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First numerical approach to a W-G model

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Summary of the talk

1. Introduction of the Wulkenhaar-Grosse model via spectral triple.

2. Introduction of the discretization scheme and numerical simulation.

3. Presentation of some results of simulation

The firsts attempts to obtain a non-commutative field theory were consisted in the direct replace of the point-wise product

$$(f\star g)(x)=\int\int d^4y\frac{d^4k}{(2\pi)^4}f(x+\frac{1}{2}\Theta\cdot k)g(x+y)e^{i\langle k,y\rangle}\ ,\ f,g\in L^2(\mathbb{R}^4)$$

This kind of models are ill-behaved they are not renormalizable due the so called UV/IR-mixing Grosse,Wulkenhaar(2004) in *renormalization as formal power series in* λ, found a NC φ^4-theory renormalizable action which develops additional marginal coupling action functional for real-valued field φ on $\mathbb{R}^4$:

$$S[\varphi]=\int d^4x\left(\frac{1}{2}\varphi\star(-\Delta+\Omega^2\tilde{x}^2+\mu^2)\star\varphi+\frac{\lambda}{4}\varphi\star\varphi\star\varphi\star\varphi\right)(x)$$

Where $\tilde{x}=2\Theta^{-1}\cdot x,\ \lambda\in R,\ \Omega\in[0,1]$, and μ is a real parameter

Does this model arise from a spectral triple?

Spectral triple ($\mathcal{A}$,$\mathcal{H}$,$\mathcal{D}$)

A collection of a Hilbert space $\mathcal{H}$, a involution unital algebra $\mathcal{A}$ represented on $\mathcal{H}$, and a selfadjoint operator $\mathcal{D}$ in $\mathcal{H}$ with compact resolvent, which satisfy the flowing requirements:

1. Dimension: n-th eigenvalue of resolvent of $\mathcal{D}$ is $\mathcal{O}(n^{-\frac{1}{p}})$ (p is called spectral dimension of the triple)

2. Order one: $[[\mathcal{D}, f], g] = 0$

3. Regularity: f and $[\mathcal{D}, f]$ belong to the domain of δk , for any $f \in A$ and $k \in Z$, where $\delta T = [|\mathcal{D}|, T]$

4. Orientability: $\exists$ Hochschild p-cycle $c \in Z_p(\mathcal{A}, \mathcal{A})$ $\pi_D(c) = 1$ for p odd,$\pi_D(c) = \gamma$ for p even with $\gamma = \gamma^*, \gamma^2 = 1, \gamma\mathcal{D} = -\mathcal{D}\gamma$

5. Finiteness and absolute continuity: $\mathcal{H}_\infty := \bigcap_k dom(\mathcal{D}^k) \subset \mathcal{H}$ is finitely generated projective $\mathcal{A}$-module,$\mathcal{H}_\infty = e\mathcal{A}^n$, with $e = e^* = e^2 \in \mathbb{M}_m(\mathcal{A})$. Hermitian structure $(\xi|a\eta) = \sum_{i=1}^n a\xi_i^*\eta_i \in A$ satisfies $\langle\xi, \eta\rangle = \int(\xi|\eta)|\mathcal{D}|^{-p} \forall f, g \in \mathcal{A}$

Theorem(Connes)

Let ($\mathcal{A}$,$\mathcal{H}$,$\mathcal{D}$) be a commutative spectral triple and assume that all endomorphisms $T \in \text{End}\mathcal{A}(H^\infty)$ are regular, the Hochschild cycle c is antisymmetric. Then there exists a compact oriented smooth manifold X such that $\mathcal{A} = C^\infty(X)$ is the algebra of smooth functions on X , and every compact oriented smooth manifold appears in this manner.

Spectral action principle (Chamseddine,Connes)

As an automorphism-invariant object, the (bosonic) action functional of physics has to be a function of the spectrum of $\mathcal{D}_A$, $S(\mathcal{D}_A) = \text{Tr}(\chi(\mathcal{D}_A))$ Where inner fluctuations can be defined as:

$$\mathcal{D} \to \mathcal{D}_A = \mathcal{D} + A, \;\; A = f[\mathcal{D}, g]$$

Proposed spectral triple.

The Dirac operator is constructed using d-dimensional bosonic and fermionic creation and annihilation operators:

$$[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0, \quad [a_\mu, a_\nu^\dagger] = \delta_{\mu\nu}$$

$$\{b_\mu, b_\nu\} = \{b_\mu^\dagger, b_\nu^\dagger\} = 0, \quad \{b_\mu, b_\nu^\dagger\} = \delta_{\mu\nu}$$

$\mu, \nu = 1, \cdots, d$. Where $a_\mu = \frac{1}{\sqrt{2\omega}}(\omega x_\mu + \partial_\mu)$, $a_\mu^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x_\mu - \partial_\mu)$

1. Dirac operator for the 4 dimensional case

$$\mathcal{D}_4 = -i\sqrt{2\omega}\delta^{\mu\nu} a_\mu^\dagger \otimes b_\nu + i\sqrt{2\omega}\delta^{\mu\nu} a_\mu \otimes b_\nu^\dagger = i\frac{d}{dx_\mu} \otimes (b_\mu + b_\mu^\dagger) + i\omega x^\mu \otimes (b_\mu - b_\mu^\dagger)$$

2. The algebra $A = S(\mathbb{R}^4)$ is determined by smoothness

3. Hilbert space for the fermionic side is defined starting from the vacuum state, by subsequent action of the fermionic creation operators $b_\nu^\dagger$ on vacuum state $b|0\rangle = 0$, $(b_1^\dagger)^{n_1} \cdots (b_d^\dagger)^{s_d}|0\rangle$: $n_\mu \in \mathbb{N}, s_\mu \in \{0, 1\}$ We call this space $\Lambda(\mathbb{C}^d)$ and therefore the complete Hilbert space is $\mathcal{H}_d = S(\mathbb{R}^d) \otimes \Lambda(\mathbb{C}^d)$.

Can be proved that all axiom of spectral triple are satisfy with minor adaptation. Flowing the Connes-Lott models, in order to implement the Higgs mechanism we consider the total spectral triple as the tensor product of the previous spectral triple $(\mathcal{A}_4, \mathcal{H}_4, \mathcal{D}_4)$ with the two point Connes-Lott like spectral triple $(\mathbb{C} \otimes \mathbb{C}, \mathbb{C}^2, M\sigma_1)$.The total Dirac operator of the product triple is:

$$\mathcal{D}_T = \mathcal{D}_4 \otimes 1 + 1 \otimes M\sigma_1 = \begin{pmatrix} \mathcal{D}_4 & M \\ M & -\mathcal{D}_4 \end{pmatrix}, \qquad \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in \mathcal{A}$$

Applying the spectral action principle to the previous spectral triple after same calculation we obtain for the commutative case:

$$\begin{aligned} S(D_A) &= \int d^4x (D^\mu \varphi \overline{D_\mu \varphi} + \frac{5}{12}(F^A_{\mu\nu} F^{\mu\nu}_A + F^B_{\mu\nu} F^{\mu\nu}_B) \\ &+ \left((|\varphi|^2)^2 - \frac{2\chi_{-1}}{\chi_0}|\varphi|^2 + 2\omega^2 ||x||^2 |\varphi|^2 \right) + \mathcal{O}(\chi_1) \end{aligned}$$

where

$$D_\mu \varphi = \partial_\mu \varphi + i(A_\mu - B_\mu)\varphi$$

1. spectral action is finite.

