On nimrep graphs associated to $SU(3)$ modular invariant partition functions

Mathew Pugh (Cardiff University)

Joint work with David E. Evans:

Ocneanu Cells for the $SU(3)$ $\mathcal{ADE}$ Graphs, Münster J. Math. **2** (2009), 95–142.
Realisation of $SU(3)$ modular invariants, Rev. Math. Phys. **21** (2009), 877–928.
A_2-planar algebras I, Quantum Topol., **1** (2010), 321-377.
A_2-planar algebras II: Planar modules. arXiv:0906.4314.
Spectral measures for nimrep graphs, Comm. Math. Phys. **295** (2010), 363–413.
Spec. measures for nimreps II: $SU(3)$, Comm. Math. Phys. **301** (2011), 771–809.
Nakayama automorphism of almost CY algebras. arXiv:1008.1003.

29 April, 2011

Overview

- $SU(3)$ $\mathcal{ADE}$ graphs as nimreps

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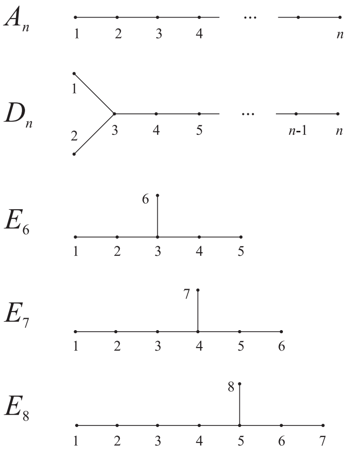
- Almost Calabi-Yau algebras and the Nakayama automorphism

ADE Graphs



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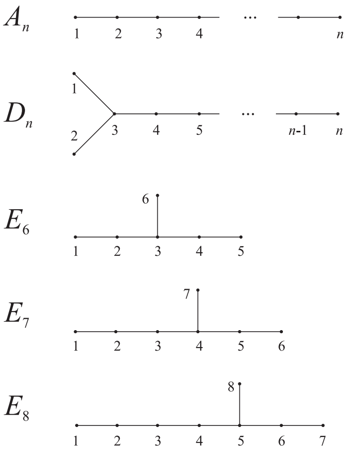

- Semisimple Lie algebras

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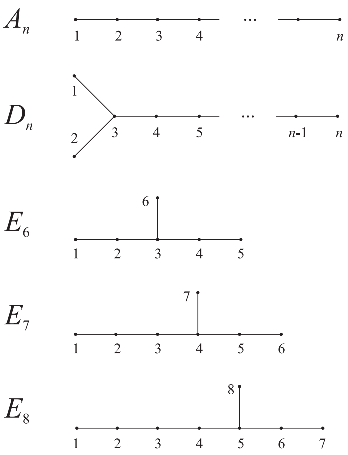

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- Non-negative integer matrices (norm < 2)

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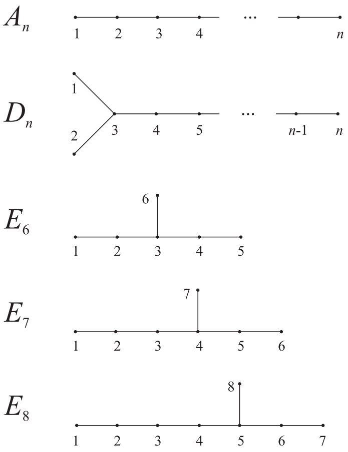

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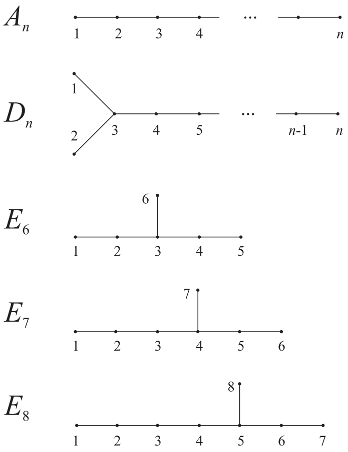

