Quantum isometry groups of the duals of certain discrete groups (and their connections to free probability)

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based on joint work with Teodor Banica and Jyotishman Bhowmick

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General framework

Groups entered mathematics as collections of symmetries of a given object (a finite set, a figure on the plane, a manifold, a space of solutions of an equation).

A symmetry group of a given object X is defined by looking at the family of 'all possible transformations' of X, usually preserving some given structure.

In modern language: we search for a *universal* object in the category of all groups acting on X.

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One can apply this idea to construct certain examples of compact quantum groups.

Compact quantum groups

Definition (S.L.Woronowicz)

A unital C^*-algebra A is the algebra of continuous functions on a compact quantum group if it admits a unital $*$-algebra homomorphism $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ such that

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta \qquad \text{(coassociativity)}$$

and

$$\overline{\Delta(\mathsf{A})(\mathsf{A} \otimes 1_{\mathsf{A}})} = \mathsf{A} \otimes \mathsf{A} = \overline{\Delta(\mathsf{A})(1_{\mathsf{A}} \otimes \mathsf{A})} \qquad \text{(quantum cancellation rules).}$$

We write $\mathsf{A} = C(\mathbb{G})$ and call $\mathbb{G}$ a compact quantum group.

A *finite-dimensional unitary representation* of $\mathbb{G}$ is a unitary matrix $U \in M_n(C(\mathbb{G}))$ such that for $i, j = 1, \ldots, n$

$$\Delta(U_{ij}) = \sum_{k=1}^{n} U_{ik} \otimes U_{kj}.$$

If a compact quantum group $\mathbb{G}$ admits $n \in \mathbb{N}$ and a unitary representation $U \in M_n(C(\mathbb{G}))$ such that $\{U_{ij} : i, j = 1, \ldots, n\}$ generates $C(\mathbb{G})$ as a C^*-algebra, then $\mathbb{G}$ is called a *compact matrix quantum group* and U a *fundamental unitary representation*.

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Actions of compact quantum groups

Definition

Let $\mathbb{G}$ be a compact quantum group and let B be a unital C^*-algebra. A map

$$\alpha : \mathrm{B} \to C(\mathbb{G}) \otimes \mathrm{B}$$

is called a (left, continuous) action of $\mathbb{G}$ on B if α is a unital *-homomorphism,

$$(\Delta \otimes \mathrm{id}_{\mathrm{B}}) \circ \alpha = (\mathrm{id}_{C(\mathbb{G})} \otimes \alpha) \circ \alpha$$

and additionally $\alpha(\mathrm{B})(C(\mathbb{G}) \otimes 1_{\mathrm{B}})$ is dense in $C(\mathbb{G}) \otimes \mathrm{B}$ (*Podleś/nondegeneracy condition*).

Category of CQGs acting on a given C^*-algebra

Consider the category $\mathfrak{C}(\mathsf{B}) := \{(\mathbb{G}, \alpha)\}$ of compact quantum groups acting on a given C^*-algebra B. A morphism in the category $\mathfrak{C}(\mathsf{B})$:

$$\gamma : (\mathbb{G}_1, \alpha_1) \to (\mathbb{G}_2, \alpha_2)$$

is a unital *-homomorphism $\gamma : C(\mathbb{G}_2) \to C(\mathbb{G}_1)$ such that

$$(\gamma \otimes \gamma) \circ \Delta_2 = \Delta_1 \circ \gamma, \quad \alpha_1 = (\gamma \otimes \mathrm{id}_{\mathsf{B}}) \circ \alpha_2.$$

We say that the category $\mathfrak{C}(\mathsf{B})$ admits a universal object, if there is $(\mathbb{G}_u, \alpha_u)$ in $\mathfrak{C}(\mathsf{B})$ such that for all $(\mathbb{G}, \alpha)$ in $\mathfrak{C}(\mathsf{B})$ there exists a unique morphism $\gamma : (\mathbb{G}, \alpha) \to (\mathbb{G}_u, \alpha_u)$. If such a universal object exists, it is unique.

