Various mathematical aspects of Φ^4 quantum field theory (QFT) on the Moyal space

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Plan

- Φ^4 theory on the Moyal space; ribbon graphs
- Renormalizability on the Moyal space (UV/IR mixing)
- Parametric representation
- Conclusions and perspectives

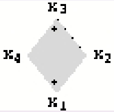
Scalar field theory on the Moyal space

Φ^4 model:

$$\mathcal{S} = \int d^4x \, [\frac{1}{2}\partial_\mu \Phi \star \partial^\mu \Phi + \frac{1}{2} m^2 \Phi \star \Phi + \frac{\lambda}{4!} \Phi \star \Phi \star \Phi \star \Phi],$$

$\int d^4x \, (\Phi \star \Phi)(x) = \int d^4x \, \Phi(x) \, \Phi(x)$
(same propagation)

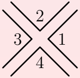
Implications of the use of the Moyal product in QFT



interaction (in position space)

$$\int d^D x \Phi^{\star 4}(x) \propto \int \prod_{i=1}^{4} d^D x_i \Phi(x_i) \delta(x_1 - x_2 + x_3 - x_4)$$
$$e^{2i \sum_{1 \le i < j \le 4} (-1)^{i+j+1} x_i \Theta^{-1} x_j}$$

$\hookrightarrow$ non-locality (oscillation $\propto$ area of parallelogram)
$\hookrightarrow$ restricted invariance: only under cyclic permutation
$\rightarrow$ ribbon graphs (relation with matrix models)



Feynman graphs in NCQFT

$\rightarrow$ clear distinction between planar and non-planar graphs

n - number of vertices,

L - number of internal lignes,

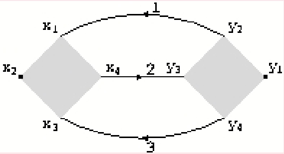
F - number of faces,

$$2 - 2g = n - L + F$$

$g \in \mathbb{N}$ - genus

$g = 0$ - planar graph

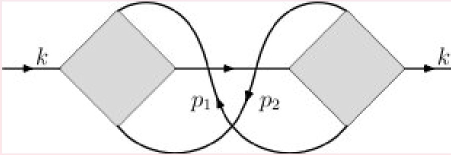
example:



$n = 2,\ L = 3,\ F = 3,\ g = 0$

$g \geq 1$ - non-planar graph

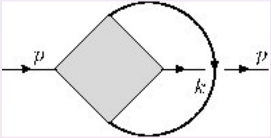
example:



$n = 2,\ L = 3,\ F = 1,\ g = 1$

Renormalization on the Moyal space

UV/IR mixing (S. Minwalla *et. al.*, *JHEP*, 2000, J. Magnen *et. al.*, *Europhys. Lett.*, 2009)



$$B = 2$$

B - number of faces broken by external lignes

$B > 1$, *planar irregular graph*

$$\lambda \int d^4k \frac{e^{ik_\mu \Theta^{\mu\nu} p_\nu}}{k^2 + m^2} \to_{|p| \to 0} \frac{1}{\theta^2 p^2}$$

$\to$ non-renormalizability!

A first solution to this problem - the Grosse-Wulkenhaar model

additional harmonic term

(H. Grosse and R. Wulkenhaar, *Comm. Math. Phys.*, 2005)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2}(\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi,$$

$$\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu} x^\nu.$$

modification of the propagator - the model becomes renormalizable

↪ various proofs (Polchinski flow equation method, BPHZ multi-scale analysis, dimensional regularization)

↪ combinatorial Hopf algebra structure of non-commutative renormalization

(A. T. and F. Vignes-Tourneret, *J. Noncomm. Geom.*, 2008)

Translation-invariant renormalizable scalar model

(R. Gurău, J. Magen, V. Rivasseau and A. T., *Commun. Math. Phys.* (in press))

the Grosse-Wulkenhaar model break translation-invariance !

the complete propagator:

$$C(p,m,\theta) = \frac{1}{p^2 + a\frac{1}{\theta^2 p^2} + \mu^2}$$

arbitrary planar irregular 2-point function: same type of $\frac{1}{p^2}$ behavior !