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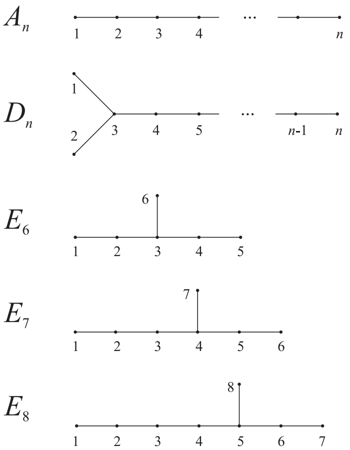

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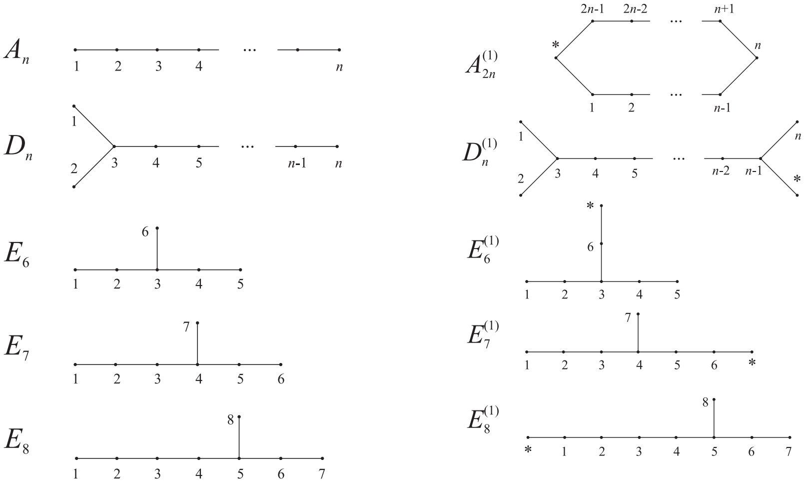
- Realisation of $SU(2)$ modular invariants by braided subfactors

ADE Graphs and Affine *ADE* Graphs



Verlinde algebra of $SU(n)$

Type III_1 factor N

Braided system ${}_N\mathcal{X}_N$ of endomorphisms

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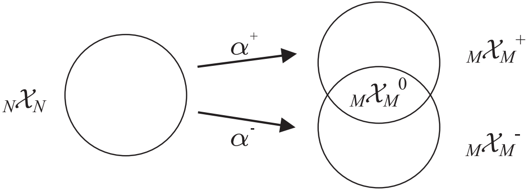
Verlinde formula

$$N_\lambda = \sum_\sigma \frac{S_{\sigma\lambda}}{S_{\sigma 1}} S_\sigma S_\sigma^*, \qquad S_\sigma = \{S_{\sigma\mu}\}_\mu$$

α-induction

Braided subfactor $N \subset M$



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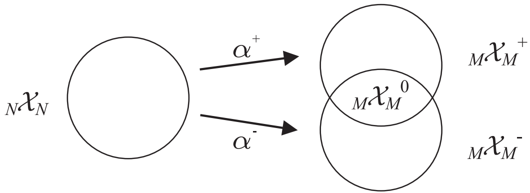

$$\alpha^{\pm} : \qquad \lambda \qquad \longrightarrow \qquad \alpha^{\pm}_{\lambda}$$

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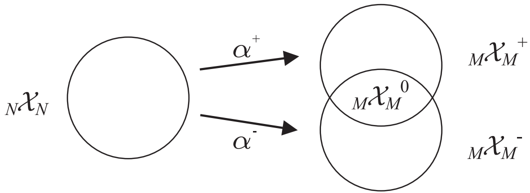

$$\alpha^\pm : \quad \lambda \quad \longrightarrow \quad \alpha^\pm_\lambda$$

$Z_{\lambda,\mu} = \langle \alpha^+_\lambda, \alpha^-_\mu \rangle$ is a modular invariant

