The Hölder inequality for KMS states and its application to thermal QFT

F. Robl[1]

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[1]Supervisor: C. Jäkel, Research funded by *Leverhulme Trust*

Overview

- The Hölder inequality for Gibbs states
- Araki's & Masuda's non-commutative L_p spaces
- The Hölder inequality for KMS states
- Application: Construction of the thermal $P(\phi)_2$ model

Schatten classes

Let $\mathcal{H}$ be a finite dimensional Hilbert space. The Schatten p-class $L_p(\mathrm{Tr})$ is defined as

$$A \in M_n(\mathbb{C}) : \quad \|A\|_{L_p(\mathrm{Tr})} = \left(\mathrm{Tr}|A|^p\right)^{1/p} < \infty$$

$A, B \in L_2(\mathrm{Tr})$ are called Hilbert Schmidt operators. $L_2(\mathrm{Tr})$ is a Hilbert space with inner product

$$(A, B)_{\mathrm{Tr}} = \mathrm{Tr}\, A^* B.$$

There holds the Hölder inequality:

$$|(A, B)_{\mathrm{Tr}}| \leq \|A\|_{L_p(\mathrm{Tr})} \|B\|_{L_q(\mathrm{Tr})} ,$$

for $1/p + 1/q = 1$.

Hölder inequality for Gibbs states

Now let $M_n(\mathbb{C}) \ni \rho \geq 0$, $\mathrm{Tr}\rho = 1$,

$$\omega_\rho(A) = \mathrm{Tr}\,\rho A,$$

and

$$\|A\|_{L_p(\omega_\rho)} := \left(\mathrm{Tr}\,|\rho^{1/2p} A \rho^{1/2p}|^p\right)^{1/p}.$$

The inner product in $L_2(\omega_\rho)$ is given by

$$(A, B)_{\omega_\rho} = \mathrm{Tr}\,\rho^{1/2} A^* \rho^{1/2} B\,.$$

Hölder inequality:

$$|(A, B)_{\omega_\rho}| \leq \|A\|_{L_p(\omega_\rho)} \|B\|_{L_q(\omega_\rho)}$$

for $1/p + 1/q = 1$.

Relative modular operators

Now for a second density matrix ν define the linear operator

$$\Delta_{\nu,\rho} A = \nu A \rho^{-1},$$

which fulfills

$$\Delta_{\nu,\rho}^{1/p} A = \nu^{1/p} A \rho^{-1/p}.$$

Applying the Hölder inequality gives

$$\begin{aligned} |(A_2 \Delta_{\nu_2,\rho}^{1/p}, A_1 \Delta_{\nu_1,\rho}^{1/q})_{\omega_\rho}| &\leq \|A_1\| \, \|A_2\| \, \|\Delta_{\nu_2,\rho}^{1/p}\|_{L_p(\omega)} \, \|\Delta_{\nu_1,\rho}^{1/q}\|_{L_q(\omega)} \\ &= \|A_1\| \, \|A_2\| \, (\mathrm{Tr}\, \nu_2)^{1/p} (\mathrm{Tr}\, \nu_1)^{1/q} \\ &= \|A_1\| \, \|A_2\| \, \omega_{\nu_2}(\mathbb{1})^{1/p} \, \omega_{\nu_1}(\mathbb{1})^{1/q} \, , \end{aligned}$$

for $1/p + 1/q = 1$. This structure is preserved for general KMS states.

Given objects

The starting point is a von Neumann algebra in standard form, i.e. let $(\mathcal{H}, \mathcal{M}, J, \mathcal{P}^\sharp)$ be given, where

- $\mathcal{H}$ is a Hilbert space,
- $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra,
- J is an anti-unitary involution on $\mathcal{H}$ and
- $\mathcal{P}^\sharp$ is a self dual cone,

such that $J\mathcal{M}J = \mathcal{M}'$, $J\Psi = \Psi$ for $\Psi \in \mathcal{P}^\sharp$; and the KMS state

$$\omega_\beta(A) = (\Omega, A\Omega), \quad A \in \mathcal{M},$$

Ω beeing a cyclic and separating vector for $\mathcal{M}$. Furthermore there are the Tomita Takesaki objects summarized by

$$SA\Omega = J\Delta^{1/2}A\Omega = A^*\Omega, \quad A \in \mathcal{M}.$$

Lastly we assume the existence of a generator L such that $\Delta^{1/2} = \mathrm{e}^{-\beta L/2}$.

