Inductive limits of projective $C^*$-algebras.

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Introduction I

Shape theory:
- a tool to study global properties of spaces
- better than homotopy theory if a space has singularities

Idea:
- approximate a space by nicer spaces (building blocks)
- study approximating system instead of original space

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<th>object:</th>
<th>commutative world</th>
<th>noncommutative world</th>
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<td>metric space $X$</td>
<td>separable $C^*$-algebra $A$</td>
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<td>absolute neighborhood retracts $X_k$</td>
<td>semiprojective $C^*$-algebras $A_k$</td>
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<td>limit (= inverse limit)</td>
<td>colimit (=inductive limit)</td>
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$X \cong \lim\limits_{\leftarrow} \{ \ldots \to X_2 \to X_1 \}$

$A \cong \lim\limits_{\rightarrow} \{ A_1 \to A_2 \to \ldots \}$
problem: Are there enough building blocks in order to approximate every space?

commutative world: Yes.
(every metric spaces is an inverse limit of ANRs)

noncommutative world: We don’t know.

Question 1.1 (Blackadar)
Which $C^*$-algebras are inductive limits of semiprojectives?

Theorem 1.2 (Sørensen, T)
$C(X)$ is semiprojective $\iff X$ is an ANR with $\dim(X) \leq 1$.

Theorem 1.3 (Loring, Shulman)
For every $C^*$-algebra $A$, the cone $CA = C_0((0,1]) \otimes A$ is an inductive limit of projective $C^*$-algebras.
Blackadar developed noncommutative shape theory for all separable $C^*$-algebras to avoid possible problems with too few building blocks, change notion of approximation:

**Definition 2.1**

A morphism $\varphi : A \to B$ is called (weakly) semiprojective, abbreviated by (W)SP, if:

- $\forall C$ with increasing sequence of ideals $J_1 \triangleleft J_2 \triangleleft \ldots \triangleleft C$, $\sigma : B \to C/\bigcup_k J_k$ (and $\varepsilon > 0$ and finite subset $F \subset A$)

- $\exists k$ and $\psi : A \to C/J_k$ such that the diagram commutes (up to $\varepsilon$ on $F$):
Definition 2.2

If in the above definition, there is always a lift \( \sigma : A \to C \), then the morphism is called (weakly) projective.

A \( C^* \)-algebra \( A \) is called (weakly) (semi-)projective, if the morphisms \( \text{id}_A : A \to A \) is.

A semiprojective:

\[
\begin{array}{ccc}
A & \xrightarrow{\sigma} & C/igcup_k J_k \\
\downarrow \psi & & \downarrow \\
C/J_k & \xrightarrow{\psi} & C
\end{array}
\]

A projective:

\[
\begin{array}{ccc}
A & \xrightarrow{\sigma} & C/J \\
\downarrow \psi & & \\
C & \xrightarrow{\psi} & C
\end{array}
\]

Theorem 2.3 (Blackadar)

Every \( C^* \)-algebras is the inductive limit of an inductive system with semiprojective connecting maps. Such a system is called shape system.
Definition 2.4

A and B are **shape equivalent**, denoted \( A \sim_{\text{Sh}} B \), if they have shape systems with intertwinings that make the following diagram commute up to homotopy:

\[
\begin{array}{ccccccc}
A_1 & \xrightarrow{\gamma_1} & A_2 & \xrightarrow{\gamma_2} & A_3 & \ldots & A \\
\downarrow{\alpha_1} & & \downarrow{\alpha_2} & & \downarrow{\beta_2} & & \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\theta_1} & \ldots & & B
\end{array}
\]

If only upper triangles commute, say A is **homotopy dominated** by B, denoted \( A \preceq_{\text{Sh}} B \).

Remark 2.5

Shape theory extends homotopy theory:
\( A \simeq B \Rightarrow A \sim_{\text{Sh}} B \); \( A \preceq B \Rightarrow A \preceq_{\text{Sh}} B \)

converses hold if \( A, B \) are SP
For $X$ a compact, connected, metric space, and $x \in X$, set:

$$C_0(X_0) := C_0(X \setminus \{x\})$$

**Example 2.6 (Dadarlat)**

If $X, Y$ are compact, connected, metric spaces, then:

$$C_0(X_0) \sim_{Sh} C_0(Y_0) \iff (X, x) \sim_{Sh} (Y, y)$$

This means: noncommutative shape theory = classical shape theory for commutative $C^*$-algebras. However:

$$C_0(X_0) \otimes K \sim_{Sh} C_0(Y_0) \otimes K \iff K^*(X, x) \cong K^*(Y, y)$$
Inductive limits of projective $C^*$-algebras I

Need criterion to decompose a $C^*$-algebra as inductive limit. For example: Given $A = \varinjlim A_k$ and $A_k = \varinjlim_A l A_k$. When is $A$ an inductive limit of some algebras $A_k$?

