

I. II. III. IV.

Non-commutative functions and some of their applications in free probability

Mihai Popa

Non-commutative functions

- J. L. Taylor (Adv. in Math. '72)
- D-V. Voiculescu (Astérisque '95, also 2008, '09) (~fully matricial functions)
- V. Vinnikov, D.S. Kaliuzhnyi-Verbovetskyi, M. P. S. Belinschi (2009 - '11)
- M. Aguiar (2011), M. Anshelevich (2011)

\mathcal{V} = vector space over \mathbb{C} ;

- the *non-commutative space* over \mathcal{V} : $\mathcal{V}_{\text{nc}} = \coprod_{n=1}^{\infty} M_n(\mathcal{V})$
- *noncommutative sets*: $\Omega \subseteq \mathcal{V}_{\text{nc}}$ such that $X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$
for all $X \in \Omega_n, Y \in \Omega_m$, where $\Omega_n = \Omega \cap \mathcal{V}^{n \times n}$
- *upper admissible sets*: $\Omega \subseteq \mathcal{V}_{\text{nc}}$ such that for all $X \in \Omega_n, Y \in \Omega_m$
and all $Z \in \mathcal{V}^{n \times m}$, there exists $\lambda \in \mathbb{C}, \lambda \neq 0$, with

$$\begin{bmatrix} X & \lambda Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Examples of upper-admissible sets:

- $\Omega = \text{Nilp } \mathcal{V} =$ the set of nilpotent matrices over \mathcal{V}
- If \mathcal{V} is a Banach space and Ω is open in the sense that $\Omega_n \subseteq \mathcal{V}^{n \times n}$ is open for all n , then Ω is upper admissible.

- $\mathcal{B} =$ unital C^* -algebra, $\mathcal{A} = *$ -algebra containing \mathcal{B} ;
 $X \in \mathcal{A}$,

$$\rho_n(X; \mathcal{B}) = \{\beta \in M_n(\mathcal{B}) : 1_n \otimes X - \beta \text{ invertible}\}$$

$$\rho(X; \mathcal{B}) = \coprod_{n=1}^{\infty} \rho_n(X; \mathcal{B}) \text{ is upper admissible}$$

- Noncommutative half-planes over \mathcal{A} :

$$\mathbb{H}^+(\mathcal{A}_{\text{nc}}) = \{a \in \mathcal{A}_{\text{nc}} : \Im a > 0\}$$

$$\mathbb{H}^-(\mathcal{A}_{\text{nc}}) = \{a \in \mathcal{A}_{\text{nc}} : \Im a < 0\}$$

$\Omega \subseteq \mathcal{V}_{nc}$ = non-commutative (upper admissible) set

Noncommutative function:

$f: \Omega \rightarrow \mathcal{W}_{nc}$ such that

- $f(\Omega_n) \subseteq M_n(\mathcal{W})$
- f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$ for all $X \in \Omega_n$, $Y \in \Omega_m$.
- f respects similarities: $f(TXT^{-1}) = Tf(X)T^{-1}$ for all $X \in \Omega_n$ and $T \in GL_n(\mathbb{C})$ such that $TXT^{-1} \in \Omega_n$.

Examples of nc-functions:

- **non-commutative polynomials**

$\mathcal{V} = \mathcal{R}^m$, $\mathcal{W} = \mathcal{N}$, where \mathcal{N} = module over a ring \mathcal{R}

$$f(X_1, \dots, X_m) = X_1 X_3 - X_3 X_1 + b_1 X_2 X_4 b_2 X_5$$

N.b.: A nc polynomial is determined uniquely by this type of nc function

- \mathbb{K} =field of characteristic zero

$$p_n = \sum_{\pi \in S_{n+1}} \text{sign}(\pi) X_1^{\pi(1)-1} X_2 \cdots X_1^{\pi(2)-1} X_2$$

$f : \Omega \in \mathbb{K}_{\text{nc}}^2 \rightarrow \mathbb{K}_{\text{nc}}$, given by

$$f(x_1, x_2) = \sum_{k=1}^{\infty} p_k(x_1, x_2).$$

For $(x_1, x_2) \in M_n(\mathbb{K}^2)$, the terms $p_k(x_1, x_2)$, $k \geq n$ all vanish.

All n -dimensional components of f are polynomials, but f is **not** a nc polynomial.