Relative modular operators

For a second vector state $\phi = (\xi, \cdot\, \xi)$, the operator defined by

$$S_{\xi,\Omega} A\Omega = A^* \xi$$

is also closable. The polar decomposition yields

$$S_{\xi,\Omega} A\Omega = J_{\xi,\Omega} \Delta_{\xi,\Omega}^{1/2} A\Omega = A^* \xi.$$

$J_{\xi,\Omega}$ is an anti-linear involution. $\Delta_{\xi,\Omega}$ is positive self-adjoint. Note the coincidences

$$S = S_{\Omega,\Omega}, \quad J = J_{\Omega,\Omega} \quad \text{and} \quad \Delta = \Delta_{\Omega,\Omega}.$$

In the finite dimensional case $\Delta_{\Omega_1,\Omega_2}$ is precisely the matrix from the first part of the talk.

Araki's generalisation

Theorem (Araki)

$$I_\alpha^{(n)} := \{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} \Re z_j \leq \alpha,\ 0 \leq \Re z_j\} ,$$

for $\alpha > 0$. *Let* $z \in I^{(n)} \equiv I_1^{(n)}$ *and* $z_m', z_m'' \in \mathbb{C}$ *be such that* $\Re z_m', \Re z_m'' > 0$, $z_m' + z_m'' = z_m$ *and*

$$\Re z_1 + \dots \Re z_{m-1} + \Re z_m'' \leq 1/2 ,$$
$$\Re z_n + \dots \Re z_{m+1} + \Re z_m' \leq 1/2 .$$

Under these conditions, for $\phi_1, \dots, \phi_n \in \mathcal{M}_*^+$, $X_0, \dots, X_n \in \mathcal{M}$ *and* $z_0 = 1 - \sum_{j=1}^n \Re z_j$

$$\left| (\Delta_{\phi_m,\Omega}^{\bar{z}_m'} X_m^* \Delta_{\phi_{m+1},\Omega}^{\bar{z}_{m+1}} \dots \Delta_{\phi_n,\Omega}^{\bar{z}_n} X_n^* \Omega, \Delta_{\phi_m,\Omega}^{z_m''} X_{m-1} \Delta_{\phi_{m-1},\Omega}^{z_{j-1}} \dots \Delta_{\phi_1,\Omega}^{z_1} X_0 \Omega) \right|$$
$$\leq \left(\prod_{j=0}^{n} \|X_j\| \right) (\Omega, \mathbb{1}\Omega)^{z_0} \left(\prod_{j=1}^{n} \phi_j(\mathbb{1})^{\Re z_j} \right) .$$

Araki's & Masuda's non-commutative L_p spaces

Definition

For $2 \leq p \leq \infty$,

$$L_p(\mathcal{M}, \Omega) \doteq \{\zeta \in \bigcap_{\xi \in \mathcal{H}} D\big(\Delta_{\xi,\Omega}^{\frac{1}{2}-\frac{1}{p}}\big) \mid \|\zeta\|_p < \infty\},$$

where

$$\|\zeta\|_p = \sup_{\|\xi\|=1} \|\Delta_{\xi,\Omega}^{\frac{1}{2}-\frac{1}{p}}\zeta\| \; .$$

For $1 \leq p < 2$, $L_p(\mathcal{M}, \Omega)$ is defined as the completion of $\mathcal{H}$ with respect to the norm

$$\|\zeta\|_p = \inf\{\|\Delta_{\xi,\Omega}^{\frac{1}{2}-\frac{1}{p}}\zeta\| \mid \|\xi\| = 1, s_{\mathcal{M}}(\xi) \geq s_{\mathcal{M}}(\zeta)\}.$$

Here $s_{\mathcal{M}}(\xi)$ denotes the smallest projection in $\mathcal{M}$, which leaves ξ invariant.