2. Additional harmonic oscillator potential for the Higgs.

3. vacuum is at $A\mu = B\mu = 0$.

Spectral action for noncommutative case

$$S(\mathcal{D}_A) =$$

$$\int d^4z \Bigg\{ \left(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{6(1+\Omega^2)^2} \right) (F^A_{\mu\nu} \star F_A^{\mu\nu} + F^B_{\mu\nu} \star F_B^{\mu\nu})$$

$$+ \left(\bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} X^A_\mu \star X^\mu_A - \frac{\chi_{-1}}{\chi_0} \right)^2 + \left(\phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} X^B_\mu \star X^\mu_B - \frac{\chi_{-1}}{\chi_0} \right)^2$$

$$+ \left(\frac{4\Omega^2}{1+\Omega^2} X^\mu_0 \star X^0_\mu - \frac{\chi_{-1}}{\chi_0} \right)^2 + 2(1+\Omega^2) D_\mu \phi \star \overline{D^\mu \phi} \Bigg\} + \mathcal{O}(\chi_1)$$

$$\begin{aligned} D_\mu \phi &= \partial_\mu \phi - i A_\mu \star \phi + i \phi \star B_\mu = \left(\phi \star X_{B\mu} - X_{A\mu} \star \phi \right) \\ F^A_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i (A_\mu \star A_\nu - A_\nu \star A_\mu) \\ X^0_\mu &= \frac{\tilde{z}}{2} \end{aligned}$$

Here the covariant coordinate $X^A_\mu(x) = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu(x)$appear with Higgs field ϕ in unified potential; potential cannot be restricted to Higgs and vacuum is non-trivial.

An other important property of the action, considering the X_μ as independent, are invariant under the translations:

$$\phi(x) \to \phi(x+a),\ X_A^\mu(x) \to X_A^\mu(x+a),\ X_B^\mu(x) \to X_B^\mu(x+a),\ X_0^\mu(x) \to X_0^\mu(x+a)$$

Which in standard ϕ^4-renormalizable theory is broken. Beside the action is invariant under $U(1) \times U(1)$ transformation:

$$\phi \to u_A \star \phi \star \overline{u_B},\ X \to u_A \star X_A^\mu \star \overline{u_A},\ X_B^\mu \to u_B \star X_B^\mu \star \overline{u_B}$$

At this point we have to choose the way of discretization: Moyal matrix base in two dimensions. Our fields can be expanded in this base as:

$$X^\mu(x) = \sum_{m_i,n_i \in \mathbb{N}} X^\mu_{\substack{m_1 n_1 \\ m_2 n_2}} f_{m_1 n_1}(x_0, x_1) f_{m_2 n_2}(x_2, x_3)$$

and

$$\phi(x) = \sum_{m_i,n_i \in \mathbb{N}} \phi_{\substack{m_1 n_1 \\ m_2 n_2}} f_{m_1 n_1}(x_0, x_1) f_{m_2 n_2}(x_2, x_3)$$

f_{nm} can be expressed with the help of Laguerre functions :

$$f_{mn}(\rho, \varphi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i\varphi(n-m)} \left(\sqrt{\frac{2}{\theta}}\rho\right)^{n-m} e^{\frac{\rho^2}{\theta}} L_m^{n-m}\left(\frac{2}{\theta}\rho^2\right)$$

This base has the nice properties:

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)$$

$$\int d^2x f_{mn}(x) = 2\pi\theta\delta_{mn}$$

Using this base we can forget the Moyal product in this way the model becomes to 9-matrix model. The $\star$-product between two fields using the previous properties can be written as

$$\begin{aligned}
\Psi(x) \star \Phi(x) &= \sum_{m_i,n_i,k_1,l_1 \in \mathbb{N}} \Psi_{\substack{m_1 n_1 \\ m_2 n_2}} \Phi_{\substack{k_1 l_1 \\ k_2 l_2}} f_{m_1 n_1}(x_0, x_1) \star f_{k_1 l_1}(x_0, x_1) \\
&\times \quad f_{m_2 n_2}(x_2, x_3) \star f_{k_2 l_2}(x_2, x_3) \\
&= \sum_{m_i, l_1 \in \mathbb{N}} \Psi\Phi_{\substack{m_1 l_1 \\ m_2 l_2}} f_{m_1 l_1}(x_0, x_1) f_{m_2 l_2}(x_2, x_3)
\end{aligned}$$

where

$$\Psi\Phi_{\substack{m_1 l_1 \\ m_2 l_2}} = \sum_{n_1, n_2 \in \mathbb{N}} \Psi_{\substack{m_1 n_1 \\ m_2 n_2}} \Phi_{\substack{m_1 l_1 \\ m_2 l_2}}$$

So the star product became a ''double'' matrix multiplication, the action and all treatment can be conducted on the infinite matrices instead directly on the continues field. Beside, due the nature of star product, there is another great simplification; the double matrix can be separated as a tensor product of two matrices or

$$\Phi(x) = \sum_{m_i, n_i \in \mathbb{N}} \Phi'_{m_1 n_1} \otimes \Phi''_{m_2 n_2} f_{m_1 n_1}(x_0, x_1) f_{m_2 n_2}(x_2, x_3)$$

So we can split the model in 2+2 dimensions in this way the calculus will be performed just on a standard matrix instead a double one. Using this base now our problem is reduced to a infinite matrix problem but in not enough to be handled numerically we have to perform a truncation in order to obtain a finite matrices, this truncation will consist in a maximum $m, n < N$ in the expansion.

Summarizing, to operate the discretization we have the flowing

rules:

$$\begin{aligned}
\phi(x) \in \mathbb{R}^4_{\ominus} \; &\rightarrow \; \hat{\phi} \in \mathbb{M}_N \\
Y^A_\mu(x) \in \mathbb{R}^4_{\ominus} \; &\rightarrow \; \hat{Y}^A_\mu \in \mathbb{M}_N \\
Y^B_\mu(x) \in \mathbb{R}^4_{\ominus} \; &\rightarrow \; \hat{Y}^B_\mu \in \mathbb{M}_N \\
\int a(x)dx \; &\rightarrow \; \mathsf{Tr}(\hat{a})
\end{aligned}$$

As a first approach to the numerical simulation, and forced by limited computation resource, we will consider the Monte Carlo simulation of the previous action around the its minimum, we translate the fields ϕ, X_μ^A, X_μ^B using the following substitution:

$$\begin{aligned}
\phi &= \psi + \sqrt{\frac{\chi_3}{\chi_4}} \cos\alpha \\
X_{A\mu} &= Y_{A\mu} + \frac{1}{2}\sqrt{\frac{\chi_3}{\chi_4}}\sqrt{\frac{2\Omega^2}{(1+\Omega^2)}} 1_\mu \sin\alpha \\
X_{B\mu} &= Y_{B\mu} + \frac{1}{2}\sqrt{\frac{\chi_3}{\chi_4}}\sqrt{\frac{2\Omega^2}{(1+\Omega^2)}} 1_\mu \sin\alpha
\end{aligned}$$