**Theorem (Dadarlat, Eilers: $AAH \neq AH$)**

There exists $A = \varinjlim A_k$ such that each $A_k$ is AH (an inductive limit of homogeneous algebras), but $A$ is not AH.

**Proposition 3.1 (T)**

$A = \varinjlim A_k$, each $A_k = \varinjlim A_k^l$, inductive limit of f.g. WSP algebras $A_k^l \Rightarrow A$ is inductive limit of some algebras $A_k^l$.

**Notation**

$A_P :=$ class of inductive limits of f.g. projective algebras

**Theorem 3.2 (Loring, Shulman)**

$A$ is f.g. $\Rightarrow$ the cone $CA = C_0((0,1]) \otimes A$ lies in $A_P$
Theorem 3.3 (T)

Let $A$ be a $C^*$-algebra. Then the following are equivalent:

1. $A$ lies in $A\mathcal{P}$
2. $A \sim_{Sh} 0$  (A has **trivial shape**)
3. $A$ is inductive limit of (f.g.) cones
4. $A$ is inductive limit of (f.g.) contractible $C^*$-algebras

Remark 3.4

This generalizes Loring, Shulman, since $C_0((0, 1]) \otimes A \simeq 0$

Corollary 3.5 (Closure properties of $A\mathcal{P}$)

$A\mathcal{P}$ is closed under countable direct sums, inductive limits, approximation by sub-$C^*$-algebras and maximal tensor products with any other $C^*$-algebra, i.e., $A \otimes_{max} B \in A\mathcal{P}$ as soon as $A \in A\mathcal{P}$
sketch of proof.

"(2) ⇒ (1)" : $A \sim_{Sh} 0$ means:

$$
\begin{array}{ccccccc}
A_1 & \xrightarrow{\gamma_1} & A_2 & \xrightarrow{\gamma_2} & \ldots & \rightarrow & A \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & \rightarrow & \ldots & \rightarrow & 0
\end{array}
$$

$\Rightarrow \gamma_k \simeq 0$, which corresponds naturally to a morphism $\Gamma_k : A_k \rightarrow CA_{k+1}$ such that $\gamma_k = ev_1 \circ \Gamma_k$

$$
\begin{array}{ccccccc}
A_1 & \xrightarrow{\gamma_1} & A_2 & \xrightarrow{\gamma_2} & \ldots & \rightarrow & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
CA_2 & \xrightarrow{\Gamma_2} & CA_3 & \rightarrow & \ldots & \rightarrow & A = \varprojlim A_k
\end{array}
$$

$$
\begin{array}{ccccccc}
CA_2 & \xleftarrow{\text{ev}_1} & A_2 & \xrightarrow{\gamma_2} & \ldots & \rightarrow & A = \varprojlim CA_k \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma_1 & & \Gamma_2 & & \\
C & & & & & & \mathcal{P}
\end{array}
$$

each $CA_k \in \mathcal{A} \Rightarrow A \in \mathcal{A} \mathcal{P}$ [by criterion for inductive limit]
Corollary 3.6

Every contractible $C^*$-algebra is an inductive limit of projective $C^*$-algebras.

Remark 3.7

This is the non-commutative analogue of the following classical result: Every contractible space is an inverse limit of ARs.

Example 3.8

$$X := \{0\} \times [-1, 1] \cup \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x \leq 1/\pi\}$$

$$X_0 := X \setminus \{(1/\pi, 0)\}$$

Then $C_0(X_0) \sim_{Sh} 0$, while $C_0(X_0) \not\cong 0$.

For every algebra $A$, $C_0(X_0, A)$ is inductive limit of projectives.
Example 3.9 (Dadarlat)

There exists a commutative $C^*$-algebra $A = C_0(X, x_0)$ such that $A \otimes K \simeq 0$ (in particular $A \otimes K \sim_{Sh} 0$), while $A \not\sim_{Sh} 0$.

Corollary 3.10

*Trivial shape does not pass to full hereditary sub-$C^*$-algebras.*

Proposition 3.11 (T)

Let $(A_k, \gamma_k)$ be an inductive system. Then there exists an inductive system $(B_k, \delta_k)$ with surjective connecting morphisms and such that $\lim_A A_k \cong \lim_B B_k$.

Moreover, we may assume $B_k = A_k \ast \mathcal{F}_\infty$, where $\mathcal{F}_\infty := C^* (x_1, x_2, \ldots | \|x_i\| \leq 1)$ is the universal $C^*$-algebra generated by a countable number of contractive generators.