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Quantum permutation groups

Both classically and in the quantum framework the simplest symmetry groups are (quantum) permutation groups, which can be viewed as the universal (quantum) groups acting on a given finite set.

Theorem (S.Wang)

The category $\mathfrak{C}(\mathbb{C}^n)$ of quantum groups acting on the n-point set admits the universal object. It is denoted S_n^+ and called the quantum permutation group of an $n-$point set.

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S_n^+ is a compact matrix quantum group, its fundamental unitary is the n by n matrix whose entries are orthogonal projections (magic unitary):

$$U = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}, \qquad (q_{ij})^2 = q_{ij}^* = q_{ij}, \quad i,j = 1,\ldots,n.$$

Quantum symmetry group of M_n – positive result

We say that the action α of $\mathbb{G}$ on B preserves a functional $\omega \in \mathsf{B}^*$ if

$$\forall_{b \in \mathsf{B}} \quad (\mathrm{id}_{C(\mathbb{G})} \otimes \omega)(\alpha(b)) = \omega(b) 1_{C(\mathbb{G})}.$$

Theorem (S.Wang)

Let D be a finite-dimensional C^*-algebra with a faithful state ω. The category $\mathfrak{C}(\mathsf{D}, \omega)$ of quantum groups acting on D and preserving the state ω admits a universal object.

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Theory of quantum isometry groups

During the last 2-3 years D.Goswami and J.Bhowmick developed a theory of quantum isometry groups of spectral triples. Here we will consider a very special example.

Spectral triples on group algebras

Γ - finitely generated discrete group with (minimal, symmetric) generating set $S := \{\gamma_1, \cdots \gamma_n\}$

$l : \Gamma \to \mathbb{N}_0$ - word-length function

Define the *Dirac* operator D on $\ell^2(\Gamma)$ by

$$D(\delta_\gamma) = l(\gamma)\delta_\gamma.$$

$C_r^*(\Gamma)$ - C^*-algebra generated in $B(\ell^2(\Gamma))$ by the left regular representation λ

$\mathbb{C}[\Gamma] := \text{Lin}\{\lambda_\gamma : \gamma \in \Gamma\}.$

Let $\widehat{D} : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ be defined by

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Category of quantum groups acting on $\hat{\Gamma}$ and preserving the length

The vector $\delta_e \in \ell^2(\Gamma)$ is cyclic and separating for $C_r^*(\Gamma)$. Let $\tau(b) := \langle \delta_e, b\delta_e \rangle$ be the usual trace on $C_r^*(\Gamma)$.

Definition

Let $\mathfrak{C}_+(\hat{\Gamma}, \mathsf{H}, D)$ be the category with objects $(\mathbb{G}, \alpha)$ such that $\mathbb{G}$ is a compact quantum group acting by α on $C_r^*(\Gamma)$ and such that

1. α is τ preserving;
2. α maps $\mathbb{C}[\Gamma]$ into $C(\mathbb{G}) \odot \mathbb{C}[\Gamma]$;
3. $\alpha\widehat{D} = (\mathrm{id}_{C(\mathbb{G})} \odot \widehat{D})\alpha$.

Existence of the quantum isometry group of $\hat{\Gamma}$

Theorem (J.Bhowmick + D.Goswami + AS)

The category $\mathfrak{C}_+(\hat{\Gamma}, \mathsf{H}, D)$ admits a universal object. It is called the quantum isometry group of $\hat{\Gamma}$ and denoted further by $\mathrm{QISO}^+(\widehat{\Gamma}, S)$ or $\mathrm{QISO}^+(\widehat{\Gamma})$.