Remark

- $L_2(\mathcal{M}, \Omega) = \mathcal{H}$, $L_\infty(\mathcal{M}, \Omega) \cong \mathcal{M}$ *and* $L_1(\mathcal{M}, \Omega) = \mathcal{M}_*$.
- $|\omega_\beta(A^* B)| \leq \|A\|_{L_p(\mathcal{M},\Omega)} \|B\|_{L_q(\mathcal{M},\Omega)}$ *for* $1/p + 1/q = 1$.

Hölder inequality for KMS states

For $A \in \mathcal{M}^+$,

$$\| A \|_p \doteq \omega_\beta \big(\underbrace{\mathrm{e}^{-\beta L/p} A \cdots \mathrm{e}^{-\beta L/p} A}_{p \text{ times}} \big)^{1/p} .$$

Theorem (J&R)

Consider a (τ, β)-KMS state ω_β over a C^*-dynamical system $(\mathcal{A}, \tau)$. Let $(z_1, \dots, z_n) \in \mathbb{C}^n$ be such, that $0 \leq \Re z_j$, $\sum_{j=1}^m \Re z_j \leq 1/2$ and $\sum_{j=m+1}^n \Re z_j \leq 1/2$, and let p_j be the smallest, positive integer such that $\frac{1}{p_j} \leq \min\{\Re z_{j+1}, \Re z_j\}$, with $z_{n+1} = z_n$ and $z_0 = z_1$. Then

$$\left| \omega_\beta \big(A_n \mathrm{e}^{-z_n \beta L} \cdots A_1 \mathrm{e}^{-z_1 \beta L} A_0 \big) \right| \leq \| A_0 \|_{p_0} \cdots \| A_n \|_{p_n} \qquad (*)$$

for all $A_0, \dots, A_n \in \mathcal{M}^+$.

Ideas of the proof I

More handy than the $L_p(\mathcal{M},\Omega)$ are the auxilliary spaces

$$\mathcal{L}_p(\mathcal{M},\Omega) := \{u\Delta_{\phi,\Omega}^{1/p} \mid u \text{ partial isometry}, \phi \in \mathcal{M}_*^+\} \qquad \text{and}$$
$$\mathcal{L}_p^*(\mathcal{M},\Omega) := \{A_0\Delta_{\phi_1,\Omega}^{z_1}A_1 \cdots \Delta_{\phi_n,\Omega}^{z_n}A_n \mid A_j \in \mathcal{M}, \phi_j \in \mathcal{M}_*^+, \sum \Re z_j \leq 1 - 1/p\},$$

for $1 \leq p < \infty$. The identification with $L_p(\mathcal{M},\Omega)$ is done via application to the distinguished vector Ω. By the invariance of the distinguished vector, $\Delta^\alpha\Omega = \Omega$, the following equivalence relation is in effect:

$$\Delta_{\Omega_1,\Omega}^{1/q}\Delta^\alpha \sim \Delta_{\Omega_1,\Omega}^{1/q} \quad \text{in } \mathcal{L}_p(\mathcal{M},\Omega)^*\,,$$

where $1 - 1/q + \alpha \leq 1/p$. Apparently, for $A \in \mathcal{M}^+$ and $1/p + 1/p' = 1$

$$\Delta^{1/2p}A\Delta^{1/2p} \in \mathcal{L}_{p'}^*(\mathcal{M},\Omega)\,.$$

Then, according to Araki and Masuda, there exists $\phi \in \mathcal{M}_*^+$ such that

$$\Delta^{1/2p}A\Delta^{1/2p} \sim \Delta_{\phi,\Omega}^{1/p} \quad \text{in } \mathcal{L}_{p'}^*(\mathcal{M},\Omega)\,.$$

Ideas of the proof II

Thusly one makes sense of the left hand side of the desired inequality and immediately can use Araki's inequality. It is left to show, that $\phi_j(\mathbb{1}) = |||A_j|||_p$.