If $A_k$ is (semi-)projective, then so is $A_k \ast \mathcal{F}_\infty$. 
Corollary 3.12

\[ A \sim_{Sh} 0 \Rightarrow A \text{ is inductive limit of projective } C^* \text{-algebra with surjective connecting morphisms.} \]

Corollary 3.13

Projectivity does not pass to full hereditary sub-\( C^* \)-algebras.

Proof.

Use example of Dadarlat: \( A \otimes K \simeq 0 \) but \( A \sim_{Sh} 0 \)
\( A \otimes K \simeq \lim_{\rightarrow} P_k \) with \( P_k \) projective and surjective connecting morphisms \( \gamma_k : P_k \rightarrow P_{k+1} \)
Consider \( Q_k := \gamma_{\infty,k}^{-1}(A) \subset P_k \). Then \( A \simeq \lim_{\rightarrow} Q_k \).
\( A \subset A \otimes K \) full hereditary \( \Rightarrow Q_k \subset P_k \) full hereditary.
If all \( Q_k \) were projective, then \( A \) would have trivial shape, a contradiction. Thus, some algebras \( Q_k \) are not projective. \( \square \)
Lemma 4.1

Given \( \alpha : A \rightarrow P, \beta : P \rightarrow A \) with \( \beta \circ \alpha = \text{id}_A \) and \( P \) projective.

\[ \Rightarrow A \text{ projective.} \]

Proof.

Given lifting problem \( \varphi : A \rightarrow C/J \), need lift \( \psi : A \rightarrow C \).

\[ P \text{ projective } \Rightarrow \text{get lift } \omega : P \rightarrow C \text{ for } \varphi \circ \beta : P \rightarrow C/J \]

Then \( \psi := \omega \circ \alpha : A \rightarrow B \) is desired lift for \( \varphi \).
Relations between the different classes II

Theorem 4.2 (T)

A projective ⇔ A semiprojective and $A \simeq 0$.

Proof.

Homotopy $\text{id}_A \simeq 0$ induces natural morphism $\phi: A \to CA$ such that $\text{id}_A = \text{ev}_1 \circ \phi$.

$\Rightarrow CA \simeq \lim P_k$ for projectives $P_k$ with surjective connecting maps [by L-S]

Semiprojectivity of $A$ gives lift $\alpha: A \to P_k$ (to some $k$) such that $(\text{ev}_1 \circ \gamma_k) \circ \alpha = \text{id}_A$. Lemma implies $A$ is projective.

this verifies a conjecture of Loring
Proposition 4.3 (Loring)

A weakly projective $C^*$-algebra has trivial shape.

WP also implies WSP. Other implication proved using that $C^*$-algebra with trivial shape is inductive limit of projectives:

Theorem 4.4

A weakly projective $\iff$ A weakly semiprojective and $A \sim_{Sh} 0$.

The above theorems are exact analogues of results in classical shape theory:

**commutative**
(for space $X$):

- $X$ is AR
  $\iff X$ is ANR and $X \sim^*$
- $X$ is AAR
  $\iff X$ is AANR and $X \sim_{Sh}^*$

**noncommutative**
(for $C^*$-algebra $A$):

- $A$ is P
  $\iff A$ is SP and $A \sim 0$
- $A$ is WP
  $\iff A$ is WSP and $A \sim_{Sh} 0$
Generalizing the above ideas, and using a mapping cylinder construction, one can prove the following:

**Theorem 4.5 (T)**

The class $\mathcal{A}_{\text{SP}}$ is closed under shape domination:

If $A \preccurlyeq_{\text{Sh}} B$ and $B$ is an inductive limit of f.g. semiprojective $C^*$-algebras, then so is $A$.

If $A \sim_{\text{Sh}} C$ and $B \sim_{\text{Sh}} D$, then $A \otimes_{\text{max}} B \sim_{\text{Sh}} C \otimes_{\text{max}} D$. Assume $B \sim_{\text{Sh}} \mathbb{C}$. Then $A$ lies in $\mathcal{A}_{\text{SP}}$ if and only if $A \otimes_{\text{max}} B$ does.

**Example 4.6**

We have $C([0, 1]^k) \preccurlyeq \mathbb{C}$. Thus $C([0, 1]^k, A)$ is a limit of semiprojectives if and only if $A$ is. For example, $C([0, 1]^k, \mathcal{O}_n)$ is a limit of semiprojectives.

**Open Problem 4.7 (Katsura)**

Is $C([0, 1], \mathcal{O}_n)$ semiprojective?