$\mathrm{QISO}^+(\widehat{\Gamma})$ is a compact matrix quantum group with a fundamental representation $[q_{t,s}]_{t,s\in S}$, where the elements $\{q_{t,s} : t, s \in S\}$ must satisfy the commutation relations implying that the prescription

$$\alpha(\lambda_\gamma) = \sum_{\gamma' \in S : l(\gamma) = l(\gamma')} q_{\gamma',\gamma} \otimes \lambda_{\gamma'}, \quad \gamma \in \Gamma$$

defines (inductively) a unital *-homomorphism from $C^*(\Gamma)$ to $C(\mathrm{QISO}^+(\widehat{\Gamma}, S)) \otimes C^*(\Gamma)$.

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Finite groups

Theorem

Let $n \in \mathbb{N} \setminus \{1, 2, 4\}$, $S = \{1, n-1\}$. *Then* $C(\mathrm{QISO}^+(\widehat{\mathbb{Z}_n})) \approx C^*(\mathbb{Z}_n) \oplus C^*(\mathbb{Z}_n)$ *(as a* C^**-algebra). In particular it is commutative (we just have classical isometries).*

Proposition

$C(\mathrm{QISO}^+(\widehat{\mathbb{Z}_4}))$ is isomorphic to $C^*(D_\infty \times \mathbb{Z}_2)$. In particular it is not commutative.

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Permutation/dihedral group

One can compute $\mathrm{QISO}^+(\widehat{S_3}, S)$ for different generating sets S. In both cases the resulting C^*-algebra is isomorphic to $C^*(S_3) \oplus C^*(S_3)$, but its actions on $C^*(S_3)$ are different (similar thing happens for the infinite dihedral group D_∞). We do not know if

$$\mathrm{QISO}^+(\widehat{S_3}, S) \approx \mathrm{QISO}^+(\widehat{S_3}, S') \text{ as compact quantum groups?}$$

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Free abelian groups

Theorem

$\mathrm{QISO}^+(\widehat{\mathbb{Z}}, \{1, -1\})$ *is isomorphic to the compact group* $\mathbb{T} \rtimes \mathbb{Z}_2$*. Its action on* $C^*(\mathbb{Z}) \cong C(\mathbb{T})$ *is given by the standard (isometric) action of the group* $\mathbb{T} \rtimes \mathbb{Z}_2$ *on* $\mathbb{T}$*.*

We do not know if $C(\mathrm{QISO}^+(\widehat{\mathbb{Z}^2}, \{\pm e_1, \pm e_2\}))$ is commutative.

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Free groups

Theorem

Consider the free group on two generators with the usual generating set. $C(\mathrm{QISO}^+(\widehat{\mathbb{F}_2}))$, is the universal C^*-algebra generated by partial isometries A, B, C, D, E, F, G, H such that if $P_A, P_B, \ldots$ denote respectively the range projections of $A, B, \ldots$ and $Q_A, Q_B, \ldots$ denote the initial projections of $A, B, \ldots$ then the matrix

$$\begin{bmatrix} P_A & P_B & P_C & P_D \\ P_E & P_F & P_G & P_H \\ Q_B & Q_A & Q_D & Q_C \\ Q_F & Q_E & Q_H & Q_G \end{bmatrix} \tag{1}$$

is a magic unitary (all entries are orthogonal projections, the sum of each row/column is equal to 1).

Theorem (continued)

The coproduct on $C(\mathrm{QISO}^+(\widehat{\mathbb{F}_2}))$ *is determined by the condition that the (unitary) matrix*

$$U = \begin{bmatrix} A & B & C & D \\ B^* & A^* & D^* & C^* \\ E & F & G & H \\ F^* & E^* & H^* & G^* \end{bmatrix}$$

is a fundamental representation. In particular the restriction of the coproduct of $C(\mathrm{QISO}^+(\widehat{\mathbb{F}_2}))$ *to the* C^**-algebra generated by the entries of the matrix in* (1) *coincides with the coproduct on Wang's* $C(S_4^+)$*.*

Liberated/free quantum groups

In recent years T.Banica, B.Collins, S.Curran, R.Speicher (and others) have initiated the study of a so-called liberation procedure. The idea can be (very informally) described as follows

- consider your favourite compact group of matrices G
- find a presentation of $C(G)$ in terms of finitely many generators, preferably coefficients of a unitary representation
- 'liberate' the generators, that is drop the assumption that they must commute
- show that the resulting family of algebraic relations determines an algebra $C(\mathbb{G})$ for a certain compact quantum group $\mathbb{G}$. We usually write $\mathbb{G} = G^+$.

