Semiclassical Einstein equations and non-commutative spacetimes

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Motivations

- At short distance the spacetime should be **non-commutative**.
- This feature should be encoded in the **“Quantum Gravity”**

No satisfactory description.

- We can get information about such a theory analyzing **some particular regimes** *[Hawking]*.
- Gravity classically Matter by quantum theory.

$$G_{ab} = 8\pi\langle T_{ab}\rangle_\omega$$

- *Doplicher, Fredenhagen and Roberts 95* use this to obtain **uncertainty relations** for the coordinates on a **flat** quantum space.

Questions

Does it work also on curved spacetimes?
Is the semiclassical equation a well posed dynamical system?

Plan of the talk

- Uncertainties for the coordinates in a generic spherically symmetric space.
- Well posedness of semiclassical Einstain equations in cosmology.
- Existence and Uniqueness of their solutions.

This talk is based on

- S. Doplicher, G. Morsella, NP, in preparation (2011)
- C. Dappiaggi, K. Fredenhagen, NP PRD **77**, 104015 (2008)
- C. Dappiaggi, NP, V. Moretti, CMP **285**, 1129-1163 (2009)
- C. Dappiaggi, NP, V. Moretti, JMP **50**, 062304 (2009)
- NP, CMP (2011)

- In *[DFR 95]* the authors find the commutation rules among the coordinates

$$[q^\mu, q^\nu] = iQ^{\mu\nu}$$

compatible with the following uncertainty relations

$$\Delta x_0 \left(\Delta x_1 + \Delta x_2 + \Delta x_3\right) \geq \lambda_P^2,$$

$$\Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_3 \Delta x_1 \geq \lambda_P^2$$

which are obtained using the following:

Minimal Principle (P0):

We cannot create a singularity just observing a system.

- Toghether with the **Heisenberg principle (HP)** (valid in Minkowksi).
- The uncertainties are tailored to the flat spacetime.
- On a curved spacetime we have to **replace** it with something else.
- We use **QFT on CST** and their comm. rel. in combination with **P0**.
- We shall perform such analysis on a spherically symmetric space.

Spherically symmetric case

- Spacetime is $\mathbb{R}^2 \times \mathbb{S}^2$, **retarded coordinates**: spanned by future null geodesic emanated from the center of the sphere

  - u the time on the worldline line of the center of $\mathbb{S}^2$
  - s affine parameter along the null geodesics with $s(0) = 0$ and $\dot{s}(0) = 1$.

$$ds^2 := -Adu^2 - 2dsdu + r^2 d\mathbb{S}^2$$

- Classical collapse of a massless **scalar field** has been studied by Christodoulou.

$$\Box\phi = 0$$

- He has given a condition for the **"energy content"** on an initial null cone $\mathcal{C}_0$ which implies the formation of a singularity inside of the cone.

$$T_{ss} = \partial_s\phi\partial_s\phi$$

Classical condition

Proposition

Consider a region of the initial null cone $\mathcal{C}_0$ contained within the two spherical sections determined by r_1 and r_2. If

$$\frac{r_2}{r_1} \in (1, 3/2) \qquad \text{and} \qquad \frac{2\,(m_2 - m_1)}{r_2} \geq 1$$

$J^+\mathcal{C}_0$, the causal future of $\mathcal{C}_0$, contains a spacelike singularity.