Now after truncating the representative matrices is convenient to operate an another substitution:

$$\begin{aligned}
Z_0 &= \widehat{Y}_0^A + i\widehat{Y}_1^A, \quad \bar{Z}_0 = \widehat{Y}_0^A - i\widehat{Y}_1^A \\
Z_1 &= \widehat{Y}_0^B + i\widehat{Y}_1^B, \quad \bar{Z}_1 = \widehat{Y}_0^B - i\widehat{Y}_1^B \\
Z_2 &= \widehat{Y}_2^A + i\widehat{Y}_2^A, \quad \bar{Z}_2 = \widehat{Y}_2^A - i\widehat{Y}_3^A \\
Z_3 &= \widehat{Y}_2^B + i\widehat{Y}_3^B, \quad \bar{Z}_3 = \widehat{Y}_2^B - i\widehat{Y}_3^B
\end{aligned}$$

For simplicity we put:

$$C = \frac{1+\Omega^2}{4\Omega^2}, \quad D = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{6(1+\Omega^2)^2}, \quad \frac{\chi_3}{\chi_4} = \mu^2$$

Implementing the previous replacements:

$$\widehat{S}_4 = \mathrm{Tr}\left(\mathcal{L}_F + \mathcal{L}_{V_0} + \mathcal{L}_{V_1} + \mathcal{L}_{D_0}\bar{\mathcal{L}}_{D_0} + \mathcal{L}_{D_1}\bar{\mathcal{L}}_{D_1} + \mathcal{L}_{D_2}\bar{\mathcal{L}}_{D_2} + \mathcal{L}_{D_3}\bar{\mathcal{L}}_{D_3}\right)$$

$$\begin{aligned}
\mathcal{L}_{4F} &= \frac{D}{2}\Big(\left[\bar{Z}_0, Z_0\right]_\star^2 + \left[\bar{Z}_1, Z_1\right]_\star^2 + \frac{1}{4}\Big(\left[Z_0 + \bar{Z}_0, Z_2 - \bar{Z}_2\right]_\star^2 - \left[Z_0 + \bar{Z}_0, Z_2 + \bar{Z}_2\right]_\star^2 \\
&+ \left[Z_0 - \bar{Z}_0, Z_2 + \bar{Z}_2\right]_\star^2 - \left[Z_0 - \bar{Z}_0, Z_2 - \bar{Z}_2\right]_\star^2 - \left[Z_1 + \bar{Z}_1, Z_3 + \bar{Z}_3\right]_\star^2 \\
&+ \left[Z_1 + \bar{Z}_1, Z_3 - \bar{Z}_3\right]_\star^2 + \left[Z_1 - \bar{Z}_1, Z_3 + \bar{Z}_3\right]_\star^2 - \left[Z_1 - \bar{Z}_1, Z_3 - \bar{Z}_3\right]_\star^2 \Big)\Big) \\
\mathcal{L}_{4V_0} &= \Big(\psi \star \bar{\psi} + \mu \cos\alpha(\psi + \bar{\psi}) + \frac{1}{2}\left(\left\{\bar{Z}_0, Z_0\right\}_\star + \left\{\bar{Z}_2, Z_2\right\}_\star\right) \\
&+ \frac{\mu \sin\alpha}{2\sqrt{C}}((-1+i)(Z_0 + Z_2) + (1+i)(\bar{Z}_0 + \bar{Z}_2))\Big)^2 \\
\mathcal{L}_{4V_1} &= \Big(\psi \star \bar{\psi} + \mu \cos\alpha(\psi + \bar{\psi}) + \frac{1}{2}\left(\left\{\bar{Z}_1, Z_1\right\}_\star + \left\{\bar{Z}_3, Z_3\right\}_\star\right) \\
&+ \frac{\mu \sin\alpha}{2\sqrt{C}}((-1+i)(Z_1 + Z_3) + (1+i)(\bar{Z}_1 + \bar{Z}_3))\Big)^2 \\
\mathcal{L}_{4D_0} &= \sqrt{2(1+\Omega^2)}\,\Big(\mu \cos\alpha(Z_1 + \bar{Z}_1 - Z_0 - \bar{Z}_0) + \psi \star (Z_1 + \bar{Z}_1) - (Z_0 + \bar{Z}_0) \star \psi\Big) \\
\mathcal{L}_{4D_1} &= \sqrt{2(1+\Omega^2)}\,\Big(\mu \cos\alpha(Z_1 - \bar{Z}_1 - Z_0 + \bar{Z}_0) + \psi \star (Z_1 - \bar{Z}_1) - (Z_0 - \bar{Z}_0) \star \psi\Big) \\
\mathcal{L}_{4D_2} &= \sqrt{2(1+\Omega^2)}\,\Big(\mu \cos\alpha(Z_3 + \bar{Z}_3 - Z_2 - \bar{Z}_2) + \psi \star (Z_3 + \bar{Z}_3) - (Z_2 + \bar{Z}_2) \star \psi\Big) \\
\mathcal{L}_{4D_3} &= \sqrt{2(1+\Omega^2)}\,\Big(\mu \cos\alpha(Z_3 - \bar{Z}_1 - Z_2 + \bar{Z}_2) + \psi \star (Z_3 - \bar{Z}_3) - (Z_2 - \bar{Z}_2) \star \psi\Big)
\end{aligned}$$

Calling (ψ, Z_i) a configuration of the fields $i = 1, \cdots, 4$, the probability to encounter this configuration is given by

$$P[(\psi, Z_i)] = \frac{e^{-S[(\psi, Z_i)]}}{\mathcal{Z}}$$

$\mathcal{Z}$ is the partition function:

$$\mathcal{Z} = D[(\psi, Z_i)]e^{-S[(\psi, Z_i)]}$$

Average value of the observable O is define by the expression:

$$\langle O \rangle = \int D[(\psi, Z_i)] \frac{e^{-S[(\psi, Z_i)]} O[(\psi, Z_i)]}{\mathcal{Z}}$$

Following Monte Carlo method will be produced a sequence of configurations $\{(\psi, Z_i)_j\}, j = 1, 2, \cdots, T_{MC}$ and evaluated the average of the observables over that set of configurations. The expectation value is approximated as

$$\langle O \rangle \approx \frac{1}{T_{MC}} \sum_{j=1}^{T_{MC}} O_j$$

The internal energy is defined as:

$$E(\Omega, \mu, \alpha) = \langle S \rangle$$

and the specific heat takes the form

$$C(\Omega, \mu, \alpha) = \langle S^2 \rangle - \langle S \rangle^2$$

It is very useful to compute separately the average values four contributions:

$$\begin{aligned} S_F(\psi, Z_i) &= \mathrm{Tr}\, \mathcal{L}_F \\ S_{V_0}(\psi, Z_i) &= \mathrm{Tr}\, \mathcal{L}_{V_0} \\ S_{V_1}(\psi, Z_i) &= \mathrm{Tr}\, \mathcal{L}_{V_1} \\ S_D(\psi, Z_i) &= \mathrm{Tr}\left(\mathcal{L}_{D_j} \bar{\mathcal{L}}_{D^j}\right) \end{aligned}$$