If $\mathbb{G}$ is a compact quantum group then the quotient of $C(\mathbb{G})$ by its commutator ideal is the algebra of functions on a certain compact group, which we will denote $\mathbb{G}_{clas}$ and call the classical version of $\mathbb{G}$.
The liberation procedure started from G should lead to a quantum group $\mathbb{G}$ such that $G = \mathbb{G}_{clas}$.

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Liberated/free quantum groups continued

The liberation procedure, though ill-defined, leads to many interesting connections between the theory of quantum groups and free probability. Up to last year it was most successful for ‘real’ compact groups. In particular categorical considerations lead naturally to four families of ‘free’ quantum groups, all defined via conditions on the entries of the fundamental representations:

- free quantum orthogonal group O_n^+ (entries in U selfadjoint);
- free quantum bistochastic group B_n^+ (entries in U selfadjoint and summing to 1 in each column)
- free quantum hyperoctahedral group H_n^+ (entries in U – partial symmetries)
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Liberated/free quantum groups continued

The liberation procedure, though ill-defined, leads to many interesting connections between the theory of quantum groups and free probability. Up to last year it was most successful for 'real' compact groups. In particular categorical considerations lead naturally to four families of 'free' quantum groups, all defined via conditions on the entries of the fundamental representations:

- free quantum orthogonal group O_n^+ (entries in U selfadjoint);
- free quantum bistochastic group B_n^+ (entries in U selfadjoint and summing to 1 in each column)
- free quantum hyperoctahedral group H_n^+ (entries in U – partial symmetries)
- free quantum symmetric group S_n^+ (entries in U – orthogonal projections)

So is there a connection?

Recall that partial symmetries are nothing but selfadjoint partial isometries. This seems to suggest that there is a connection between H_n^+ and $\mathrm{QISO}^+(\widehat{\mathbb{F}_2})$. Indeed, it turns out that we can construct and study a two-parameter family of compact quantum groups which includes both the families above and $\mathrm{QISO}^+(\widehat{\mathbb{F}_2})$.

General setup

Let $p \in \mathbb{N}$. Put $\mathcal{F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, let $F_p \in M_{2p}$ be the matrix given by

$$F_p = \begin{pmatrix} \mathcal{F} & 0 & \cdots & 0 \\ 0 & \mathcal{F} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathcal{F} \end{pmatrix}$$

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Write $\mathcal{J}_p = \{i\alpha : i \in \{0,1\}, \alpha \in \{1,\dots,p\}\}$

Let $A_h(p)$ denote the universal unital C^*-algebra generated by elements $\{U_{z,y} : z, y \in \mathcal{J}_p\}$ such that

1. $U := (U_{z,y})_{z,y \in \mathcal{J}_p}$ is a unitary;
2. $$U = F_p \bar{U} F_p;$$
3. each $U_{z,y}$ is a partial isometry.

We could have also replaced the last condition by for example:

$$\text{(b)} \qquad \sum_{z \in \mathcal{J}_p} U_{z,y} = \sum_{z \in \mathcal{J}_p} U_{y,z} = 1 \quad \text{for all } y \in \mathcal{J}_p$$

$$\text{(s)} \qquad \text{each } U_{z,y} \text{ is an orthogonal projection}$$

These conditions, together with (i)-(ii) lead (via the usual procedure) to three families of compact quantum groups, denoted respectively by $H^+(p)$, $B^+(p)$ and $S^+(p)$.

