Bounded groupoid cocycles

Jean Renault

Université d'Orléans

April 27, 2010

Bounded groupoid cocycles

Jean Renault

Université d'Orléans

April 27, 2010

Introduction

D. Coronel, A. Navas, M. Ponce

[Bounded orbits versus invariant sections for cocycles of affine isometries over a minimal dynamics, *Math arXiv*:**1101.3523v2**, (2011)]

have recently given a generalization of the classical Gottschalk-Hedlund theorem to affine isometric actions on a Hilbert space.

I will show that their result has a natural generalization in the groupoid setting. The main difficulty is to pass from a constant Hilbert bundle to a continuous field of Hilbert spaces.

Introduction

D. Coronel, A. Navas, M. Ponce

[Bounded orbits versus invariant sections for cocycles of affine isometries over a minimal dynamics, *Math arXiv*:**1101.3523v2**, (2011)]

have recently given a generalization of the classical Gottschalk-Hedlund theorem to affine isometric actions on a Hilbert space.

I will show that their result has a natural generalization in the groupoid setting. The main difficulty is to pass from a constant Hilbert bundle to a continuous field of Hilbert spaces.

The Gottschalk-Hedlund theorem

Let us first recall the Gottschalk-Hedlund theorem.

Theorem

Let T be a minimal continuous map on a compact space X. For a continuous function $f : X \to \mathbb{R}$, the following conditions are equivalent:

1. *there exists a continuous function $g : X \to \mathbb{R}$ such that for all $x \in X$, $f(x) = g(x) - g(Tx)$;*
2. *there exists $x_0 \in X$ such that the sums $\sum_{k=0}^{n-1} f(T^k x_0)$ are bounded;*
3. *for all $x \in X$, the sums $\sum_{k=0}^{n-1} f(T^k x)$ are bounded.*

Give a sketch of the proof!

The Gottschalk-Hedlund theorem

Let us first recall the Gottschalk-Hedlund theorem.

Theorem

Let T be a minimal continuous map on a compact space X. For a continuous function $f : X \to \mathbb{R}$, the following conditions are equivalent:

1. *there exists a continuous function $g : X \to \mathbb{R}$ such that for all $x \in X$, $f(x) = g(x) - g(Tx)$;*
2. *there exists $x_0 \in X$ such that the sums $\sum_{k=0}^{n-1} f(T^k x_0)$ are bounded;*
3. *for all $x \in X$, the sums $\sum_{k=0}^{n-1} f(T^k x)$ are bounded.*

Give a sketch of the proof!

The Gottschalk-Hedlund theorem

Let us first recall the Gottschalk-Hedlund theorem.

Theorem

Let T be a minimal continuous map on a compact space X. For a continuous function $f : X \to \mathbb{R}$, the following conditions are equivalent:

1. *there exists a continuous function $g : X \to \mathbb{R}$ such that for all $x \in X$, $f(x) = g(x) - g(Tx)$;*
2. *there exists $x_0 \in X$ such that the sums $\sum_{k=0}^{n-1} f(T^k x_0)$ are bounded;*
3. *for all $x \in X$, the sums $\sum_{k=0}^{n-1} f(T^k x)$ are bounded.*

Give a sketch of the proof!

Groupoids and cocycles

There exist easy generalizations of this theorem. One appeared in my thesis and uses the language of groupoids and cocycles.

In the previous theorem, the function f defines a cocycle on the groupoid

$$G(X,T)=\{(x,m-n,y): x,y\in X, m,n\in\mathbb{N}\quad T^n x = T^m y\}$$

according to

$$c(x,m-n,y)=\sum_{k=0}^{n-1} f(T^k x)-\sum_{k=0}^{m-1} f(T^k x)$$

and it is of the form $\quad f=g-g\circ T\quad$ if and only if c is a coboundary, i.e. of the form $\quad c=g\circ r-g\circ s\quad$, where $r(x,k,y)=x$ and $s(x,k,y)=y$.

Groupoids and cocycles

There exist easy generalizations of this theorem. One appeared in my thesis and uses the language of groupoids and cocycles.