$$\begin{aligned}\phi_j(\mathbb{1}) &= (\xi_j, \mathbb{1}\xi_j) = (J_{\xi_j,\Omega}\Delta_{\xi_j,\Omega}^{1/2}\Omega, J_{\xi_j,\Omega}\Delta_{\xi_j,\Omega}^{1/2}\Omega) \\ &\leq ((\Delta_{\xi_j,\Omega}^{1/p})^{p/2}\Omega, (\Delta_{\xi_j,\Omega}^{1/p})^{p/2}\Omega) = ((\Delta^{1/2p}A\Delta^{1/2p})^{p/2}\Omega, (\Delta^{1/2p}A_j\Delta^{1/2p})^{p/2}\Omega) \\ &= \omega_\beta(A_j \mathrm{e}^{-\beta L/p} \cdots \mathrm{e}^{-\beta L/p} A_j),\end{aligned}$$

as $J^*_{\xi_j,\Omega}J_{\xi_j,\Omega}$ is a projection. □

Remark

- (*) *is uniform in* $\Im z_j$
- $|||\cdot|||_p$ *norms are "better" than* $||\cdot||$.

The thermal $P(\phi)_2$ model

Define $Q := \mathcal{S}'_{\mathbb{R}}(S_\beta \times \mathbb{R})$ and for $f, g \in \mathcal{S}(S_\beta \times \mathbb{R})$

$$C(f,g) := (f, (-\Delta + m^2)^{-1} g).$$

In this context the bidual embedding $\phi(f) : Q \to \mathbb{R}$, $q \mapsto \langle q, f \rangle$ is called the Euclidean quantum field. For the free Gaussian measure there holds

$$\int_Q \phi(f)\, \phi(g)\ \mathrm{d}\phi_C = C(f,g).$$

More interestingly, the interacting measure is defined by

$$\mu := \lim_{l \to \infty} \mathrm{e}^{\int_{S_\beta \times [-l,l]} :P(\phi(\alpha,x)):_C\ \mathrm{d}x\ \mathrm{d}\alpha}\ \mathrm{d}\phi_C,$$

where P is a bounded below polynomial. μ is translation invariant.

Interacting Schwinger functions

For $0 \leq \alpha_1 < \ldots < \alpha_n < \beta$

$$\mathcal{S}_\beta(\alpha_1, x_1, \ldots, \alpha_n, x_n) := \int \phi(\delta_{\alpha_1} \otimes \delta_{x_1}) \ldots \phi(\delta_{\alpha_n} \otimes \delta_{x_n}) \, \mathrm{d}\mu$$

$$= \int \phi(\delta_0^{(2)}) U(\alpha_2 - \alpha_1, x_2 - x_1) \cdots \phi(\delta_0^{(2)}) U(\alpha_n - \alpha_{n-1}, x_n - x_{n_1}) \phi(\delta_0^{(2)}) \, \mathrm{d}\mu,$$

where $U(\alpha, x)$ implements translations and rotations on the cylinder. The second line above follows from translation invariance of μ. $\mathcal{S}_\beta$ only depends on the relative coordinates, so for the purpose of this talk we sloppily write

$$\mathcal{S}_\beta(\alpha_1, x_1, \ldots, \alpha_{n-1}, x_{n-1}).$$

Osterwalder Schrader reconstruction

The Osterwalder Schrader reconstruction for thermal fields is due to Klein & Landau. Aim: Construct

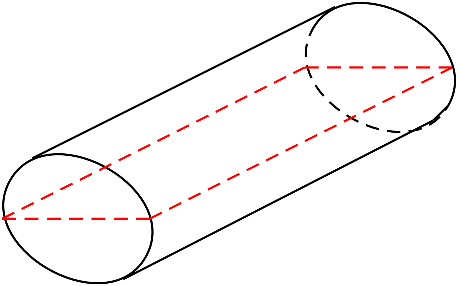
- Hilbert space $\mathcal{H}_\beta$,
- field operators ϕ_β,
- a distinguished (vacuum) vector Ω_β,
- a generator of time translations (Liouvillean) L,

such that one can define for $f \in \mathcal{S}(S_\beta \times \mathbb{R})$

$$\mathcal{W}_\beta(f_1, \ldots, f_n) = (\Omega_\beta, \phi_\beta(f_1) \ldots \phi_\beta(f_n)\Omega_\beta)$$

and there holds

$$\mathcal{S}_\beta(\alpha_1, x_1, \ldots, \alpha_{n-1}, x_{n-1}) = \mathcal{W}_\beta(-i\alpha_1, x_1, \ldots, -i\alpha_{n-1}, x_{n-1}).$$



Prametrize zylinder by (α, x) for $\alpha \in (-\beta/2, \beta/2]$, $x \in \mathbb{R}$ and define the reflection map R:

$$(R\phi)(\alpha, x) := \phi(-\alpha, x)$$

For $0 \leq \gamma \leq \beta$ we denote by $\Sigma_{[0,\gamma]}$ the σ-algebra generated by the functions $\phi(f)$ with $\operatorname{supp} f \subset [0, \gamma] \times \mathbb{R}$.

