Precosheaves of C^*-algebras and their representations, and applications to conformal nets

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Outline

1. Motivation
2. Definitions
3. Net bundles
4. Nets
5. The example of nets over S^1
6. Comment

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- Precosheaves of C^*-algebras arise naturally in AQFT, as the set observables localized within a suitable class of regions of a spacetime (called the observable net. When the space has a nontrivial topology (is not simply connected), this class of regions is not directed under inclusions.
- To deal with these situtations, [Fredenhagen 1990] shows the existence of the colimit C^*-algebra of a precosheaf of C^*-algebras, and shows some key properties of the colimit in the theory of superselection sectors over S^1. The colimit is characterized by the properties that the representation of the precocheaf (those that we shall call Hilbert space representations) extend to the colimit.
- Recent works [Carpi, Kawahigashi, Longo 2008], [Brunetti,R 2009], [Brunetti,Franceschini,Moretti 2009] show that the colimit does not econde all the physical information of the observable net. In particular [Brunetti,R 2009] have shown the existence of charged superselection sectors of the observable net over a spacetime which are affected by the topology, when not trivial, of the spacetime. These sectors are described by representations which does not extend to a representation of the colimit.

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Nets

Net of C^*-algebras $(\mathcal{A}, \jmath)_K$ over a poset K:

- *fibres*: $\mathcal{A} = \{\mathcal{A}_o \ , \ o \in K\}$ of unital C^*-algebras
- *inclusion maps*: $\jmath = \{\jmath_{\tilde{o}o} : \mathcal{A}_o \to \mathcal{A}_{\tilde{o}} \ , \ o \leq \tilde{o}\}$ unital monomorphisms satisfying the net relations

$$\jmath_{o''o'} \circ \jmath_{o'o} = \jmath_{o''o} \quad , \qquad o'' \geq o' \geq o \ .$$

If the inclusion maps are isomorphisms we talk of a **C^*-net bundle**.

Morphism of nets $(\rho, \mathrm{f}) : (\mathcal{A}, \jmath)_K \to (\mathcal{B}, i)_P$ where :

$\mathrm{f} : K \to P$ is a poset morphism, i.e. $\mathrm{f}(o') \geq \mathrm{f}(o)$, $\quad o' \geq o$

$\rho := \{\rho_o : \mathcal{A}_o \to \mathcal{B}_{\mathrm{f}(o)}\}$ unital morphisms, such that

$$i_{\mathrm{f}(o')\,\mathrm{f}(o)} \circ \rho_o = \rho_{o'} \circ \jmath_{o'o} \ , \qquad o' \geq o \ .$$

- faithful on the fibres if ρ_o is a monomorphism for any o;
- isomorphism when both ρ and f are isomorphisms

A net is trivial if it is isomorphic to a constant net $(\mathcal{C}, id)_K$, where $\mathcal{C}_o \simeq \mathcal{F}$ for a fixed C^*-algebra $\mathcal{F}$ and any $id_{\tilde{o}o}$ is the identity of $\mathcal{F}$.

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Observations

- We are interested in the cases where K is *not upward directed*. In this case $(\mathcal{A}, \jmath)_K$ is a pre-cosheaf. We adopt the convention used in AQFT that call these objects nets.
- Example arise in AQFT. If X is a globally hiperbolic spacetime, One considers a good subbase K for the topology of X ordered under inclusion. Good means relatively compact, connected and simply connected open set of X. The mapping associating the C^*-algebra $\mathcal{A}_o$, generated by the observables localized within o, to any o of K gives a net. K is not upward directed if X is not simply connected.
- Examples con be constructed for any poset (use of the algebraic topology of the poset).
- The previous definition can be generalize to include symmetry groups G: G-covariant net, and continuous G-covariant nets.
- The definition of net can be given for other categories: groups, Hilbert spaces...

Representations

Representation of a net is a pair (π, V)

- $\pi := \{\pi_o : \mathcal{A}_o \to \mathcal{B}\mathcal{H}\}$ representations on fixed Hilbert space $\mathcal{H}$;
- *inclusion operator*: $V := \{V_{o'o} : \mathcal{H} \to \mathcal{H} \,|\, o' \geq o\}$ unitary operators satisfying net relations

$$V_{o''o'} V_{o'\tilde{o}} = V_{o''o} \ , \qquad o'' \geq o' \geq o \ ;$$

and

$$\mathrm{Ad}_{V_{o'o}} \circ \pi_o = \pi_{o'} \circ \jmath_{o'o} \ , \qquad o' \geq o.$$

It is faithful if π_o is a monomorphism for any o

A **Hilbert space representation** is a representation of the form $(\pi, 1)$, i.e. $V_{o'o} = 1_{\mathcal{H}}$ for any $o' \geq o$.