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Examples of nc-functions:

- free holomorphic functions (G. Popescu)

$$f(X_1, \dots, X_n) = \sum_{m=1}^{\infty} \sum_{\mathbf{i}=(i_1, \dots, i_m)} A_{\mathbf{i}} X_{i_1} \cdots X_{i_m}$$

where X_{i_1}, \dots, X_{i_n} are free elements in some operator algebra

- the generalized moment series of $\mu \in \Sigma_{\mathcal{B}: \mathcal{D}}$

$\phi : \mathcal{A} \rightarrow \mathcal{D}$ cp \mathcal{B} -bimodule map

$\tilde{\mu}((\mathbb{1} - \mathcal{X}b)^{-1}) = M_{\mu} = (M_{n, \mu})_n$, where $\tilde{\mu} = (1_n \otimes \mu)_n$ id the fully matricial extension of μ .

$$M_{n, \mu}(b) = \sum_{k=0}^{\infty} (1_n \otimes \mu)([\mathcal{X} \cdot b]^k),$$

M_X **indeed** encodes all the moments of X , not only the symmetric ones, since for

$$b = \begin{bmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in M_{n+1}(\mathcal{B})$$

we have that

$$(1_{N+1} \otimes \phi)([X \cdot b]^n) = \begin{bmatrix} 0 & \dots & 0 & \phi(Xb_1 \cdots Xb_n) \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Examples of nc-functions:

- the full \mathcal{B} -resolvent of X

\mathcal{B} = unital C^* -algebra, \mathcal{A} =*-algebra containing \mathcal{B} ;
 $X \in \mathcal{A}$,

$$\rho_n(X : \mathcal{B}) = \{\beta \in M_n(\mathcal{B}) : 1_n \otimes X - \beta \text{ invertible}\}$$

on each $\rho(1_n \otimes X; M_n(\mathcal{B}))$ consider the analytic function
 $\beta \mapsto (1_n \otimes X - \beta)^{-1}$

- the generalized Cauchy transform of X : \mathcal{G}_X

$\phi : \mathcal{A} \rightarrow \mathcal{D}$ cp \mathcal{B} -bimodule map

$\mathcal{G}_X = (G_X^{(n)})_n$, where

$$G_X^{(n)} : \mathbb{H}^+(M_n(\mathcal{B})) \ni b \mapsto G_X^{(n)}(b) = \phi_n[(b - X \otimes 1_n)^{-1}] \in \mathbb{H}^-(M_n(\mathcal{D}))$$

Difference-differential calculus

Nc functions admit a nice differential calculus. The difference-differential operators can be calculated directly by evaluation on block-triangular matrices.

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} f(X) & \Delta_R f(X, Y)(Z) \\ 0 & f(Y) \end{bmatrix}$$

The operator $Z \mapsto \Delta_R f(X, Y)(Z)$ is linear and

$$f(Y) = f(X) + \Delta_R(X, Y)(X - Y)$$

Particularly, if $\mathcal{B} \subset \mathcal{A}$, and $X \in \mathcal{A}$, a QD-bialgebra structure is induced by

$$\delta_X : \mathcal{B}\langle X \rangle \otimes \mathcal{B}\langle X \rangle \longrightarrow \mathcal{B}\langle X \rangle$$

where $\delta_{X|\mathcal{B}} = 0$ and $\delta_X(X) = 1 \otimes 1$.

For $f \in \mathcal{B}\langle X \rangle$, if $\delta_X(f) = \sum f_1 \otimes f_2$, with $f_1, f_2 \in \mathcal{B}\langle X \rangle$, then

$$\Delta_R f(X, X)(b) = \sum f_1 \cdot b \cdot f_2.$$

The relations for higher order derivatives is similar.

The Taylor-Taylor expansion:

If $f : \Omega \rightarrow \mathcal{W}_{nc}$ is a non-commutative function, Ω =upper-admissible set, $X \in \Omega_n$. Then for each N and $X \in \Omega_n$ we have that

$$f(Y) = \sum_{k=0}^N \Delta_R^k f(\underbrace{X, \dots, X}_{k+1 \text{ times}}) \underbrace{(X - Y, \dots, X - Y)}_{k \text{ times}} \\ + \Delta_R^{N+1} f(\underbrace{X, \dots, X, Y}_{N+1 \text{ times}}) \underbrace{(X - Y, \dots, X - Y)}_{N+1 \text{ times}}$$

XY

Moreover, if $0 \in \Omega$ then for $X \in \Omega$ we have

$$f(X) = \sum_{k=0}^{\infty} \Delta_R^k \widetilde{f}(\underbrace{0, \dots, 0}_{k+1}) \underbrace{(X, \dots, X)}_k$$

where $\widetilde{\Delta}_R^k f(\underbrace{0, \dots, 0}_{k+1})$ are the fully matricial extension of the multilinear maps $\Delta_R^k f(\underbrace{0, \dots, 0}_{k+1}) : \mathcal{V}^k \rightarrow \mathcal{W}$.