In the previous theorem, the function f defines a cocycle on the groupoid

$$G(X,T) = \{(x, m-n, y) : x, y \in X, m, n \in \mathbb{N} \quad T^n x = T^m y\}$$

according to

$$c(x, m-n, y) = \sum_{k=0}^{n-1} f(T^k x) - \sum_{k=0}^{m-1} f(T^k x)$$

and it is of the form $f = g - g \circ T$ if and only if c is a coboundary, i.e. of the form $c = g \circ r - g \circ s$, where $r(x,k,y) = x$ and $s(x,k,y) = y$.

Groupoids and cocycles

There exist easy generalizations of this theorem. One appeared in my thesis and uses the language of groupoids and cocycles.

In the previous theorem, the function f defines a cocycle on the groupoid

$$G(X,T)=\{(x,m-n,y):x,y\in X,m,n\in\mathbb{N}\quad T^n x=T^m y\}$$

according to

$$c(x,m-n,y)=\sum_{k=0}^{n-1}f(T^k x)-\sum_{k=0}^{m-1}f(T^k x)$$

and it is of the form $\quad f=g-g\circ T\quad$ if and only if c is a coboundary, i.e. of the form $\quad c=g\circ r-g\circ s\quad$, where $r(x,k,y)=x$ and $s(x,k,y)=y$.

Groupoids and cocycles

There exist easy generalizations of this theorem. One appeared in my thesis and uses the language of groupoids and cocycles.

In the previous theorem, the function f defines a cocycle on the groupoid

$$G(X,T) = \{(x, m-n, y) : x, y \in X, m, n \in \mathbb{N} \quad T^n x = T^m y\}$$

according to

$$c(x, m-n, y) = \sum_{k=0}^{n-1} f(T^k x) - \sum_{k=0}^{m-1} f(T^k x)$$

and it is of the form $f = g - g \circ T$ if and only if c is a coboundary, i.e. of the form $c = g \circ r - g \circ s$, where $r(x, k, y) = x$ and $s(x, k, y) = y$.

A groupoid version of the G-H theorem

Theorem (R, 1980)

Let G be a topological groupoid on a compact space X and let A be a topological abelian group endowed with trivial G-action. Assume that G is minimal and that A has no compact subgroups. For a continuous cocycle $c : G \longrightarrow A$, the following conditions are equivalent:

1. *c is a continuous coboundary;*
2. *there exists $x \in X$ such that $c(G_x)$ is relatively compact;*
3. *$c(G)$ is relatively compact.*

The space of coefficients

The above setting is unsatisfactory: the natural data for continuous groupoid cohomology consist of:

- a topological groupoid G over a topological space X,
- a space of coefficients (or G-module) A, which is a continuous bundle of topological abelian groups A_x over X endowed with a continuous G-action, i.e. G acts by isomorphisms $L(\gamma) : A_{s(\gamma))} \rightarrow A_{r(\gamma)}$ and the action map $G * A \rightarrow A$ is continuous.

The above theorem only covers the case when A is a constant bundle endowed with a trivial action.

The space of coefficients

The above setting is unsatisfactory: the natural data for continuous groupoid cohomology consist of:

- a topological groupoid G over a topological space X,
- a space of coefficients (or G-module) A, which is a continuous bundle of topological abelian groups A_x over X endowed with a continuous G-action, i.e. G acts by isomorphisms $L(\gamma) : A_{s(\gamma))} \to A_{r(\gamma)}$ and the action map $G * A \to A$ is continuous.

The above theorem only covers the case when A is a constant bundle endowed with a trivial action.

The space of coefficients

The above setting is unsatisfactory: the natural data for continuous groupoid cohomology consist of:

- a topological groupoid G over a topological space X,
- a space of coefficients (or G-module) A, which is a continuous bundle of topological abelian groups A_x over X endowed with a continuous G-action, i.e. G acts by isomorphisms $L(\gamma) : A_{s(\gamma))} \longrightarrow A_{r(\gamma)}$ and the action map $G * A \longrightarrow A$ is continuous.

The above theorem only covers the case when A is a constant bundle endowed with a trivial action.