Suppose $\frac{s_2}{s_1} < 3/2$, then it holds that

$$\frac{m_2 - m_1}{r_2} \geq \pi \int_{s_1}^{s_2} s\partial_s\phi\partial_s\phi \; ds = \pi \int_{s_1}^{s_2} sT_{ss} \; ds$$

thus, if

$$\int_{s_1}^{s_2} s\partial_s\phi\partial_s\phi \; ds \geq \frac{1}{2\pi}$$

a singularity appears in the interior of $\mathcal{C}_0$

Quantum constraint

Take a quantum state ω, after measuring $\phi(f)$ the state is

$$\omega_f(A) := \frac{\omega\left(\phi(f)\ A\ \phi(f)\right)}{\omega(\phi(f)\phi(f))}\ .$$

Expectation values change accordingly

$$\langle Q \rangle_{f,0} := \omega_f(Q) - \omega(Q)\ .$$

in particular we get

$$\langle \phi(x)\phi(x) \rangle_{f,0} \geq \frac{|E(f,x)|^2}{\omega_2(f,f)}\ .$$

- Quantize the **"initial conditions"** and use the **"classical dynamics"**.
- The restriction of a quantum theory on a null cone $\mathcal{C}_0$ is well defined.
- It is a sort of **"Minimal"** semiclassical description.

On $\mathcal{C}_0$ using only the from of E and the $|\omega_2(f,f)| \leq \|f\|_2 \|\partial f\|_2$.

$$\int_{s_1}^{s_2} s \langle T_{ss} \rangle_{f,0} ds \geq \frac{\lambda_P^2}{{s_2}^2}$$

Combining it with the classical results, we have BH formation if

$$\frac{\lambda_P^2}{{s_2}^2} \geq \frac{1}{2\pi}$$

Notice that the support of f in the **detector** $\phi(f)$ on $\mathcal{C}_0$ extends up to s_2.

Proposition

If we use a detector supported in a sphere determined by s_2 we create a BH

It is an estimate of the **detector resolution**.
Now using **(P0)** denying the proposition we get

$$\Delta s \gtrsim \lambda_P$$

which is compatible with the results in *[DFR]* valid for the flat case

$$\Delta s^2 \gtrsim \Delta t \Delta r \gtrsim \lambda_P^2$$

see also *[Tomassini Viaggiu]*

Semiclassical equations in cosmology

To improve the results treat the backreaction on the whole spacetime.
We shall analyze this problem assuming: **homogeneity** and **isotropy**.

- a spacetime $M = (I \times S, g)$
  - $I \subset \mathbb{R}$ *"cosmological time"*
  - S is a 3d manifold: the *"space"*.
- **Friedmann Robertson Walker** metric

$$g = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\mathbb{S}^2(\theta,\varphi)\right].$$

- Dynamical degree of freedom $a(t)$.
- Recent observations:
  - $\kappa \simeq 0 \Longrightarrow$ Conformally Flat.
  - $a(t) = e^{Ht}$ the **Hubble parameter** H (very small, positive, almost constant).
- Once an initial condition is fixed, Einstein eq. are equivalent to

$$-R = 8\pi\langle T\rangle, \qquad \nabla^a\langle T_{ab}\rangle = 0\ .$$

- Init. cond. $G_{00} - 8\pi\langle T_{00}\rangle = 0$ is satisfied up to some radiation.

Let us assume this point of view and consider a very simple matter model

$$P\varphi = 0 \,, \qquad P = -\Box + \xi R + m^2 \,.$$

- The **quantization** is solved once you give $\mathcal{A}(M)$ and a state ω, described by the class of $n-$point functions (correlation functions).

$$\langle \varphi(x_1) \ldots \varphi(x_n) \rangle_\omega = \omega_n(x_1, \ldots, x_n) \qquad \omega_n \in \mathcal{D}'(M^n).$$

- T_{ab} shows products of fields evaluated at the same point **(divergent)**.
- In Minkowksi these are "cured" subtracting the **vacuum**.

Question

What plays the role of the vacuum in a curved spacetime?