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logo

$$\begin{aligned}
 & f \left(\begin{bmatrix} X & Z_1 & 0 & \cdots & 0 \\ 0 & X & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & X & Z_k \\ 0 & \cdots & \cdots & 0 & Y \end{bmatrix} \right) \\
 = & \begin{bmatrix} f(X) & \Delta_R f(X, X)(Z_1) & \cdots & \cdots & \Delta_R^k f(X, \dots, X, Y)(Z_1, \dots, Z_k) \\ 0 & f(X) & \ddots & & \Delta_R^{k-1} f(X, \dots, X, Y)(Z_2, \dots, Z_k) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & f(X) & \Delta_R f(X, Y)(Z_k) \\ 0 & \cdots & \cdots & 0 & f(Y) \end{bmatrix} .
 \end{aligned}$$

\mathcal{B} = unital C^* -algebra

$\mathcal{B} \subset \mathcal{A}$ inclusion of unital C^* ($*$)-algebras

$\phi : \mathcal{A} \rightarrow \mathcal{B}$ positive conditional expectation

$X = X^* \in \mathcal{A}$

$$\phi(Xb_1Xb_2X) \neq \phi(X^3)b_1b_2$$

moments = multilinear maps

$$\mathcal{B}^n \ni (b_1, \dots, b_n) \mapsto \phi(Xb_1 \cdots Xb_nX)$$

“Distributions”

$\mathcal{B}\langle \mathcal{X} \rangle$ = the $*$ -algebra of non-commutative polynomials in the self-adjoint variable \mathcal{X} and with coefficients in \mathcal{B}

$\Sigma_{\mathcal{B}} = \{ \nu : \mathcal{B}\langle \mathcal{X} \rangle \rightarrow \mathcal{B} : \text{positive conditional expectation} \}$

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Example:

Free Central Limit Thm.: \mathcal{B} -Semicircular Law (D-V. Voiculescu, '95, R. Speicher '97):

$b \mapsto \eta(b) = \text{cp map}$

$s(\mathcal{X}b_1\mathcal{X}\cdots b_{2n}\mathcal{X}) = 0$

$$s(\mathcal{X}b_1\mathcal{X}b_2\cdots\mathcal{X}b_5\mathcal{X}) = \eta(b_1)b_2\eta(b_3)b_4\eta(b_5) + \eta(b_1)b_2\eta(b_3\eta(b_4)b_5) + \\ \eta(b_1\eta(b_2)b_3)b_4\eta(b_5) + \eta(b_1\eta(b_2)b_3\eta(b_4)b_5) + \eta(b_1\eta(b_2\eta(b_3)b_4)b_5)$$

easy using non-crossing pair partitions



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Free Independence: ($\mathcal{A}_1, \mathcal{A}_2 =$ subalgebras of \mathcal{A})

$$\phi(a_1 a_2 \cdots a_n) = 0$$

whenever $a_i \in \mathcal{A}_{k(i)}$ with $\phi(a_k) = 0$ and $a_k \in \mathcal{A}_{\varepsilon(k)}$ with $k(i) \neq k(i+1)$ and $\varphi(a_k) = 0$.

$$\boxplus : B\langle \mathcal{X} \rangle \times B\langle \mathcal{X} \rangle \longrightarrow B\langle \mathcal{X} \rangle$$

given by: X, Y free over B , then $\mu_{X+Y} = \mu_X \boxplus \mu_Y$

No literature about infinite divisibility in Σ_B except R. Speicher,
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Infinite divisibility wrt classical (additive) convolution: back to Kolmogorov, Hincin, P. Levy

Levy-Hincin formula:

$$\mathcal{F}_\nu(t) = \exp \left[i\alpha t + \int (e^{itx} - 1 - itx) \frac{x^2+1}{x^2} d\rho(x) \right]$$

Infinite divisibility wrt free additive convolution: H. Bercovici, D-V. Voiculescu ('92, '93), H. Bercovici, V. Pata ('95, '99)

Free Levy-Hincin formula:

$$\Phi_\nu(z) = \gamma + \int \frac{1+tz}{z-t} d\sigma(t) \quad \text{for } \Phi_\nu(z) = \left(\frac{1}{G_\mu(z)} \right)^{\langle -1 \rangle} - z$$

or, equivalently

$$\frac{1}{z} R_\nu(z) = \alpha + \int_{\mathbb{R}} \frac{z}{1-tz} d\rho(t) \quad \text{where } G_{\mu_X} \circ \left[\frac{1}{z} R_X(z) + \frac{1}{z} \right] = z$$

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Bercovici - Pata bijection

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Bercovici - Pata bijection

The non-commutative R -transform of $\nu \in \Sigma_{\mathcal{B}}$:

$$M_{\nu}(b) - \mathbf{1} = R_{\nu}(bM_{\nu}(b))$$

The equation is always meaningful for $b \in \text{Nilp}(\mathcal{B})$;
if $\mathcal{A} = C^*$ -algebra, then also for b in a non-commutative ball around 0.