Bockenhauer-Evans-Kawahigashi

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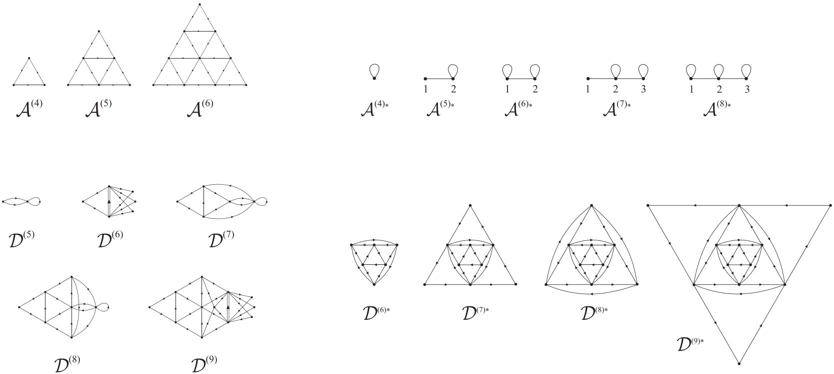
$SU(2)$: nimrep gives classification of modular invariants at level k:

Capelli-Itzykson-Zuber

For $N \subset M \to Z_{\mathcal{G}}$ ($\mathcal{G}$ an ADE graph)

$$G_\rho = \mathcal{G}$$

$SU(3)$ $\mathcal{ADE}$ Graphs



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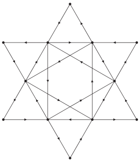
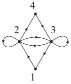
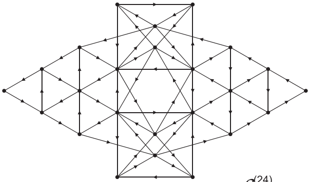


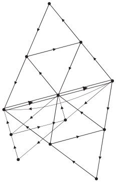
$\mathcal{E}^{(8)}$



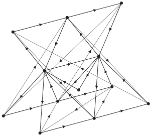
$\mathcal{E}^{(8)*}$



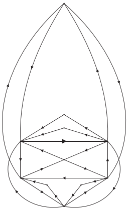
$\mathcal{E}^{(24)}$



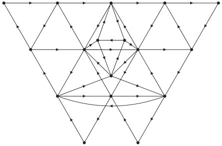
$\mathcal{E}_1^{(12)}$



$\mathcal{E}_2^{(12)}$



$\mathcal{E}_4^{(12)}$



$\mathcal{E}_5^{(12)}$

Subgroups of $SU(3)$

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$\mathcal{ADE}$ graph	Subgroup $\Gamma \subset SU(3)$
(ADE)	B: finite subgroups of $SU(2) \subset SU(3)$
$\mathcal{A}^{(n)}$	A: $\mathbb{Z}_{n-2} \times \mathbb{Z}_{n-2}$
-	A: $\mathbb{Z}_m \times \mathbb{Z}_n \quad (m \neq n \neq 3)$
$\mathcal{D}^{(n)} \quad (n \equiv 0 \bmod 3)$	C: $\Delta(3(n-3)^2) = (\mathbb{Z}_{n-3} \times \mathbb{Z}_{n-3}) \rtimes \mathbb{Z}_3$
$\mathcal{D}^{(n)} \quad (n \not\equiv 0 \bmod 3)$	-
-	C: $\Delta(3n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$, $(n \not\equiv 0 \bmod 3)$
-	D: $\Delta(6n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$
$\mathcal{A}^{(n)*}$	-
$\mathcal{D}^{(n)*} \quad (n \geq 7)$	A: $\mathbb{Z}_{\lfloor (n+1)/2 \rfloor} \times \mathbb{Z}_3$
$\mathcal{E}^{(8)}$	E $= \Sigma(36 \times 3) = \Delta(3.3^2) \rtimes \mathbb{Z}_4$
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$\mathcal{E}_1^{(12)}$	F $= \Sigma(72 \times 3)$
$\mathcal{E}_2^{(12)}$	G $= \Sigma(216 \times 3)$
$\mathcal{E}_3^{(12)}$	B $\times \mathbb{Z}_3$: $BD_4 \times \mathbb{Z}_3$
$\mathcal{E}_4^{(12)}$	L $= \Sigma(360 \times 3) \cong TA_6$
$\mathcal{E}_5^{(12)}$	K $\cong TPSL(2,7)$
$\mathcal{E}^{(24)}$	-
-	H $= \Sigma(60) \cong A_5$
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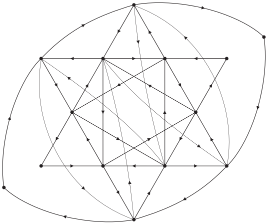