The corresponding expectation values are:

$$\begin{aligned} E_F(\Omega, \mu, \alpha) &= \langle S_F \rangle \\ E_{V_0}(\Omega, \mu, \alpha) &= \langle S_{V_1} \rangle \\ E_{V_1}(\Omega, \mu, \alpha) &= \langle S_{V_1} \rangle \\ E_D(\Omega, \mu, \alpha) &= \langle S_D \rangle \end{aligned}$$

Now in order to measure the contributions of different modes to the configuration ϕ we need a control parameter. This turns out to be the sums on l and m of $|\psi_{lm}|, |Z_{ilm}|^2$, this quantity is called the full power of the field and it represents the norm of the field,it can be calculate as:

$$\begin{aligned} \varphi_a^2 &= \mathrm{Tr}(|\psi|^2) \\ Z_{ia}^2 &= \mathrm{Tr}(|Z_i|^2) \end{aligned}$$

To distinguish the contributions from the different modes we define the quantity:

$$\begin{aligned} \varphi_0^2 &= \sum_{n=0}^{N} |a_{nn}|^2 \\ Z_{i0}^2 &= \sum_{n=0}^{N} |z_{inn}|^2 \end{aligned}$$

Referring to the radial base it is easy to see that a such parameters are connected to the pure spherical contribution.

We can generalize the previous quantity defining some parameters φ_l in such a way they form a decomposition of the full power of the fields.

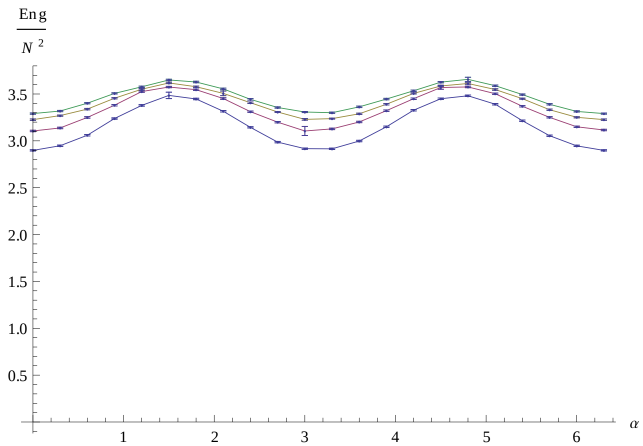
$$\varphi_a^2 = \varphi_0^2 + \sum_l \varphi_l^2, \;\; Z_{ia}^2 = Z_{i0}^2 + \sum_l Z_{il}^2$$

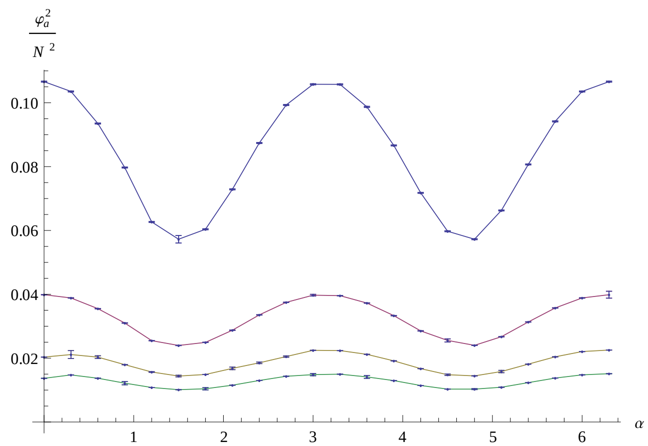
Higher modes are defined by the quantity:

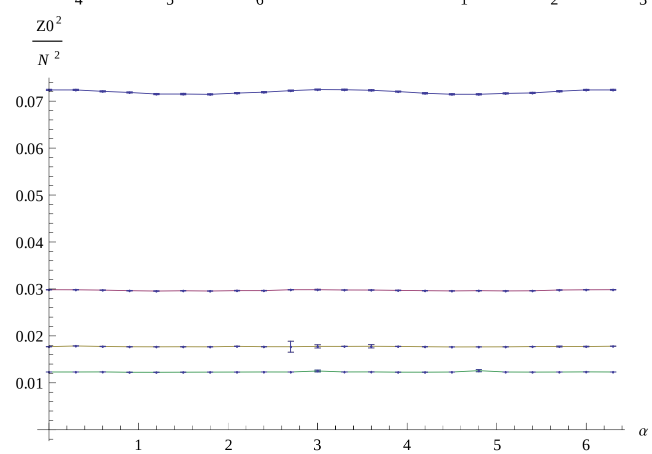
$$\varphi_l := \sqrt{\sum_{n,m=0}^{l} |a_{nm}(1-\delta_{nm})|^2}$$

In the simulations are used the quantities related to $l = 0$ and the for $l = 1$ as representative of those contribution where the rotational symmetry is broken.