- Remnant of the **Spectrum Condition** ("no $k_0 < 0$ in $\widehat{\omega}_2(\mathbf{k})$")
- It has to be a **local and covariant condition**
- **Idea:** look at the directions in T^*M^2 of **non rapid decrease** common to **all localized distributions** $\widehat{\omega_2 f_{x_1,x_2}}(k_1, k_2)$
- Formalized using Hörmander **microlocal analysis**

Microlocal spectrum condition

Definition

$\omega_2 \in \mathcal{D}'(M^2)$ satisfies the **microlocal spectrum condition** (μSC) if

$$\mathrm{WF}(\omega_2) = \left\{(x_1, x_2, k_1, k_2) \in T^*M^2 \setminus \{0\} \mid (x_1, k_1) \sim (x_2, k_2), k_1 \triangleright 0\right\}.$$

like the Minkowski vacuum

The Hadamard parametrix

Theorem (Radzikowski)

A state ω_2 satisfies $\mu SC \iff \omega_2$ is of Hadamard form:

$$\omega_2 = \mathcal{H} + W \qquad \mathcal{H} = \lim_{\epsilon \to 0^+} \frac{U}{\sigma_\epsilon} + V \log\left(\frac{\sigma_\epsilon}{\mu^2}\right)$$

$\mathcal{H}$ depends on the **local geometry** *and on μ only.*

Regularization: subtract $\mathcal{H}$ from ω_2.

Regularization and stress tensor

We shall use the following stress tensor (with $\xi = 1/6$) *[Moretti 03]*

$$T_{ab} := \partial_a\varphi\partial_b\varphi - \frac{1}{6}\left[g_{ab}\left(\partial_c\varphi\partial^c\varphi + \left(m^2 + R\right)\varphi^2\right) - R_{ab}\varphi^2 + \nabla_a\partial_b\varphi^2\right]$$

Hence, on some Hadamard state ω

$$\langle T_{ab}\rangle_\omega = \lim_{x\to y} D_{ab}(\omega_2 - \mathcal{H})$$

In the considered procedure there is some freedom

- $\mathcal{H}$ is **not** uniquely defined, it is known **up to some smooth terms**
- Local fields are **not** invariant $\Longrightarrow$ determined up to local counterterms **ambiguities** (**renormalization freedom**).
- The ambiguities have been studied and classified by *[Hollands Wald]*

Conservation equations for T_{ab} are satisfied: $\nabla_a \langle T^a{}_b \rangle_\omega = 0$ but (un)-fortunately the **trace** is different from the classical one.

$$\langle T \rangle_\omega := \frac{2[v_1]}{8\pi^2} + \left(-3 \left(\frac{1}{6} - \xi \right) \Box - m^2 \right) \langle \varphi^2 \rangle_\omega .$$

More precisely ($\xi = 1/6$) *[Wald 1978]*

$$2[v_1] = \frac{1}{360} \left(C_{ijkl} C^{ijkl} + R_{ij} R^{ij} - \frac{R^2}{3} + \Box R \right) + \frac{m^4}{4} .$$

The renormalization freedom for T is

$$\langle T' \rangle_\omega = \langle T \rangle_\omega + \alpha\, m^2 R + \beta\, m^4 + \gamma\, \Box R\ .$$

- In $\langle T \rangle_\omega$, three contributions: $T_{anomalies} + T_{ren.freedom} + T_{state}$.
- Cancel $\Box R$ from the trace $\Longrightarrow$ Wald's fifth axiom holds for T.
- We can **not** cancel $T_{anomalies}$ completely.
- $T_{anomalies}$ is **not** a mixture of perfect fluids: $\rho = H^4$
- Similarities with $f(R)$ gravity. **But** $f(R) \Longrightarrow$ unstable solutions.