The space of coefficients

The above setting is unsatisfactory: the natural data for continuous groupoid cohomology consist of:

- a topological groupoid G over a topological space X,
- a space of coefficients (or G-module) A, which is a continuous bundle of topological abelian groups A_x over X endowed with a continuous G-action, i.e. G acts by isomorphisms $L(\gamma) : A_{s(\gamma))} \to A_{r(\gamma)}$ and the action map $G * A \to A$ is continuous.

The above theorem only covers the case when A is a constant bundle endowed with a trivial action.

Continuous groupoid cohomology

Definition

Let G be a topological groupoid and let A be a continuous G-module. We define $H^1(G,A)$ as the group of isomorphism classes of G-equivariant A-principal bundles over $G^{(0)}$.

- A continuous cocycle is a continuous map $c: G \to A$ such that $c(\gamma) \in A_{r(\gamma)}$ and $c(\gamma\gamma') = c(\gamma) + L(\gamma)c(\gamma')$. It defines the A-principal bundle $A(c) = A$, where G acts on the left by $\gamma z = L(\gamma)z + c(\gamma)$.
- c is a continuous coboundary if and only if $A(c)$ is trivial (equivalently, there exists an equivariant continuous section).
- Replacing G by an equivalent groupoid, each G-equivariant A-principal bundle can be realized by a continuous cocycle.

Continuous groupoid cohomology

Definition

Let G be a topological groupoid and let A be a continuous G-module. We define $H^1(G, A)$ as the group of isomorphism classes of G-equivariant A-principal bundles over $G^{(0)}$.

- A continuous cocycle is a continuous map $c : G \to A$ such that $c(\gamma) \in A_{r(\gamma)}$ and $c(\gamma\gamma') = c(\gamma) + L(\gamma)c(\gamma')$. It defines the A-principal bundle $A(c) = A$, where G acts on the left by $\gamma z = L(\gamma)z + c(\gamma)$.
- c is a continuous coboundary if and only if $A(c)$ is trivial (equivalently, there exists an equivariant continuous section).
- Replacing G by an equivalent groupoid, each G-equivariant A-principal bundle can be realized by a continuous cocycle.

Continuous G-Hilbert modules

We are interested in the case when the space of coefficients is a G-Hilbert module $A = H$.

We denote by

- H the total bundle space;
- $\pi : H \to X$ the projection;
- $\mathcal{H} = C(X, H)$ the space of its continuous sections. Recall that $\mathcal{H}$ is a C*-module over $C(X)$.

Definition

A G-Hilbert module is a continuous field of Hilbert spaces $(H_x)_{x \in X}$ on which G acts by isometries $L(\gamma) : H_{s(\gamma))} \to H_{r(\gamma)}$ and such that the action map $G * H \to H$ is continuous.

Bounded cocycles are coboundaries

Theorem

Let G be a minimal topological groupoid on a compact space X. Let $c : G \to H$ be a continuous cocycle, where H is a G-Hilbert module and let $H(c)$ be the associated affine bundle. Then the following conditions are equivalent:

1. *c is a continuous coboundary;*
2. *$H(c)$ admits an equivariant continuous section;*
3. *$H(c)$ admits a bounded orbit;*
4. *$\|c\|$ is bounded.*

Special cases

- When G is a group, this is a well-known result which goes back to B. Johnson 1967. In fact the result is true for a much larger class of Banach spaces than Hilbert spaces (we still assume that the action is isometric). U. Bader, T. Gelander and N. Monod have recently shown in [A fixed point theorem for L^1 spaces, *Math arXiv*:**1012.1488v1**, (2010)] that it is true for Banach spaces which are L-embedded.
- When G is the groupoid associated with a group action and when H is a constant bundle over X, this is the recent result by D. Coronel, A. Navas, M. Ponce [Bounded orbits versus invariant sections for cocycles of affine isometries over a minimal dynamics, *Math arXiv*:**1101.3523v2**, (2011)] quoted earlier.