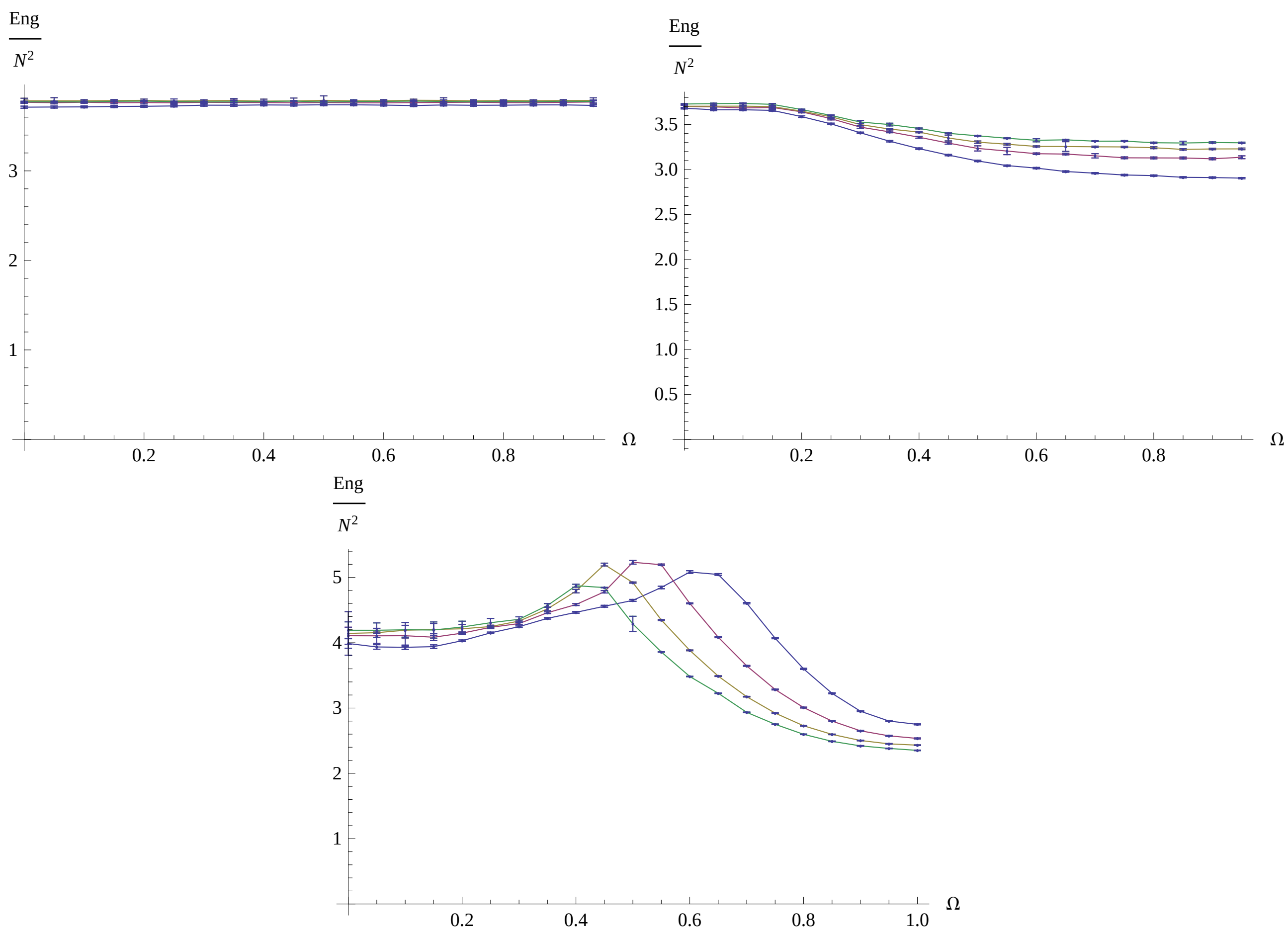
Energy density, φ_a^2, Z_{0a}^2 (from the left to right) for $\mu = 0, \Omega = 1$ and varying α from 0 to 2π.



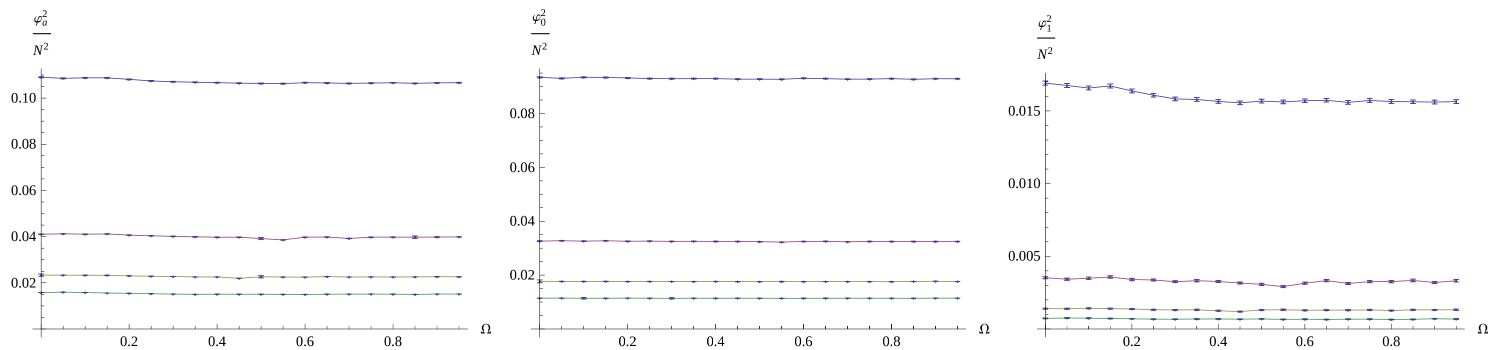




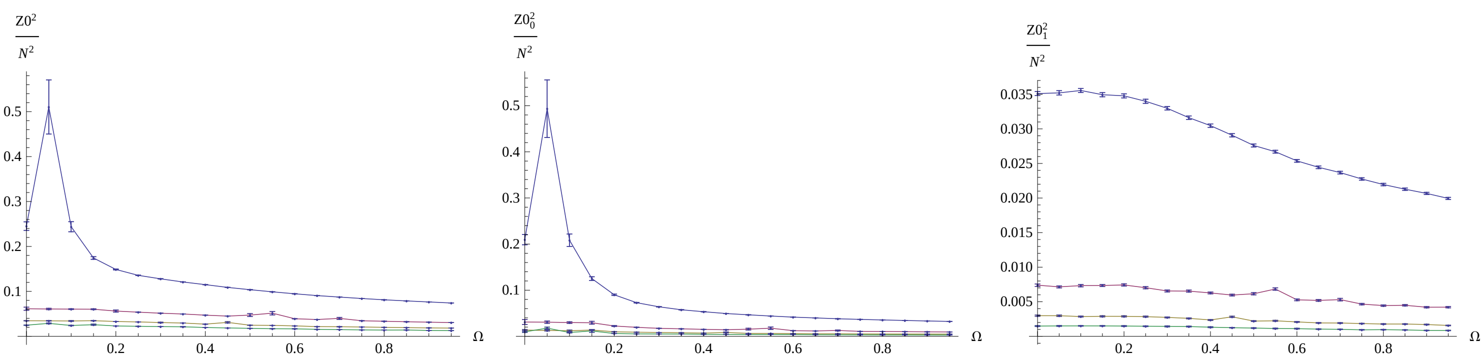
Energy density for $\mu = 0, 1, 3$ (from the left to right) and $\alpha = 0$ varying Ω from 0 to 1.