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Identifications of $B^+(p)$ and $S^+(p)$

Theorem

We have

$$B^+(p) \approx O^+_{2p-1},$$

$$S^+(p) \approx H^+_p,$$

$$H^+(p) \approx \mathrm{QISO}^+(\widehat{\mathbb{F}_p})$$

In fact we can also consider the two-parameter versions $G^+(p,q)$, combining the usual liberated quantum groups with twisted ones. This leads to further nontrivial examples.

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Classical versions

We can compute classical versions of all quantum groups above. In particular

$$H^{+}(p)_{clas} \approx \mathbb{T}^{p} \rtimes H_{p}.$$

Note that $\mathbb{T}^p \rtimes H_p$ is the classical isometry group of $\mathbb{T}^p$; this agrees very well with the isomorphism $H^+(p) \approx \mathrm{QISO}^+(\widehat{\mathbb{F}_p})$.

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Definitions of $G^+(p)$ via intertwiners

It turns out that each $G^+(p)$ (and more generally, $G^+(p,q)$) introduced above can be naturally described via the intertwiners of the tensor powers of the fundamental unitaries. We provide just a sample:

Theorem

The algebra $C(H^+(p))$ is the universal C^*-algebra generated by the entries of a unitary $2p$ by $2p$ matrix U such that the vector ξ is a fixed vector for $U^{\otimes 2}$ and the map $e_{i\alpha} \to e_{i\alpha} \otimes e_{\bar{i}\alpha} \otimes e_{i\alpha}$ defines a morphism in $\mathrm{Hom}(U; U^{\otimes 3})$ ($\bar{0} = 1$, $\bar{1} = 0$).

Recall that

$$\mathrm{Hom}(U^{\otimes k}; U^{\otimes l}) = \{T \in B((\mathbb{C}^{2p+q})^{\otimes k}; (\mathbb{C}^{2p+q})^{\otimes l}) : (T \otimes 1)U^{\otimes k} = U^{\otimes l}(T \otimes 1)\}.$$

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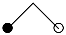
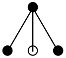
Such results (plus some free probability techniques) facilitate the description of Tannakian categories of representations of $G^+(p,q)$ via (coloured) partitions. For example for the case of $H^+(p)$ the generating partitions are of the form

$\pi_1 =$  , $\pi_2 =$  .

Example of application

If $\mathbb{G}_1$ and $\mathbb{G}_2$ are compact quantum groups and $\pi : C(\mathbb{G}_1) \to C(\mathbb{G}_2)$ is a unital injective *-homomorphism intertwining the respective coproducts, then $\mathbb{G}_2$ is said to be *a quantum group extension* of $\mathbb{G}_1$.

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The quantum group $H^+(p)$ is an extension of S^+_{2p}.

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The quantum group $H^+(p)$ is an extension of S_{2p}^+.

Using the techniques of the similar type we can also compute $\mathrm{QISO}(\widehat{\mathbb{Z}_s^{*n}})$ for $s > 5$, $n \in \mathbb{N}$. This leads to new categories of coloured partitions and to interesting (free) probabilistic interpretations.

Theorem

Let $s \geq 5$, $n \geq 2$. The main character of the quantum isometry group $\mathrm{QISO}(\widehat{\mathbb{Z}_s^{*n}})$ decomposes as

$$\chi = \sum_{k=1}^{s} 2\cos\left(\frac{2k\pi}{s}\right)\alpha_k$$

where $\alpha_1, \ldots, \alpha_s$ are free Poisson variables of parameter $1/(2s)$, free.

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Let us finish by listing a few open problems:

- to what extent does $\mathrm{QISO}^{+}(\widehat{\Gamma}, \mathrm{S})$ depend on the generating set S?
- can one deduce some properties of $\mathrm{QISO}^{+}(\widehat{\Gamma})$ directly from the properties of Γ?
- can one construct examples of non-compact locally compact quantum groups as liberations or quantum symmetry groups?

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