Properties:

- R_{ν} is a non-commutative function
- $R_{\mu \boxplus \nu} = R_{\mu} + R_{\nu}$
- The op-valued semicircular law is characterized by

$$R_s(b) = \eta(b)b$$

for some cp map $\eta : \mathcal{B} \rightarrow \mathcal{B}$. ●

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Theorem (V.Vinnikov, M.P., '10)

$\mu \in \Sigma_{\mathcal{B}}$ is free infinitely divisible if and only if there exist some selfadjoint $\alpha \in \mathcal{B}$ and some completely positive (\mathbb{C} -)linear map $\nu : \mathcal{B}\langle \mathcal{X} \rangle \rightarrow \mathcal{B}$ such that

$$R_{\mu}(b) = [\alpha \cdot \mathbf{1} + \tilde{\nu}(b(\mathbf{1} - \mathcal{X}b)^{-1})] b.$$

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The case of c. p. maps

C-freeness:

$\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ subalgebras containing \mathcal{B} .

$\Phi : \mathcal{A} \rightarrow \mathcal{D}$ a \mathcal{B} -bimodule map, $\psi : \mathcal{A} \rightarrow \mathcal{B}$ conditional expectation

$\mathcal{A}_1, \mathcal{A}_2$ are c-free if:

- $\mathcal{A}_1, \mathcal{A}_2$ are free w.r.t. ψ
- $\Phi(a_1 \cdots a_n) = \Phi(a_1) \cdots \Phi(a_n)$ whenever $\psi(a_j) = 0$ and $a_j \in \mathcal{A}_{\epsilon(j)}$ with $\epsilon(j) \neq \epsilon(j+1)$

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b

- F. Boca (JFA, '91) - "amalgamated free product of cp maps"
- R. Speicher, M. Bozejko (Pac. J. Math, '96) - scalar case, "c-freeness", cR -transform, limit laws
- K. Dykema, E. Blanchard (Pac. J. Math., '01) - reduced free products and embeddings of free products of von Neumann algebras
- M. P., J-C Wang ('08, to appear in Trans. AMS) - multiplicative properties, c-free S -transform
- op-valued cR -transform: W. Mlotkowski ('03), M.P('08)

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$\Sigma_{\mathcal{B}:\mathcal{D}} = \{ \mu : \mathcal{B}\langle \mathcal{X} \rangle \rightarrow \mathcal{D} \text{ } \mathcal{B}\text{-bimodule maps, c.p.} \}$

$\Sigma_{\mathcal{B}} = \{ \nu : \mu : \mathcal{B}\langle \mathcal{X} \rangle \rightarrow \mathcal{B} \text{ positive conditional expectations} \}$

$\mathcal{A} \ni X = X^* \leftrightarrow (\mu_X, \nu_X) \in \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$

c-freeness induces

$\boxed{\mathbf{C}}$: $(\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}) \times (\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}) \rightarrow \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$

$((\mu_X, \nu_X), (\mu_Y, \nu_Y)) \mapsto (\mu_{X+Y}, \nu_{X+Y}), \text{ where } X, Y = \text{c-free}$

Op-valued Boolean independence:

$\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ are Boolean independent if

$$\theta(a_1 a_2 \cdots a_n) = \theta(a_1) \cdots \theta(a_n) \text{ whenever } a_i \in \mathcal{A}_{\epsilon(i)} \text{ with } \epsilon(i) \neq \epsilon(i+1).$$

$$\uplus : \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}:\mathcal{D}} \longrightarrow \Sigma_{\mathcal{B}:\mathcal{D}}$$

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\uplus is linearized by $\mu \longrightarrow B_\mu$,

$$B_\mu(b) \cdot M_\mu(b) = M_\mu(b) - \mathbb{1}$$

Op-valued Bernoulli law is given by $B_{\text{Ber}} = \eta(b)b$;

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Theorem (M.P. 10):