Special cases

- When *G is a group*, this is a well-known result which goes back to B. Johnson 1967. In fact the result is true for a much larger class of Banach spaces than Hilbert spaces (we still assume that the action is isometric). U. Bader, T. Gelander and N. Monod have recently shown in [A fixed point theorem for L^1 spaces, *Math arXiv*:**1012.1488v1**, (2010)] that it is true for Banach spaces which are L-embedded.
- When *G is the groupoid associated with a group action* and when *H is a constant bundle* over X, this is the recent result by D. Coronel, A. Navas, M. Ponce [Bounded orbits versus invariant sections for cocycles of affine isometries over a minimal dynamics, *Math arXiv*:**1101.3523v2**, (2011)] quoted earlier.

Idea of the proof

We follow the proof that Coronel, Navas and Ponce give for the infinite-dimensional case. It decomposes into two parts. A compactness argument gives the existence of a weakly continuous equivariant section. Then, using minimality and and a finite dimensional approximation property of C*-modules, one shows that a weakly continuous equivariant section is necessarily continuous.

The weak topology

In what follows, $\pi : H \to X$ is a continuous Hilbert bundle over a compact space X. We denote by $\mathcal{H} = C(X, H)$ the $C(X)$-module of continuous sections.

We define the weak topology on H as follows: we embed H into $X \times \mathcal{H}^*$ via the natural evaluation map. The weak topology is the subspace topology when $\mathcal{H}^*$ is endowed with the $*$-weak topology. We write H_w to specify the weak topology.

One can observe that the space of weakly continuous sections $C(X, H_w)$ agrees with the bounded $C(X)$-linear maps from $C(X, H)$ to $C(X)$, where the section $x \mapsto \xi(x)$ defines the $C(X)$-linear map $\eta \mapsto <\xi, \eta>$, where $<\xi, \eta> (x) = <\xi(x), \eta(x)>_x$.

Existence of a weakly compact invariant convex subset

We assume now that H is a continuous G- Hilbert bundle and that $c : G \longrightarrow H$ is a continuous cocycle. We endow H with the corresponding affine action. Our assumption is that there is a bounded orbit under this action. One deduces the existence of a non-empty weakly compact invariant convex subset. By Zorn, we have the existence of a minimal weakly compact invariant convex subset M.

The set M is the graph of a section

By minimality of G, $\pi(M) = X$. It remains to show that $M_x = M \cap \pi^{-1}(x)$ has exactly one element for all $x \in X$. The proof is a classical trick which uses the uniform convexity of the Hilbert spaces H_x (with a constant uniform convexity constant). Thus, M is the graph of a section. Since M is weakly compact, this section is weakly continuous. Since M is invariant under G, this section is equivariant.

Details of the proof

Let $R = \sup_{\xi \in M} \|\xi\|_{\pi(\xi)}$ and $\epsilon > 0$. Choose $\zeta \in M$ such that $\|\zeta\| > (1 - \delta^2)R$ where $\delta = \delta(\epsilon)$ is the uniform convexity module. Choose $\eta \in H_z$ (where $z = \pi(\zeta)$) such that $\|\eta\| = 1$ and $| < \zeta, \eta > | > (1 - \delta^2)R$. Choose $f \in C(X, H)$ such that $f(z) = \eta$. Let $V = \{y \in X : \|f(y)\| < 1 + \delta\}$.

Let $x \in X$ and $\xi_1, \xi_2 \in M_x$. We are going to show that $\xi_1 = \xi_2$. Let m their midpoint. Then m belongs to M. Let us show that its orbit meets the open set

$$U = \{\xi \in H : \pi(\xi) \in V, \quad | < \xi, f \circ \pi(\xi) > | > (1 - \delta^2)R\}.$$

If not it would be contained in the closed convex set $M \setminus U$ and this would contradict the minimality of M. Let $\gamma \in G$ be such that $\gamma m \in U$. Then $\|\gamma m\| > (1 - \delta)R$. The uniform convexity inequality implies $\|\gamma\xi_1 - \gamma\xi_2\| < \epsilon R$, hence $\|\xi_1 - \xi_2\| < \epsilon R$. Hence $\xi_1 = \xi_2$.

