon P \to A$ with $\beta \circ \alpha = \mathrm{id}_A$ and P projective.
$\Rightarrow$ A projective.

Proof.

Given lifting problem $\varphi\colon A \to C/J$, need lift $\psi : A \to C$.



P projective $\Rightarrow$ get lift $\omega\colon P \to C$ for $\varphi \circ \beta\colon P \to C/J$
Then $\psi := \omega \circ \alpha : A \to B$ is desired lift for φ □

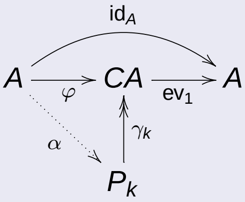
Relations between the different classes II

Theorem 4.2 (T)

A projective $\Leftrightarrow$ A semiprojective and $A \simeq 0$.

Proof.

homotopy $\mathrm{id}_A \simeq 0$ induces natural morphism $\varphi\colon A \to CA$ such that $\mathrm{id}_A = \mathrm{ev}_1 \circ \varphi$.
$\Rightarrow CA \cong \varinjlim P_k$ for projectives P_k with surjective connecting maps [by L-S]
Semiprojectivity of A gives lift $\alpha : A \to P_k$ (to some k) such that $(\mathrm{ev}_1 \circ \gamma_k) \circ \alpha = \mathrm{id}_A$. Lemma implies A is projective.



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- this verifies a conjecture of Loring

Relations between the different classes III

Proposition 4.3 (Loring)

A weakly projective C^*-algebra has trivial shape.

WP also implies WSP. Other implication proved using that C^*-algebra with trivial shape is inductive limit of projectives:

Theorem 4.4

A weakly projective $\Leftrightarrow$ *A weakly semiprojective and* $A \sim_{Sh} 0$.

The above theorems are exact analogues of results in classical shape theory:

commutative (for space X):

- X is AR
  $\Leftrightarrow X$ is ANR and $X \simeq *$
- X is AAR
  $\Leftrightarrow X$ is AANR and $X \sim_{Sh} *$

noncommutative (for C^*-algebra A):

- A is P
  $\Leftrightarrow A$ is SP and $A \simeq 0$
- A is WP
  $\Leftrightarrow A$ is WSP and $A \sim_{Sh} 0$

Inductive limits of semiprojectives I

Generalizing the above ideas, and using a mapping cylinder construction, one can prove the following:

Theorem 4.5 (T)

The class $\mathcal{ASP}$ is closed under shape domination: If $A \precsim_{Sh} B$ and B is an inductive limit of f.g. semiprojective C^*-algebras, then so is A.

If $A \sim_{Sh} C$ and $B \sim_{Sh} D$, then $A \otimes_{\max} B \sim_{Sh} C \otimes_{\max} D$. Assume $B \sim_{Sh} \mathbb{C}$. Then A lies in $\mathcal{ASP}$ if and only if $A \otimes_{\max} B$ does.

Example 4.6

We have $C([0,1]^k) \simeq \mathbb{C}$. Thus $C([0,1]^k, A)$ is a limit of semiprojectives if and only if A is.
For example, $C([0,1]^k, \mathcal{O}_n)$ is a limit of semiprojectives.

Open Problem 4.7 (Katsura)

Is $C([0,1], \mathcal{O}_n)$ semiprojective?