Scalar product:

$$\forall F \in L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu) : \quad (F, F) := \int_Q R(\overline{F}) F \mathrm{d}\mu \geq 0.$$

By factoring out the kernel $\mathcal{N}$ of $(\cdot, \cdot)$, we can define the physical Hilbert space.

$$\mathcal{H}_\beta := \overline{L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu)/\mathcal{N}}.$$

Let $\mathcal{V} : L^2(Q, \Sigma_{[0,\beta/2]}, \mathrm{d}\mu) \to \mathcal{H}_\beta$ denote the canonical projection, then

$$\Omega_\beta := \mathcal{V}(1)$$

is called the distinguished (vacuum) vector. The field ϕ_β, on $\mathcal{H}_\beta$ acts as

$$\phi_\beta(\delta \otimes g)\mathcal{V}(F) = \mathcal{V}(\phi(\delta \otimes g)F),$$

for $F \in L^2(Q, \Sigma_{[0,\beta/2]}, \mu)$.

Define $\mathcal{D}_\gamma := \mathcal{V}(L^2(Q, \Sigma_{[0,\beta/2-\gamma]}, \mu)) \subset \mathcal{H}_\beta$ for $0 \leq \gamma \leq \beta/2$. For $0 \leq \alpha \leq \gamma$ the operators $P(\alpha)$ on $\mathcal{D}_\gamma$ defined by

$$P(\alpha)\mathcal{V}(\psi) := \mathcal{V}(U(\alpha)\psi), \quad \psi \in L^2(Q, \Sigma_{[0,\beta/2-\gamma]}, \mu),$$

form a *local symmetric semigroup*, i.e.

- $\mathcal{D}_{\alpha_2} \subset \mathcal{D}_{\alpha_1}$ for $0 \leq \alpha_1 \leq \alpha_2 \leq \beta/2$ and

  $$\bigcup_{0 \leq \alpha \leq \beta/2} \mathcal{D}_\alpha$$

  is dense in $\mathcal{H}_\beta$;
- $P(\alpha)$ is linear;
- $P(0) = \mathbb{1}$, $P(\alpha)\mathcal{D}_\gamma \subset \mathcal{D}_{\gamma-\alpha}$ for $0 \leq \alpha \leq \gamma \leq \beta/2$, and

  $$P(\alpha)P(\gamma) = P(\alpha + \gamma);$$
- $P(\alpha)$ is symmetric;
- $P(\alpha)$ is weakly continuous.

Theorem (Klein & Landau and independently Fröhlich)

For every local symmetric semigroup $(P(\alpha), \mathcal{D}_\alpha, \beta/2)$ on a Hilber space $\mathcal{H}$, there exists a generator L, which fulfills

$$P(\alpha)\psi = \mathrm{e}^{-\alpha L}\psi, \quad \psi \in \mathcal{D}_\alpha.$$

Therefore it is possible to define

$$\begin{aligned} &\mathcal{W}_\beta(t_1 - i\alpha_1, x_1, \dots, t_n - i\alpha_n, x_n) \\ &\quad = (\Omega_\beta, \phi_\beta(\delta)\mathrm{e}^{-it_1 L}\mathrm{e}^{-\alpha_1 L}\mathrm{e}^{ix_1 P}\phi(\delta)\dots\mathrm{e}^{-it_n L}\mathrm{e}^{-\alpha_n L}\mathrm{e}^{ix_n P}\phi_\beta(\delta)\Omega_\beta) \end{aligned}$$

for $\alpha_j > 0$ and $\sum_j \alpha_j \leq \beta/2$. Then there holds

$$\mathcal{S}_\beta(\alpha_1, x_1, \dots, \alpha_n, x_n) = \mathcal{W}_\beta(-i\alpha_1, x_1, \dots, -i\alpha_n, x_n).$$