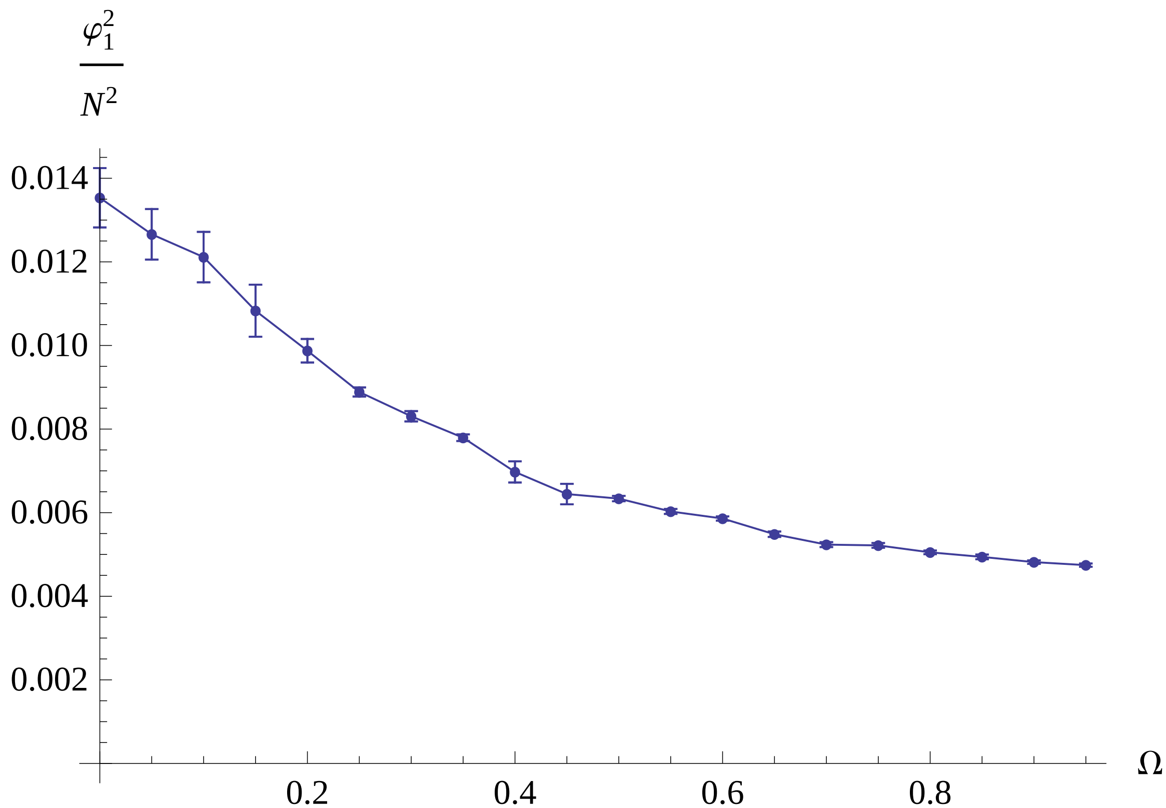
φ_a^2, φ_0^2, φ_1^2 density for $\mu = 1$ and $\alpha = 0$ varying Ω from 0 to 1.



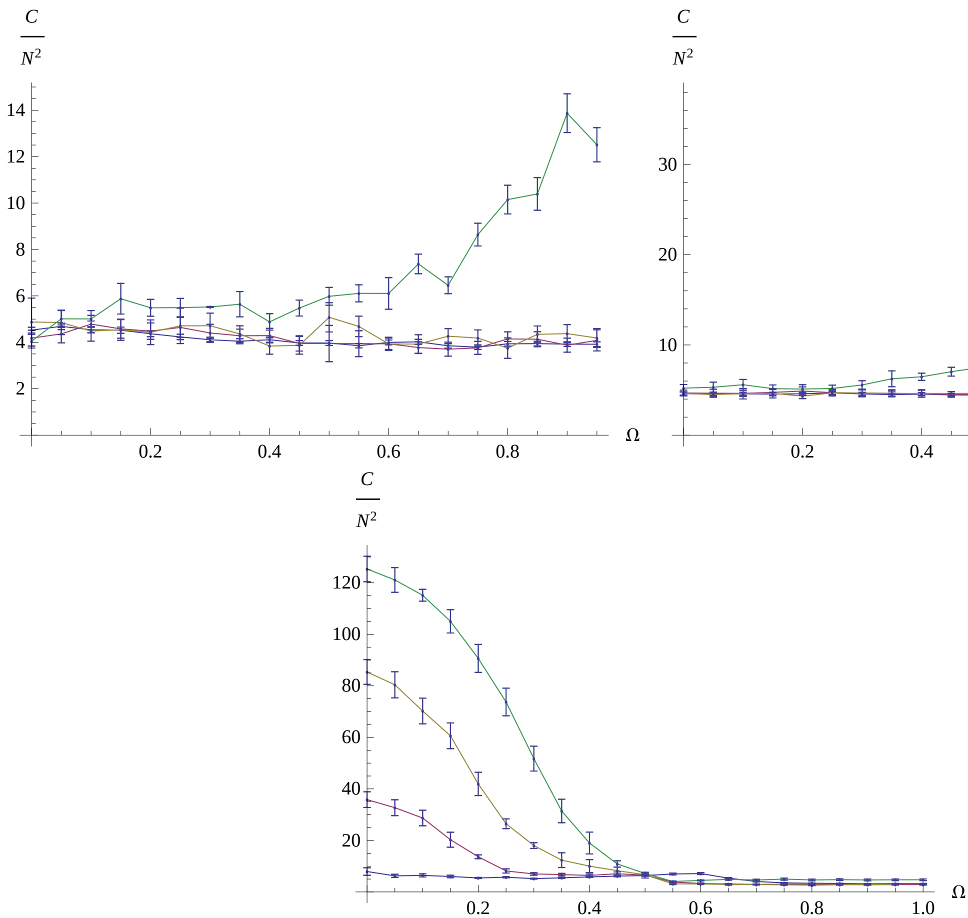
Z_{0a}^2, Z_{00}^2, Z_{01}^2 density for $\mu = 1$ and $\alpha = 0$ varying Ω from 0 to 1.



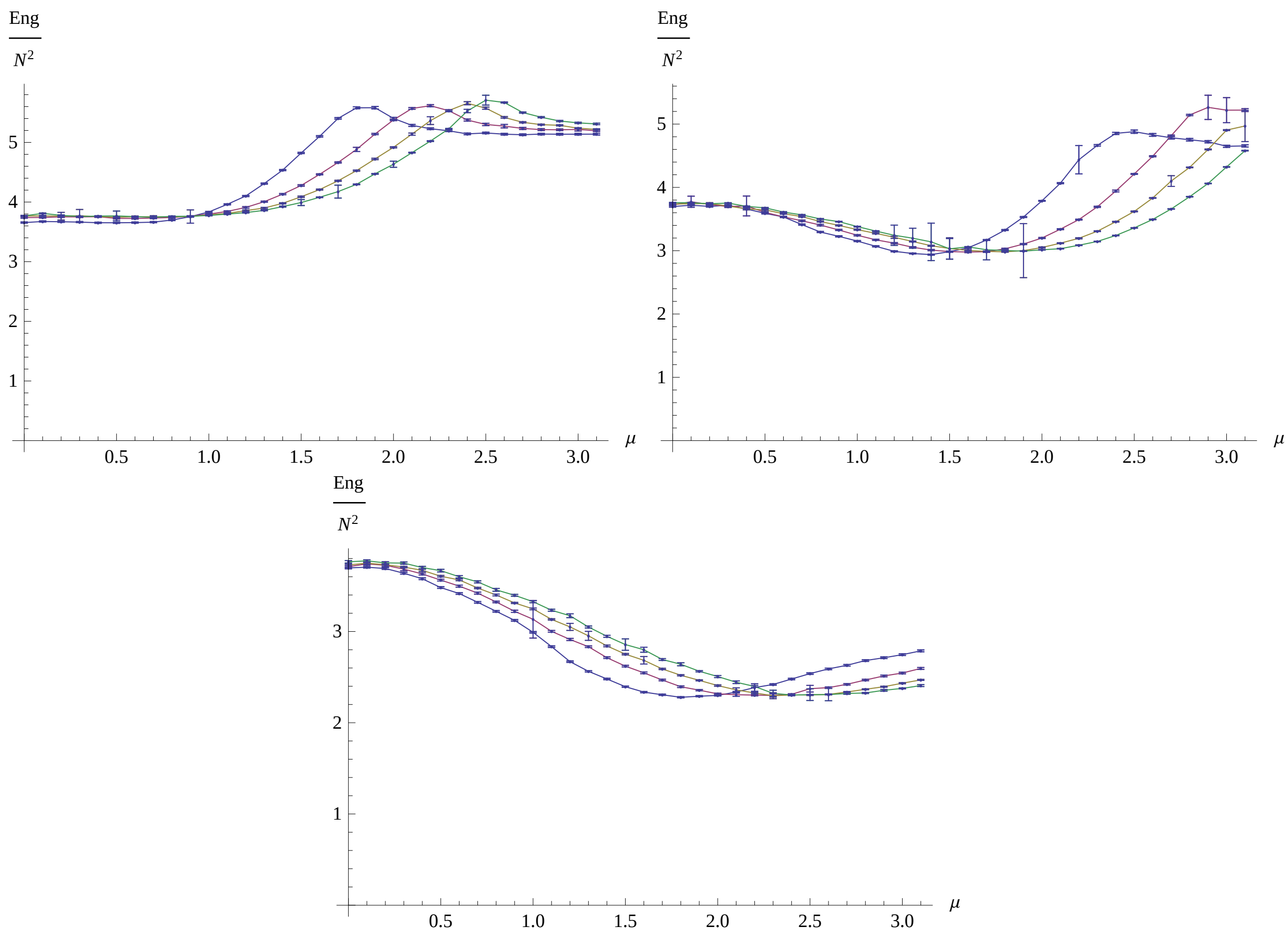
Z_{00}^2 for $\mu = 1$, $\alpha = 0$ and $N = 15$ varying Ω from 0 to 1.