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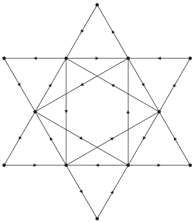

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A_2-Temperley-Lieb algebra: $U_i \in (\bigotimes_{\mathbb{N}} M_3)^{SU(3)_k}$

$$U_i^2 = \delta U_i, \qquad U_i U_j = U_j U_i, \ \ |i-j| > 1,$$

$$U_i U_{i\pm1} U_i - U_i = U_{i\pm1} U_i U_{i\pm1} - U_{i\pm1}$$

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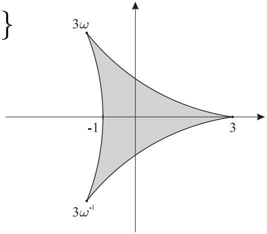

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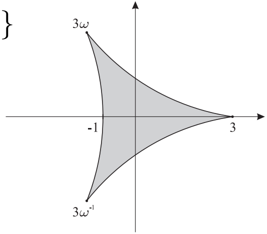

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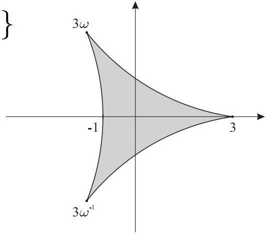

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Fundamental domain C of $\mathbb{T}^2/S_3$ in $\mathbb{T}^2$:

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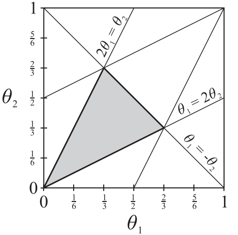

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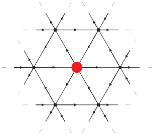

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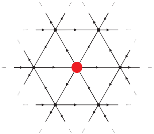

$$\dim(\otimes^k M_3)^{SU(3)} = \langle |\Delta|^{2k}\Omega, \Omega\rangle$$
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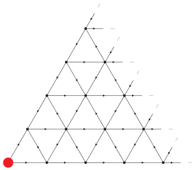

$$\Delta = S \otimes 1 + 1 \otimes S^* + S^* \otimes S \text{ on } \ell^2(\mathbb{N} \times \mathbb{N})$$

Graph $\mathcal{A}^{(h)}$

Jacobian $J := \det(\partial(x,y)/\partial(\theta_1,\theta_2))$ for change of variables

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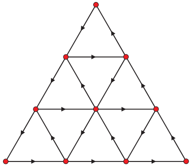

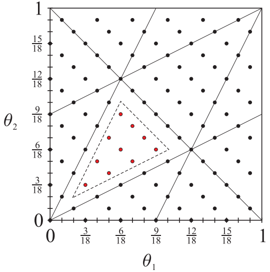

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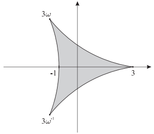

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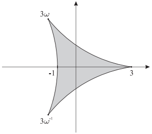

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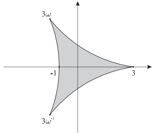

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$$\omega_1^3 - z\omega_1^2 + \overline{z}\omega_1 - 1 \quad = \quad 0$$
$$\omega_2^3 - \overline{z}\omega_2^2 + z\omega_2 - 1 \quad = \quad 0$$

$$\omega = z - P + (z^3 - 3\overline{z})/3P$$
$$P = ([27 - 9z\overline{z} + 2z^3 + 3\sqrt{3}\sqrt{27 - 18|z|^2 + 4z^3 + 4\overline{z}^3 - |z|^4}]/2)^{1/3}$$

$$z \rightarrow (\omega, \overline{\omega'})$$

Overview

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$\mathcal{V}_{s_1,s_2}$, vector space with basis given by A_2-planar tangles

generated by 

subject to the Kuperberg relations: $\delta \in \mathbb{R}, \quad \alpha = \delta^2 - 1$