The relative norm function of a section

We are going to show that a weakly continuous equivariant section is necessarily continuous. This will be done by showing that its oscillation is zero. However, how can we define the oscillation of a section f, since its values $f(x)$ and $f(y)$ live in different spaces?

Definition

Let $f : X \to E$ be a section of a Banach bundle $E \to X$. Its relative norm function $N_f : E \to \mathbb{R}_+$ is the scalar function defined by $N_f(e) = \|e - f \circ \pi(e)\|$.

Proposition

Let f be a section of E. Then the following conditions are equivalent

- $f : X \to E$ *is continuous [resp. continuous at x] and*
- $N_f : E \to \mathbb{R}_+$ *is continuous [resp. continuous at $f(x)$].*

The relative norm function of a section

We are going to show that a weakly continuous equivariant section is necessarily continuous. This will be done by showing that its oscillation is zero. However, how can we define the oscillation of a section f, since its values $f(x)$ and $f(y)$ live in different spaces?

Definition

Let $f : X \to E$ be a section of a Banach bundle $E \to X$. Its relative norm function $N_f : E \to \mathbb{R}_+$ is the scalar function defined by $N_f(e) = \|e - f \circ \pi(e)\|$.

Proposition

Let f be a section of E. Then the following conditions are equivalent

- $f : X \to E$ *is continuous [resp. continuous at x] and*
- $N_f : E \to \mathbb{R}_+$ *is continuous [resp. continuous at $f(x)$].*

The relative norm function of a section

We are going to show that a weakly continuous equivariant section is necessarily continuous. This will be done by showing that its oscillation is zero. However, how can we define the oscillation of a section f, since its values $f(x)$ and $f(y)$ live in different spaces?

Definition

Let $f : X \to E$ be a section of a Banach bundle $E \to X$. Its relative norm function $N_f : E \to \mathbb{R}_+$ is the scalar function defined by $N_f(e) = \|e - f \circ \pi(e)\|$.

Proposition

Let f be a section of E. Then the following conditions are equivalent

- $f : X \to E$ *is continuous [resp. continuous at x] and*
- $N_f : E \to \mathbb{R}_+$ *is continuous [resp. continuous at $f(x)$].*

Continuity of equivariant sections

Suppose now that E is a G-Banach bundle.

Proposition

Let f be a section of E. Then the following conditions are equivalent

- $f : X \to E$ *is equivariant;*
- $N_f : E \to \mathbb{R}_+$ *is invariant.*

Thus it suffices to study the continuity of the invariant scalar function N_f. Its oscillation is also invariant; moreover it is upper semi-continuous. One deduces that the set of points of continuity of an equivariant section of a G-Banach bundle is an intersection of open invariant subsets.

Weak continuity ⇒ norm continuity

Since we assume that G is minimal, an intersection of open invariant subsets of X is either the empty set or X itself. Thus in order to show that an equivariant section is continuous, it suffices to show that it has at least one point of continuity. We show now that a weakly continuous section has at least one point of continuity.

Proposition

Let $f : X \to H$ be a weakly continuous section of a separable Hilbert bundle H. Then the set of its points of continuity is a dense G_δ.

Proof. This results from a well-known approximation property (e.g. [D. Blecher, A new approach to C*-modules, 1995]) of C*-module $\mathcal{H}$ over a C*-algebra A.

Weak continuity $\Rightarrow$ norm continuity

Since we assume that G is minimal, an intersection of open invariant subsets of X is either the empty set or X itself. Thus in order to show that an equivariant section is continuous, it suffices to show that it has at least one point of continuity. We show now that a weakly continuous section has at least one point of continuity.

Proposition

Let $f : X \to H$ be a weakly continuous section of a separable Hilbert bundle H. Then the set of its points of continuity is a dense G_δ.

Proof. This results from a well-known approximation property (e.g. [D. Blecher, A new approach to C*-modules, 1995]) of C*-module $\mathcal{H}$ over a C*-algebra A.

Weak continuity ⇒ norm continuity

Since we assume that G is minimal, an intersection of open invariant subsets of X is either the empty set or X itself. Thus in order to show that an equivariant section is continuous, it suffices to show that it has at least one point of continuity. We show now that a weakly continuous section has at least one point of continuity.