Massive model

With $\kappa = 0$ and $\xi = 1/6$, the equation $-R = 8\pi\langle T\rangle$ becomes

$$-6\left(\dot{H} + 2H^2\right) = -8\pi m^2 \langle \varphi^2 \rangle_\omega - \frac{1}{30\pi}\left(\dot{H}H^2 + H^4\right) + \frac{m^4}{4\pi}$$

Important: The quantum state enters in the equations via $\langle \varphi^2 \rangle_\omega$

Physical input: We would like to use **"vacuum states"** i.e. $\langle \varphi^2 \rangle_\omega = 0$

Impossible: Adiabatic states, have similar properties

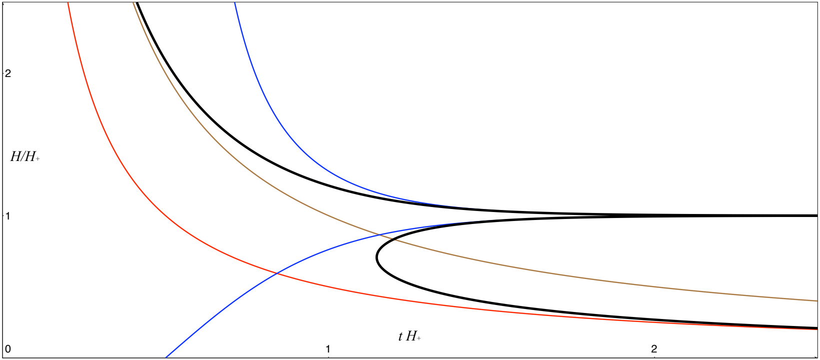
[Parker, Parker Fulling, Lüders Roberts, Junker Schrohe, Olbermann]

Assume (for the moment) $T_{state} = 0$

We have only $T_{anomalies}$ and $T_{ren.freedom} = \alpha R + \beta m^2$

The differential equation is an ordinary one $\Longrightarrow$ it can be solved

With some choice of α and β $H = 0$ and $H = H_+$ are stable solutions.



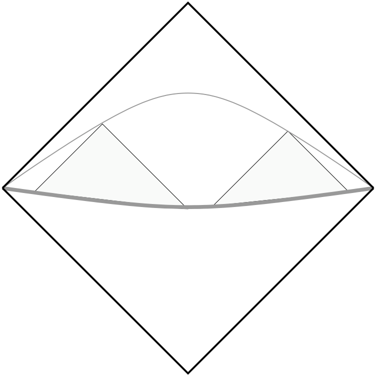
- ($m = 0$) a length scale is introduced (proportional to G). **Two** fixed points instead of one. *[Wald 80, Starobinsky 80, Vilenkin 85]*
- Quantum effects are **not negligible** at least in the past.
- ($m \neq 0$) H_+ is a renormalization constant.

Form of the initial singularity

Question

Where is the singularity t_0 in the Penrose diagram?

$$ds^2 = a^2\left(-d\tau^2 + d\mathbf{x}^2\right).$$



- **Classical solution**
  Radiation dominated:
  $\tau = \tau_0 + A(t - t_0)^{1/2} \to \tau_0$
  for $t \to t_0$
  Horizon problem.
- **Quantum Corrections**
  $\rho = 1/a(t)^2 \quad :$
  $\tau = \tau_0 + \log(t - t_0) \to -\infty$
  for $t \to t_0$
  Singularity is light like.

Power law inflation with
Null Big Bang $\Im^- \cup i^-$

Fix the quantum state out of the asymptotic structure

- Consider the state $\omega_{1,0}$ which looks like a vacuum on $\Im^-$ asymptotically.
- It is **pure**, **homogeneous** and **isotropic** *[Lüders Roberts]*

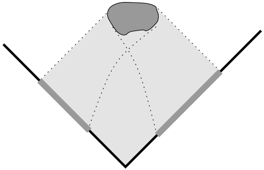
Def.

Theorem

If M has a null Big Bang and

$$(1+(\tau+r)^2)(1+(\tau-r)^2)m^2a(\tau)^2$$

can be smoothly extended over $\Im^- \cup i^-$, then $\omega_{1,0}$ satisfies the μSC.