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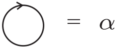
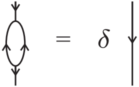


K2:



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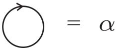

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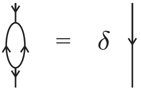

K3:

$\{e_1, e_2, e_3\}$ basis of $\mathbb{C}^3$

 $\in \mathrm{Hom}(\overline{\rho} \otimes \rho, \mathbb{C})$:
$\overline{e_i} \otimes e_j \rightarrow \delta_{ij}$

 $\in \mathrm{Hom}(\mathbb{C}, \overline{\rho} \otimes \rho)$:
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 $= \alpha$

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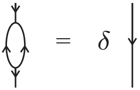

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A_2-planar algebras

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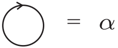

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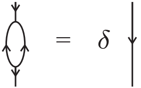


K3:

A_2-planar algebras

 $\longrightarrow U_i \in (\bigotimes_{\mathbb{N}} M_3)^{SU(3)_k}$, A_2-Temperley-Lieb operator:

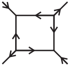
K1:

 $= \alpha$

K2:

 $= \delta$ 

K3:

 $=$  $+$ 

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$$U_i^2 = \delta U_i, \qquad U_i U_j = U_j U_i, \;\; |i-j| > 1,$$

$$U_i U_{i\pm1} U_i - U_i = U_{i\pm1} U_i U_{i\pm1} - U_{i\pm1}$$

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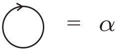
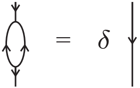


K2:



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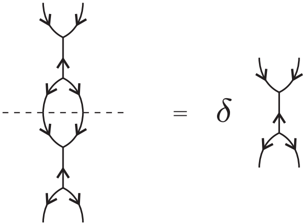

From A_2-TL to almost Calabi-Yau algebras

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- $F(f^m_{(1,0)}) = \otimes^m(\mathbb{C}\mathcal{G})_1 = (\mathbb{C}\mathcal{G})_m$

Define graded algebra:
$\bigoplus_{k=0}^{\infty} F(f_{(k,0)}) \cong \mathbb{C}\mathcal{G}/\{\sum_{b,b'} W(\triangle^{(a,b,b')})bb'\} = A(\mathcal{G}, W)$

Cooper

Almost Calabi-Yau algebras

Hilbert series of graded algebra: $H(t) = \sum H_n t^n$

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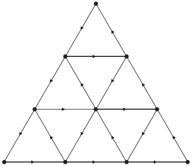
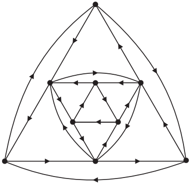
Nakayama automorphism for $A(\mathcal{G}, W)$

γ: automorphism of graph given by clockwise rotation by $2\pi/3$ ($\gamma^3 = \mathrm{id}$)

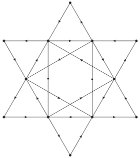
$$\beta = \begin{cases} \gamma^2 & \text{for} \quad \mathcal{A}^{(n)},\ n \geq 4, \\ \gamma^{2n} & \text{for} \quad \mathcal{D}^{(n)*},\ n \geq 5, \\ \gamma & \text{for} \quad \mathcal{E}^{(8)}, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$



$\mathcal{A}^{(n)}$



$\mathcal{D}^{(n)*}$



$\mathcal{E}^{(8)}$

Hochschild (co)homology and Cyclic homology

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- Hochschild (co)homology and cyclic homology of $A \rightsquigarrow$ invariants for subfactor $N \subset M$ realised by pair $(\mathcal{G}, W)$

T. Banica and D. Bisch, Spectral measures of small index principal graphs, Comm. Math. Phys. **269** (2007), 259281.

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