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Aim of the talk

Aim:i introduce a new object associated with nets, and analyzing, in terms of this object, the question of the existence of nontrivial representations of a net. Prove that any net over S^1 has faithful representations

A well known example. The triangle of groups is a net $(G, y)_T$ of groups where T is the poset

[Gersten, Stallings, 1990] S sum of the groups angles: $S \leq \pi$, then the net embeds faithfully into the colimit. $S > \pi$, examples of triangle of groups whose colimit is trivial. The same result applies to the corresponding net $(\mathrm{C}^*(G), \mathrm{C}^*(y))_T$ of group C^*-algebras. [Bildea, 2006] some results on triangle of C^*-algebras.

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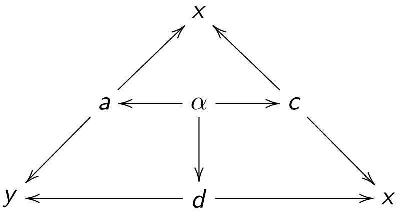

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The fundamental group of a poset

Graph of a poset K.

- Vertices : elements of K
- Edges : for any inclusion $o \leq a$ there are two edges:

$$ao : o \to a \quad , \quad \overline{ao} : a \to o \ ,$$

In particular, for $a = o$ we have $e_a : a \to a$ and $\overline{e}_a : a \to a$.

Connectedness A path p is a finite composition of edges:

$$p = b_n * b_{n-1} * \cdots * b_1 \text{ where } T(b_i) = S(b_{i+1}) \ .$$

We write $p : o \to a$ if $S(b_1) = o$ and $T(b_n) = a$. A loop over o is a path $p : o \to o$.

- The posets K we consider are pathwise connected: for any $a, o \in K$ there is a path $p : a \to o$.

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The fundamental group is the edge path group of the poset: i.e.the quotient

$$\pi_1^o(K) := \{p : o \to o\}/\sim$$

where

- $e_o * p \sim p$ and $p * \overline{p} \sim e_o$ for any $p : o \to o$
- $p_1 * a''a' * a'a * q_1 \sim p_1 * a''a * q_1$, for any $a'' \geq a' \geq a$ and $q_1 : o \to a$, $p_1 : a'' \to o$.

Sometimes we write $\pi_1(K)$, since the isomorphism class of $\pi_1^o(K)$ does not depend on o. K is simply connected if $\pi_1(K)$ is trivial.

▷ If the poset is either upward or downward directed then it is simply connected.

▷ If K is a good subbase for the topology of a space X then $\pi_1(K) \cong \pi_1(X)$.

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The rôle of topology:the holonomy dynamical system

Let $(\mathcal{A}, \jmath)_K$ be a C^*-net bundle. As the inclusion maps are isomorphisms define

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So, $\jmath$ extends to paths: $\jmath_p := \jmath_{b_n} \circ \jmath_{b_{n-1}} \circ \cdots \jmath_{b_1}$. Note $\jmath_p : \mathcal{A}_o \to \mathcal{A}_a$ is an isomorphism for any $p : o \to a$.

Net relations imply $\jmath_p = \jmath_q$ if $p \sim q$. Hence, fix $o \in K$,

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gives an action of $\pi_1^o(K)$ on $\mathcal{A}_o$. We call $(\mathcal{A}_o, \pi_1^o(K), \jmath_*)$ the holonomy dynamical system of the net bundle.

Lemma

- *the holonomy dynamical system is a complete invariant of the net bundle.*
- *the representations of a net bundle are in bijective correspondence to the covariant representation of its holonomy dynamical system.*
- *any net bundle over a simply connected poset is trivial*

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The enveloping net bundle of a net $(\mathcal{A}, \jmath)_K$

The enveloping net bundle of a net $(\mathcal{A}, \jmath)_K$ is a C^*-net bundle $(\overline{\mathcal{A}}, \bar{\jmath})_K$, which comes equipped with a morphism, the canonical embedding,

$$\epsilon : (\mathcal{A}, \jmath)_K \to (\overline{\mathcal{A}}, \bar{\jmath})_K \ ,$$

satisfying the following universal property: given morphisms with values in C^*-net bundles, $(\varphi, \mathrm{h}), (\theta, \mathrm{h}) : (\overline{\mathcal{A}}, \bar{\jmath})_K \to (\mathcal{C}, y)_P$ and $(\psi, \mathrm{f}) : (\mathcal{A}, \jmath)_K \to (\mathcal{B}, \imath)_S$, then

$$\begin{cases} (\varphi, \mathrm{h}) \circ \epsilon = (\theta, \mathrm{h}) \circ \epsilon \ \Rightarrow \ \varphi = \theta \ , \\ \exists ! \ (\psi^\uparrow, \mathrm{f}) \text{ such that } (\psi^\uparrow, \mathrm{f}) \circ \epsilon = (\psi, \mathrm{f}) \ , \end{cases}$$

where $(\psi^\uparrow, \mathrm{f}) : (\overline{\mathcal{A}}, \bar{\jmath})_K \to (\mathcal{B}, \imath)_S$ is the pullback.