J. Magnen *et. al., Europhys. Lett.*, 2009

$\hookrightarrow$ other modification of the action:

$$S = \int d^4p\, [\frac{1}{2}p_\mu \phi \star p^\mu \phi + \frac{1}{2}a\frac{1}{\theta^2 p^2}\phi\star\phi + \frac{1}{2}m^2\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi].$$

renormalizability at any order in perturbation theory !

Scales - renormalization group

definition of the RG scales:

- locus where $C^{-1}(p)$ is big
- locus where $C^{-1}(p)$ is low

$$C^{-1}_{\mathrm{comm}}(p) = p^2$$

$$C^{-1}_{GW} = p^2 + \Omega^2 x^2$$

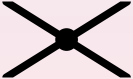
$$C^{-1}(p) = p^2 + \frac{a}{\theta^2 p^2}$$

mixing of the UV and IR scales - key of the renormalization

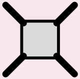
Renormalizability of NCQFT: locality $\rightarrow$ "Moyality"

QFT $\rightarrow$ NCQFT

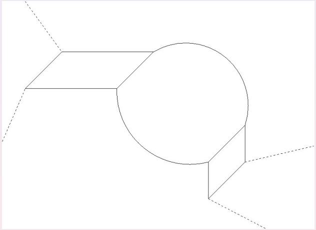
locality $\rightarrow$ "Moyality"

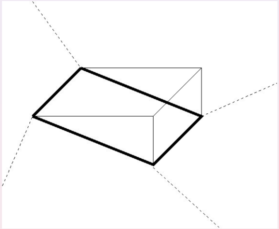


$\rightarrow$



The principle of "Moyality" - Feynman graph level





$\hookrightarrow$ study of the non-commutative combinatorial Dyson-Schwinger equations (*i. e.* the Hochschild cohomolgy of the non-commutative Connes-Kreimer Hopf algebra)

(A. T. and D. Kreimer, *J. Noncomm. Geom.* (in press)

Parametric representation for commutative QFT

introduction of the Schwinger parameters α:

$$\frac{1}{p^2 + m^2} = \int d\alpha \, e^{-\alpha(p^2+m^2)}.$$

$$\mathcal{A}(p) = \int_0^\infty \frac{e^{-V(p,\alpha)/U(\alpha)}}{U(\alpha)^{\frac{D}{2}}} \prod_{\ell=1}^{L} (e^{-m^2\alpha_\ell} d\alpha_\ell)$$

U, V - polynomials in the parameters α_ℓ

$$U \;\; = \;\; \sum_{\mathcal{T}} \prod_{\ell \notin \mathcal{T}} \alpha_\ell \; ,$$

$\mathcal{T}$ - a (spanning) tree of the graph

Parametric representation of the noncommutative model

((A. T., J. Phys. **A** (2009), T. Krajewski *et. al. J. Noncomm. Geom.* (2010))

$$\mathcal{A}^\star(p) = \int_0^\infty \frac{e^{-V^\star(p,\alpha)/U^\star(\alpha)}}{U^\star(\alpha)^{\frac{D}{2}}} \prod_{\ell=1}^{L} (e^{-m^2\alpha_\ell} d\alpha_\ell)$$

Définition:

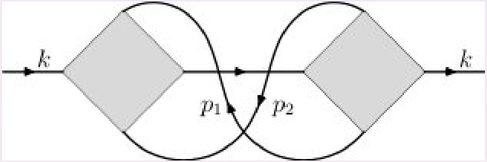
A $\star-$tree (or quasi-trees) $\mathcal{T}^\star$ is a subgraph with only one face.

$$U^\star(\alpha) = \left(\frac{\theta}{2}\right)^b \sum_{\mathcal{T}^\star} \prod_{\ell \notin \mathcal{T}^\star} 2\frac{\alpha_\ell}{\theta}$$

$b = F - 1 + 2g.$

$\hookrightarrow$ similar formula for the $V^\star$ polynomial

A simple example



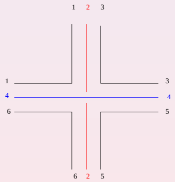
the $\star-$trees: $\{1\}$, $\{2\}$, $\{3\}$ and $\{1,2,3\}$

$$U^\star(\alpha_1,\alpha_2,\alpha_3) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \frac{\theta^2}{4}.$$

$\hookrightarrow$ rich topological and combinatorial structures

Conclusion and perspectives

- applications of these techniques for the renormalizability study of loop quantum gravity models (A. T., *Class. Quant. Grav.* 2010)
- generalization to tensor models (appearing in recent approaches (group field theory) for a theory of quantum gravity)



various non-trivial topological and combinatorial structures:

- topological graph polynomials

  (R. Gurău, *Annales Henri Poincaré*, A. T., arXiv:1012.1798)
- combinatorial Hopf algebras structures

Thank you for your attention!