Construction of the algebra $\mathcal{M}$

$L^\infty(Q, \Sigma_{\{0\}}, \mu)$ leaves $L^2(Q, \Sigma_{[0,\beta/2]}, \mu)$ and $\mathcal{N}$ invariant. Therefore one can define a representation of $L^\infty(Q, \Sigma_{\{0\}}, \mu)$ on $\mathcal{H}_\beta$ by

$$\pi_\beta(A)\mathcal{V}(F) = \mathcal{V}(AF),$$

where $A \in L^\infty(Q, \Sigma_{\{0\}}, \mu)$ and $F \in L^2(Q, \Sigma_{[0,\beta/2]}, \mu)$. Then $\mathcal{M}$ is defined to be the von Neumann algebra generated by

$$\mathrm{e}^{itL}\pi_\beta(A)\mathrm{e}^{-itL}.$$

Ω_β is cyclic and separating for $\mathcal{M}$. Naturally,

$$\omega_\beta(A) := (\Omega_\beta, A\Omega_\beta), \quad A \in \mathcal{M}.$$

Remark

Same construction for $L^\infty(Q, \Sigma_{\{\beta/2\}}, \mu)$ results in $\mathcal{M}'$.

Tomita Takesaki objects

The Tomita Takesaki objects can be constructed quite explicitly from operations on $L^2(Q, \Sigma_{[0,\beta/2]}, \mu)$.

- Modular operator: $\Delta^{1/2} = \mathrm{e}^{-\beta L/2}$.
- Modular conjugation J: Induced action of $j := \overline{(R_{\beta/4} \cdot)}$ on $\mathcal{H}_\beta$.

Obviously $J\mathcal{M}J = \mathcal{M}'$.

How can we see, that $J\Delta^{1/2}A\Omega_\beta = A^*\Omega_\beta$? Remarkable result by Klein & Landau:

$$\mathcal{H}_\beta = L^2(Q, \Sigma_{\{0,\beta/2\}}, \mu).$$

Proof is based on Markov property. But on $L^2(Q, \Sigma_{\{0,\beta/2\}}, \mu)$ the $*$-operation is just complex conjugation.

Theorem

For $f_j \in \mathcal{S}(\mathbb{R}^2)$, $j \in \{1, \ldots, n\}$, *the following limit exists,*

$$\mathcal{W}_\beta(f_1, \ldots, f_n) := \lim_{\alpha_j \to 0} (\Omega_\beta, \phi(f_1)\mathrm{e}^{-\alpha_1 L} \cdots \phi(f_{n-1})\mathrm{e}^{-\alpha_{n-1} L}\phi(f_n)\Omega_\beta).$$

Remark

We were not able to prove the existence of the Wightman functions for time-zero fields. Up to now the Wightman functions also have to be smeared out in time.

Outline of proof

At first approximate the time-zero field operators in $\mathcal{M}$, for $h_j \in \mathcal{S}(\mathbb{R})$,

$$\phi_\beta^{(\ell)}(h_j) \to \phi_\beta(\delta \otimes h_j).$$

Then we can directly apply the Hölder inequality:

$$(\Omega_\beta,\, \phi_\beta^{(\ell)}(h_1)\,\mathrm{e}^{-(\alpha_1+it_1)L} \cdots \phi_\beta^{(\ell)}(h_{n-1})\,\mathrm{e}^{-(\alpha_{n-1}+it_{n-1})L}\,\phi_\beta^{(\ell)}(h_n)\,\Omega_\beta)$$

$$\leq |||\phi_\beta^{(\ell)}(h_1)|||_{p_1(\alpha_1)} \cdots |||\phi_\beta^{(\ell)}(h_n)|||_{p_n(\alpha_n)}\,,$$

where p_j is the smallest, positive integer such that $\frac{1}{p_j} \leq \min\{\Re\alpha_{j+1}, \Re\alpha_j\}$. Now there holds the inequality (without proof)

$$|||\phi_\beta(h_j)|||_{p(\alpha_j)} \leq \frac{p(\alpha_j)}{2}\,|h|_{\mathcal{S}},$$

where $|\cdot|_{\mathcal{S}}$ is some Schwarz norm. Polynomial groth is good enough for convergence in the sense of distributions. □

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