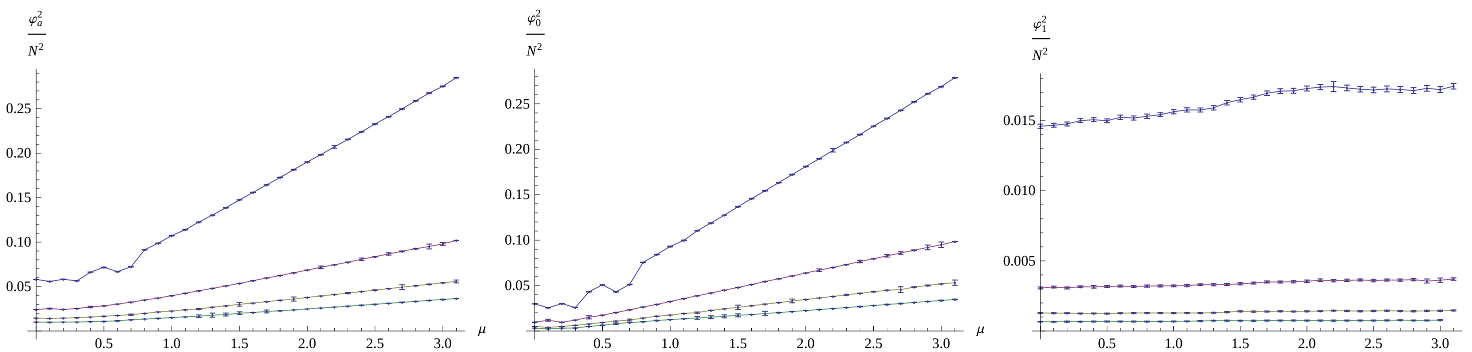
Specific heat density for $\mu = 0, 1, 3$ (from the left to right) and $\alpha = 0$ varying Ω from 0 to 1.



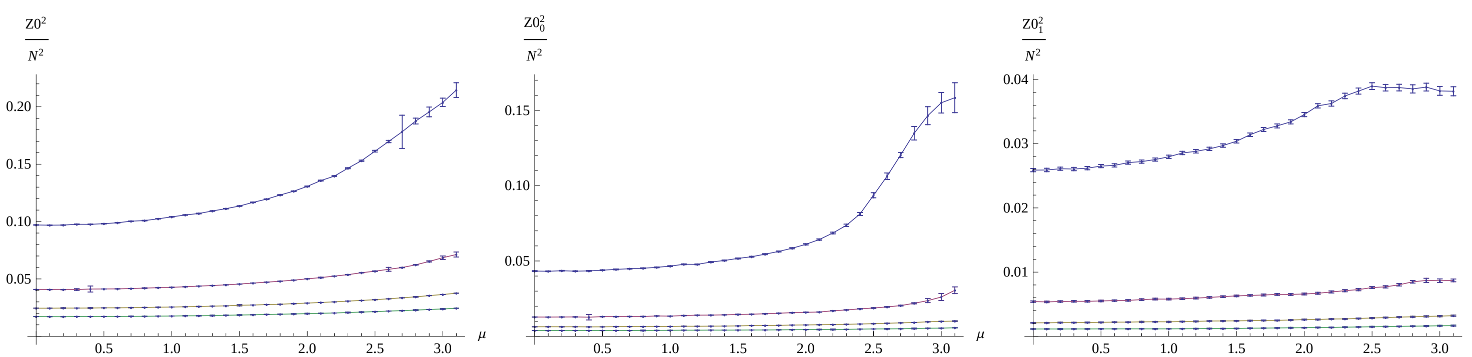
Energy density for $\Omega = 0, 0.5, 1$ (from the left to right) and $\alpha = 0$ varying μ from 0 to 3.