Glimpse of the mathematical setup

the Moyal space

The *Moyal algebra* is the linear space of smooth and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^{\mathcal{D}})$ equipped with the

$$(f \star g)(x) = \int \frac{d^D k}{(2\pi)^D} d^D y \, f(x + \frac{1}{2}\Theta \cdot k) g(x+y) e^{ik \cdot y}.$$

$\star$ - Moyal product (non-local, noncommutative product)

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}, \tag{2}$$

$$\Theta = \begin{pmatrix} \Theta_2 & 0 \\ 0 & \Theta_2 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

BPHZ renormalization scheme
renormalization conditions

$$\Gamma^4(0,0,0,0) = -\lambda_r,\ \ G^2(0,0) = \frac{1}{m^2},\ \ \frac{\partial}{\partial p^2} G^2(p,-p)|_{p=0} = -\frac{1}{m^4}. \quad (3)$$

where Γ^4 and G^2 are the connected functions and
$0 \to p_m$ (the minimum of $p^2 + \frac{a}{\theta^2 p^2}$)

power counting theorem:

$\hookrightarrow$ 2− and 4−point planar functions *(primitively divergent)*

- planar regular 2−point function: wave function and mass renormalization
- planar regular 4−point function: coupling constant renormalization
- planar irregular 2−point function: renormalization of the constant *a*
- planar irregular 4−point function: convergent

non-planar tadpole graphs insertions - IR convergence (equivalent proof for these particular graphs

D. Blaschke, F. Gieres, E. Kronberger, T. Reis, M. Schweda, R. I.P. Sedmik *et. al.*, *JHEP* '08)

Comparison with other models

	the "naive" model		*GW model*		*model* (1)	
	2P	4P	2P	4P	2P	4P
planar regular	ren.	ren.	ren.	ren.	ren.	ren.
planar irregular	UV/IR	log UV/IR	conv.	conv.	finite ren.	conv.
non-planar	IR div.	IR div.	conv.	conv.	conv.	conv.

Decomposition of the propagator

(J. Ben Geloun and A. T., *Lett. Math. Phys.* '08, 0806.3886 [math-ph])

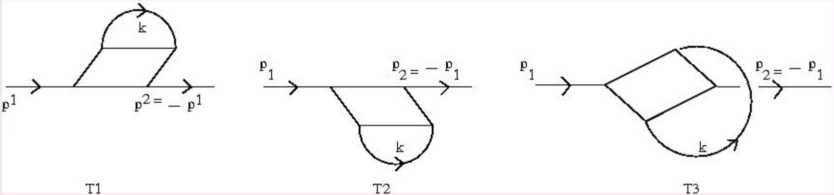
$$C(p,m,\theta) = \frac{1}{p^2 + m^2 + \frac{a}{\theta^2 p^2}}.$$

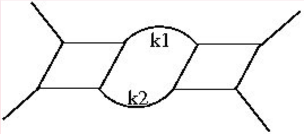
$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A+B},$$

$$A = p^2 + m^2, \quad B = \frac{a}{\theta^2 p^2}.$$

$$\Rightarrow C(p,m,\theta) = \frac{1}{p^2+m^2} - \frac{a}{\theta^2} \frac{1}{p^2+m^2} \frac{1}{p^2+m_1^2} \frac{1}{p^2+m_2^2}$$