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- Any $\mu \in \Sigma_{\mathcal{B}:\mathcal{D}}$ is infinitely divisible with respect to boolean convolution.
- For any $\mu \in \Sigma_{\mathcal{B}:\mathcal{D}}$, there exists a selfadjoint $\alpha \in \mathcal{D}$ and a cp \mathbb{C} -linear map $\sigma : \mathcal{B}\langle \mathcal{X} \rangle \rightarrow \mathcal{D}$ such that

$$B_\mu(b) = [\alpha \cdot \mathbb{1} + \tilde{\sigma}(b(\mathbb{1} - \mathcal{X}b)^{-1})] \cdot b. \quad (1)$$

BP

logos

Non-commutative cR -transform for $(\mu, \nu) \in \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$:

$$(M_{\mu}(b) - \mathbb{1}) \cdot {}^cR_{\mu,\nu}(bM_{\nu}(b)) = (M_{\mu}(b) - \mathbb{1}) \cdot M_{\mu}(b)$$

- cR is a non-commutative function
- if $X, Y \in \mathcal{A}$ are c -free w.r.t. (Φ, ψ) , then ${}^cR_{X+Y} = {}^cR_X + {}^cR_Y$

Theorem(M.P., V. Vinnikov '10)

A pair $(\mu, \nu) \in \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$ is c -free infinitely divisible iff ν is free infinitely divisible and there exist some selfadjoint $\alpha \in \mathcal{B}$ and some completely positive (\mathbb{C} -)linear map $\tilde{\nu} : \mathcal{B}\langle \mathcal{X} \rangle \rightarrow \mathcal{D}$ such that

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r bp

The Non-commutative Boolean-to-Free Bercovici-Pata bijection:

$\mathcal{BP} : \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}} \rightarrow \boxed{\mathbb{C}}$ -infinitely divisible elements of $\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$

(μ, ν)

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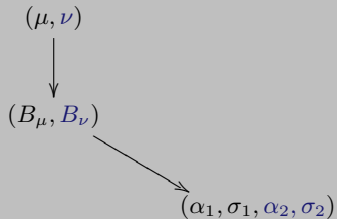
$$\begin{array}{c} (\mu, \nu) \\ \downarrow \\ (B_\mu, B_\nu) \end{array}$$

B-transf

logo

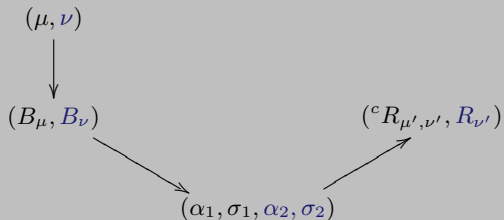
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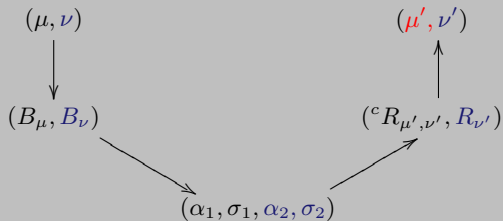


R

logo

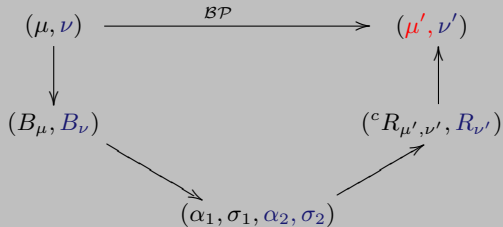
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Idea: μ determines $\rho_\mu : \mathcal{B}\langle \mathcal{X} \rangle \rightarrow \mathcal{B}$ via the moment-free cumulant recurrence relations. On $\mathcal{B}\langle \mathcal{X} \rangle_0 = \mathcal{B}\langle \mathcal{X} \rangle \setminus \mathcal{B}$ consider the positive pairing $\langle f(\mathcal{X}), g(\mathcal{X}) \rangle = \rho_\mu(g(\mathcal{X})^* f(\mathcal{X}))$ and the self-adjoint operator $T : f(\mathcal{X}) \mapsto \mathcal{X}f(\mathcal{X})$.

Then consider the self-adjoint operator V on the full Fock \mathcal{B} -bimodule over $\mathcal{B}\langle \mathcal{X} \rangle_0$, which \mathcal{B} -valued distribution with respect to the ground \mathcal{B} -state coincides with μ , given by

$$V = a_X + a_X^* + \tilde{T} + \mu(X)\text{Id}.$$

The terms from the Taylor-Taylor development of \tilde{R}_μ are the moments of \tilde{T} with respect to the mapping $\langle \cdot, \mathcal{X} \rangle$ and the conclusion follows using the additivity property of \tilde{R} and some cb-norm inequalities. 1

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BP