- Assigning the enveloping net bundle is a functor in the category of nets.
- If the poset is simply connected the fibres of the enveloping net bundle are isomorphic to the colimit C^*-algebra.
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satisfying the following universal property: given morphisms with values in C^*-net bundles, $(\varphi, \mathrm{h}), (\theta, \mathrm{h}) : (\overline{\mathcal{A}}, \bar{\jmath})_K \to (\mathcal{C}, y)_P$ and $(\psi, \mathrm{f}) : (\mathcal{A}, \jmath)_K \to (\mathcal{B}, \imath)_S$, then

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A classification of nets

A net is said to be

- degenerate if its enveloping net bundle vanish, and nondegenerate otherwise
- injective if it is nondegenerate and the canonical embedding is injective.
- realizable if the colimit does not vanish and the embedding is injective.

Lemma

- ▷ *A net is nondegerate iff admits nontrivial representations*
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Inductive system of nets $\{(\mathcal{A}, \jmath)_K \, , \, (\psi, \mathrm{f})\}_S$ where

- S is an *upward directed* poset
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The inductive limit net $\lim_S(\mathcal{A}^\alpha, \jmath^\alpha)_{K^\alpha}$ exists and is defined over the inductive limit poset.

Theorem

- ▷ *The functor assigning the enveloping net bundle preserves inductive limits.*
- ▷ *If the nets of the inductive systems are injective and the linking morphisms are monomorphisms, then the inductive limit net is injective.*

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Nets over S^1

The *poset* $\mathcal{I}$ is the set of connected open interval of S^1 having a proper closure, ordered under inclusion. Since $\mathcal{I}$ is a base for the topology of S^1, its homotopy group is $\mathbb{Z}$

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Idea of the proof

Step 1: Inductive limit

Let $\{x_n\}$ be a dense sequence of points of S^1. Define

$$\mathcal{I}_n := \cup_{i=1}^n \mathcal{I}(x_i) \ , \qquad n \in \mathbb{N} \ ,$$

where $\mathcal{I}(x) := \{o \in \mathcal{I} \ , \ x \notin cl(o)\}$.

Note that $\mathcal{I}_n \subset \mathcal{I}_{n+1}$ and that, since $\{x_n\}$ is dense, any $o \in \mathcal{I}$ belongs eventually to the sequence $\mathcal{I}_n$. This implies that $\mathcal{I}$ is the inductive limit poset of $\{\mathcal{I}_n\}$.

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Step 2: finite approximations and reduction

Assume that the first n elements of $\{x_n\}$ verify

$$x_1 <^+ x_2 <^+ x_3 <^+ \cdots <^+ x_n \ ,$$

where $x <^+ y <^+ z$ means that starting from x, y preceeds z with respect the clockwise orientation.

The poset C_n. The elements of C_n are n^2 intervals

$$(x_i, x_j) := \{x \in S^1 \ , \ x_i <^+ x <^+ x_j\} \ , \qquad i, j \in \{1, \ldots, n\}.$$

With respect to the inclusion : there are n maximal elements $(x_i, x_i) = S^1 \setminus \{x_i\}$ for $i = 1, \ldots n$; and n minimal elements (x_i, x_{i+1}) for $i = 1, \ldots, n-1$ and (x_n, x_1).

The embedding. $(\mathcal{A}, \jmath)_{\mathcal{I}}$ induces a net $(\hat{\mathcal{A}}, \hat{\jmath})_{C_n}$, and there is a morphism $(\psi, \mathrm{f}) : (\mathcal{A}, \jmath)_{\mathcal{I}_n} \to (\hat{\mathcal{A}}, \hat{\jmath})_{C_n}$ faithful on the fibres. In particular: $\mathrm{f} : \mathcal{I}_n \to C_n$ is an epimorphism defined

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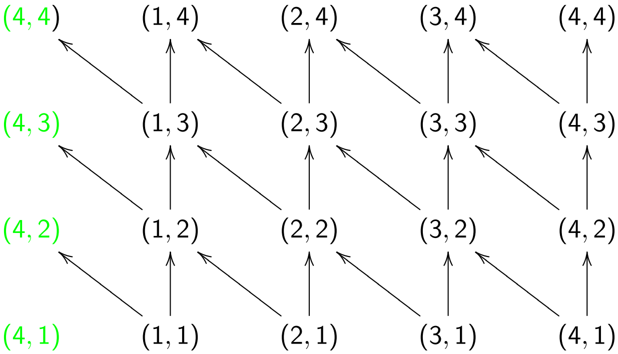
Step3: the n-cylinder and injectvity

Actually any net over C_n is injective. This is so because the poset C_n, called n-cylinder, has an important regularity that can be seen using an equivalent description of C_n in terms of a graph. We present the case $n = 4$



Outline

1. Motivation
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- *Does a net over S^1 have faithful Hilbert space representations, i.e. is realizable?* This is equivalent to saying that the holonomy dynamical system $(\overline{\mathcal{A}}_o, \pi_o^1(K), \bar{\jmath}_*)$ of the enveloping net bundle has invariant representations

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