Proposition

Let $f : X \to H$ be a weakly continuous section of a separable Hilbert bundle H. Then the set of its points of continuity is a dense G_δ.

Proof. This results from a well-known approximation property (e.g. [D. Blecher, A new approach to C*-modules, 1995]) of C*-module $\mathcal{H}$ over a C*-algebra A.

Factorisation maps

He proves that a Banach module $\mathcal{E}$ over a C*-algebra A is a C*-module iff there exists a directed set I, a net of integers (n_i) and nets of contractive A-linear maps $\varphi_i : \mathcal{E} \to A^{n_i}$ and $\psi_\lambda : A^{n_i} \to \mathcal{E}$ such that for all $e \in \mathcal{E}$, $\psi_i \circ \varphi_i(e)$ tends to e.

We only use the easy part $\Rightarrow$ which is the fact that the C*-algebra of compact operators $\mathcal{K}(\mathcal{H})$ has an approximate unit of the form $e_i = \sum < \xi_k, \xi_k >$. If $\mathcal{H}$ is countably generated, one can choose $I = \mathbb{N}$.

End of the proof

Let f be a weakly continuous section of H. It is the pointwise limit of the sequence of the continuous sections $f_i = \psi_i \circ \varphi_i \circ f$. Indeed $\varphi_i \circ f$ is continuous since the weak and the norm topology agree on the finite-dimensional vector bundle $X \times \mathbb{C}^{n_i}$ and so is f_i. This proves the proposition.

One deduces from the above discussion that the weakly continuous equivariant section we found in the first part is continuous.

Remark. The proof relies of an approximation property which is specific to C*-modules, hence to Hilbert bundles. I do not know if the result is still true for Banach bundles, even in the constant bundle case.

End of the proof

Let f be a weakly continuous section of H. It is the pointwise limit of the sequence of the continuous sections $f_i = \psi_i \circ \varphi_i \circ f$. Indeed $\varphi_i \circ f$ is continuous since the weak and the norm topology agree on the finite-dimensional vector bundle $X \times \mathbb{C}^{n_i}$ and so is f_i. This proves the proposition.

One deduces from the above discussion that the weakly continuous equivariant section we found in the first part is continuous.

Remark. The proof relies of an approximation property which is specific to C*-modules, hence to Hilbert bundles. I do not know if the result is still true for Banach bundles, even in the constant bundle case.

End of the proof

Let f be a weakly continuous section of H. It is the pointwise limit of the sequence of the continuous sections $f_i = \psi_i \circ \varphi_i \circ f$. Indeed $\varphi_i \circ f$ is continuous since the weak and the norm topology agree on the finite-dimensional vector bundle $X \times \mathbb{C}^{n_i}$ and so is f_i. This proves the proposition.

One deduces from the above discussion that the weakly continuous equivariant section we found in the first part is continuous.

Remark. The proof relies of an approximation property which is specific to C*-modules, hence to Hilbert bundles. I do not know if the result is still true for Banach bundles, even in the constant bundle case.

Some questions

1) The result should be valid for more general Banach bundles. However our proof only works for Hilbert bundles.

2) It seems reasonable to define a property (FH) (existence of a continuous equivariant section for isometric affine actions on Hilbert bundles) for topological groupoids extending the classical notion for groups. Such a property has been defined and studied by C. Anantharam-Delaroche for ergodic measured groupoids [Cohomology of property T groupoids and applications, *Ergod. Th. & Dynam. Sys*, **25** (2005), 465–471].

Some questions

1) The result should be valid for more general Banach bundles. However our proof only works for Hilbert bundles.

2) It seems reasonable to define a property (FH) (existence of a continuous equivariant section for isometric affine actions on Hilbert bundles) for topological groupoids extending the classical notion for groups. Such a property has been defined and studied by C. Anantharam-Delaroche for ergodic measured groupoids [Cohomology of property T groupoids and applications, *Ergod. Th. & Dynam. Sys*, **25** (2005), 465–471].