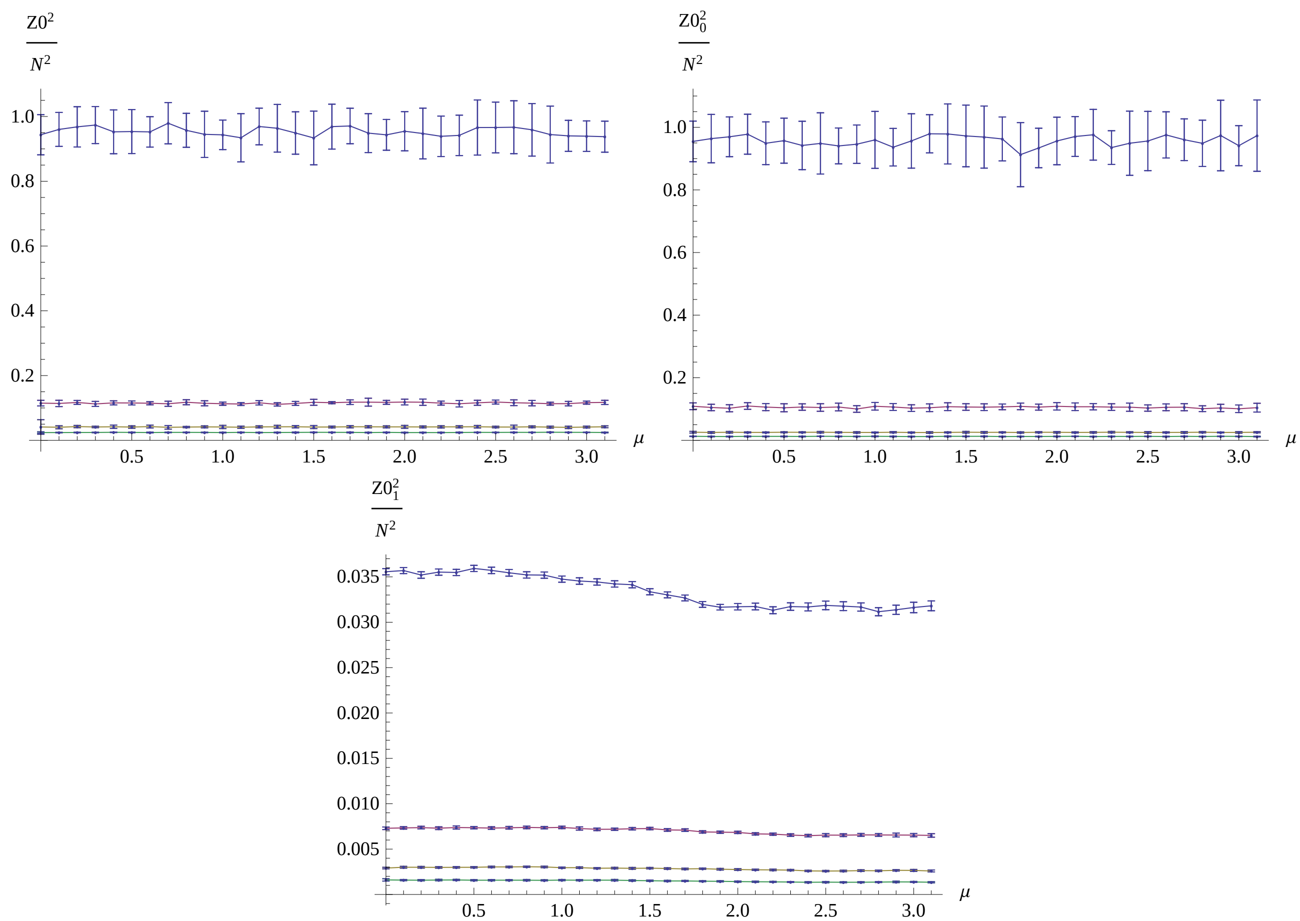
φ_a^2, φ_0^2, φ_1^2 density for $\Omega = 0.5$ and $\alpha = 0$ varying μ from 0 to 3.



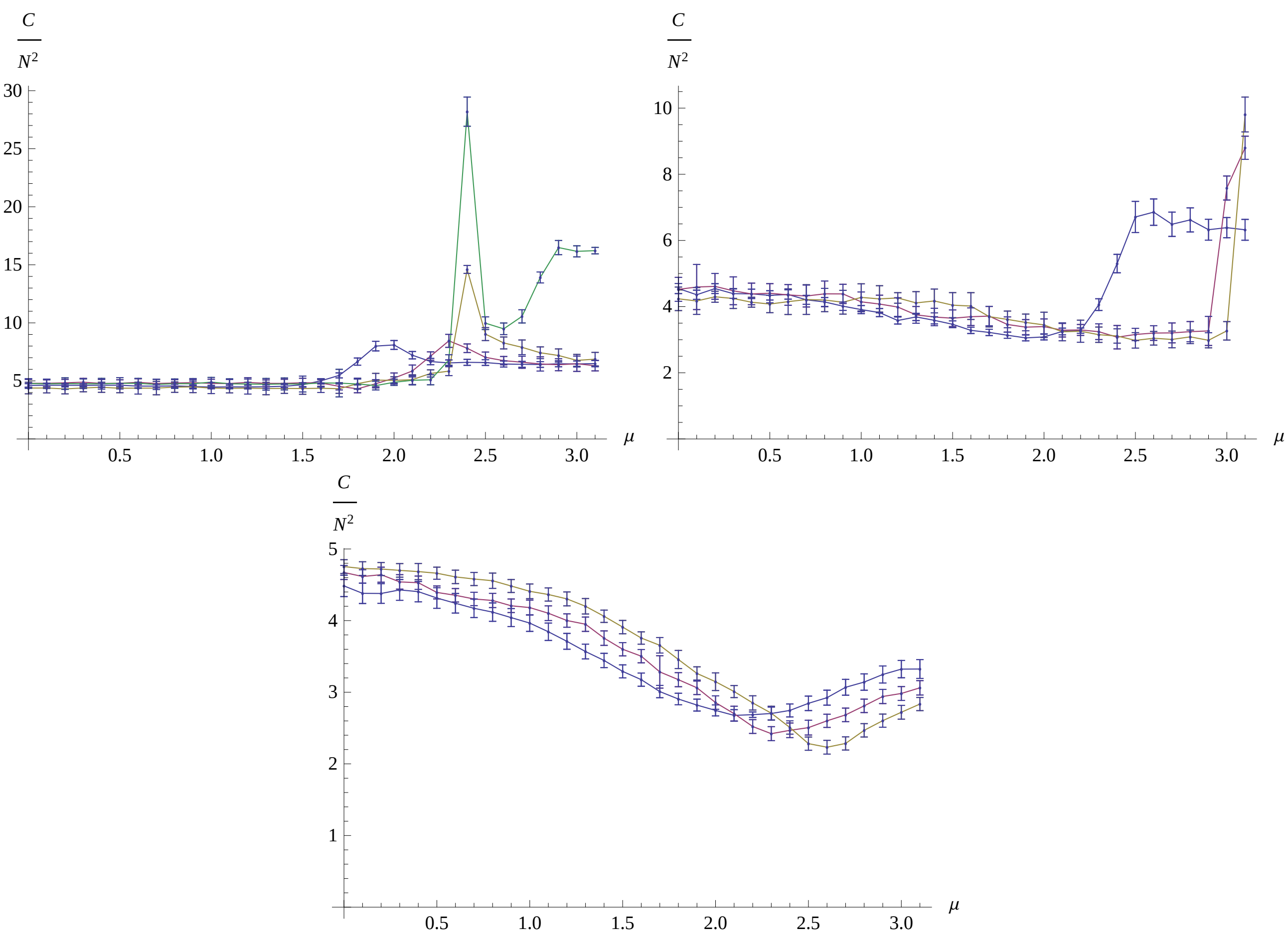
Z_{0a}^2, Z_{00}^2, Z_{01}^2 density for $\Omega = 0.5$ and $\alpha = 0$ varying μ from 0 to 3.



Z_{0a}, Z_{00}, Z_{01} density (from the left to right) for $\Omega = 0$ and $\alpha = 0$ varying μ from 0 to 3.



Specific heat density for $\Omega = 0, 0.5, 1$ (from the left to right) and $\alpha = 0$ varying μ from 0 to 3.



Conclusions

1. Monte Carlo simulation of such model seams feasible .

2. Peaks in specific heat appears increasing the matrix size $\rightarrow$ phases transitions.

3. The defined order parameters crucially depends on Ω .

Prospectives

1. Characterization of the phase transitions (transitions order).

2. Computation of transition curve and characterization of the phases regions.

3. Extension of the parameters space $\mu^2 < 0$, $\Omega > 1$.