Renormalization group flow





Wave function renormalization - γ function

$$\Sigma(p) = \Sigma_{\rm plr}(p) + \Sigma_{\rm pli}(p)$$

$\Sigma(p)$ - self-energy

$$Z = 1 - \frac{\partial}{\partial p^2}\Sigma_{\rm plr}(p)$$

the noncommutative correction is irrelevant

(it leads to a convergent integral)

$$\Sigma_{\rm plr}(p) = \int d^4k \left(\frac{1}{k^2+m^2} - \frac{a}{\theta^2}\frac{1}{k^2+m^2}\frac{1}{k^2+m_1^2}\frac{1}{k^2+m_2^2} \right).$$

$$\Rightarrow Z = 1 + \mathcal{O}(\lambda^2)$$

$$\Rightarrow \gamma = 0 + \mathcal{O}(\lambda^2)$$

mass renormalization:

$$-\frac{\Sigma_{\text{plr}}}{Z}, \quad \beta_m \propto \beta_m^{\text{comutativ}}.$$

renormalization of the parameter a:

$$\beta_a = 0.$$

coupling constant renormalization:

$$-\frac{\Gamma^4}{Z^2},$$

Γ^4 - 4−point funtion
the noncommutative correction is irrelevant
(it leads to a convergent integral)

$$\lambda^2 \int d^4k \left(\frac{1}{k^2+m^2} - \frac{a}{\theta^2} \frac{1}{k^2+m^2} \frac{1}{k^2+m_1^2} \frac{1}{k^2+m_2^2} \right)^2.$$

$$\beta_\lambda \propto \beta_\lambda^{\text{comutativ}}.$$

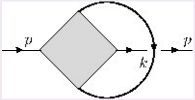
the noncommutative corrections of the propagators $\rightarrow$ irrelevant contribution to the RG flow at any order in perturbation theory

explicit quantum corrections at $1-$loop level - use of Bessel functions (D. Blaschke *et. al.*, *JHEP* '08, 0807.3270 [hep-th])

Commutative limit

(J. Magnen, V. Rivasseau and A. T., submitted to *J. High Energy Phys.*, 0807.4093 [hep-th])

$\hookrightarrow$ crucial issue in NCQFTs



"non-planar" tadpole behavior:

$$\int d^4k \frac{e^{ik_\mu \Theta^{\mu\nu} p_\nu}}{k^2 + m^2}$$

- if $\theta \neq 0$, $\frac{1}{\theta^2 p^2}$ (for $|p| \to 0$)
- if $\theta \to 0$, $\int d^4k \frac{1}{k^2+m^2}$
  (the usual wave function and mass renormalization)

arbitrary planar irregular 2-point function: same type of behavior !

$$\delta_m = \delta_{m'} + \delta_{m''} + \delta_{m'''}.$$

"The amount of theoretical work one has to cover before being able to solve problems of real practical value is rather large, but this circumstance is [...] likely to become more pronounced in the theoretical physics of the future."
P.A.M. Dirac, *"The principles of Quantum Mechanics"*, 1930

Thank you for your attention

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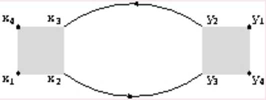
$$(f \star g)(x) = \int \frac{d^D k}{(2\pi)^D} d^D y \, f(x + \frac{1}{2}\Theta \cdot k) g(x + y) e^{ik \cdot y}$$

power counting:

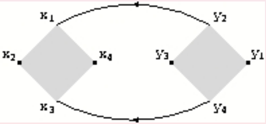
$$\omega = 4g + \frac{N-4}{2} + (B-1)$$

B - number of broken faces

improved factor in the broken faces
example:



$n = 2,\ L = 2,\ F = 2,\ B = 1$



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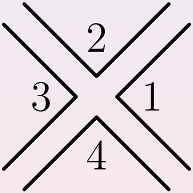
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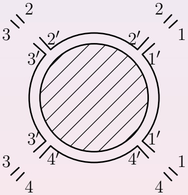
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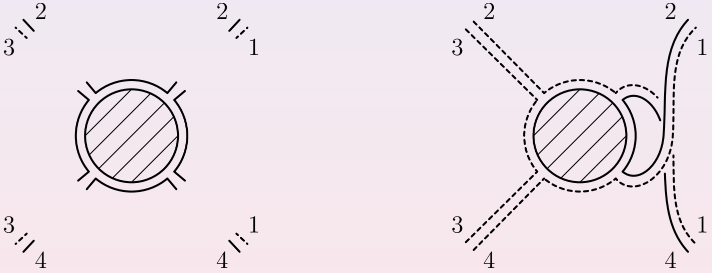
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