Proof: $\omega_{1,0} = (\Delta \restriction \Im^- \otimes \Delta \restriction \Im^-) \circ \lambda$

Exsistence and uniqueness of solutions at small time

- $\omega_{1,0}$ is an **asymptotic vacuum**. (Initial conditions of the problem)
- We search for solutions of $-R = 8\pi\langle T\rangle_{\omega_{1,0}}$ near **NBB** $\Im^-$.
- Indicating by $X := H^{-1}$, we rewrite the equation as:

$$\frac{dX}{dt} = 1 - \frac{X^2}{X_c^2 - X^2} + m^2 \frac{C\, X^4}{X_c^2 - X^2}\langle\varphi^2\rangle_{\omega_{1,0}} \,.$$

- It is **not** an ordinary differential equation.
- $\langle\varphi^2\rangle_{\omega_{1,0}}$ is a functional of $X = H^{-1}$.
- To get existence of sol. $\Longrightarrow$ show that $X := \mathcal{T}(X)$. A **fixed point** for

$$\mathcal{T}(X) = \int_0^t \left[\frac{X_c^2 - 2X^2}{X_c^2 - X^2} + Cm^2 \frac{X^4}{X_c^2 - X^2}\langle\varphi^2\rangle_{\omega_{1,0}}\right] dt'$$

- Prove that $\mathcal{T}$ is a **contraction map** on $B_c \subset \mathcal{B}$ then use **Banach fixed point theorem**. Def.

On $B_c \subset \mathcal{B}$ we have a well posed initial value problem

Theorem

For a sufficiently small t_0, $\mathcal{T}$ is a contraction on B_c. Thus it exists one and only one X in B_c for which

$$X = \mathcal{T}(X)$$

- Proof: We have to better analyze $\langle\varphi^2\rangle_{\omega_{1,0}}$ and its first func. derivative

$$\langle\varphi^2\rangle_{\omega_{1,0}} := \frac{1}{2\pi^2 a^2}\int_0^\infty k^2 dk \left[\overline{\chi}_k \chi_k - \Theta(k-ma)\left(\frac{1}{2k} - \frac{m^2a^2}{4k^3}\right)\right]$$

$$\chi_k'' + (k^2 + m^2a^2)\chi = 0$$

Sketch of proof

Some comments

- The found solution is C^2.
- **But** there are smooth spacetimes as close as you want to that solution.
- The existence **does not depend on the state**, in the sense the theorem holds also for other initial conditions on $\Im^-$ provided the state is Hadamard.
- All these solutions show a typical phase of **power law inflation** which is then **state independent**.
- When smeared on constant time surfaces $\Delta_{\omega_{1,0}} T = 0$. ▸ Proof

Summary

- Semiclassical Backreaction can be used to constraint the **non commutativity**.
- Semiclassical solutions of Einstein's equations **can be found**.
- Some of their physical properties **do not depend** on the homogeneous state

Open Questions

- Is it possible to combine both results?
- Can we say something for the generic case?

Thanks a lot for your attention!

Homogeneous Hadamard states in cosmological spacetime

- The **pure**, **homogeneous** and **isotropic** *[Lüders Roberts]*

$$\omega_2(x,y) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\overline{T_k}(x_0)}{a(x_0)}\frac{T_k(y_0)}{a(y_0)} e^{i\mathbf{k}\cdot(x-y)} d\mathbf{k}\ ,$$

T_k is a smooth function of τ, such that $\overline{T_k}T_k' - \overline{T_k}'T_k = i$ and

$$T_k''(\tau) + (m^2 a(\tau)^2 + k^2)T_k(\tau) = 0.$$

- Consider **incoming plane waves** $\chi_k(\tau) \sim \frac{e^{-ik\tau}}{\sqrt{2k}}$ for $\tau \to -\infty$.
- Every $T_k := A(k)\chi_k + B(k)\overline{\chi}_k$ with $|A(k)|^2 - |B(k)|^2 = 1$.
- The state depends upon A and B. We indicate it as ω_{AB}.

back

Fix an interval $(0, t_0)$ then impose the following initial conditions

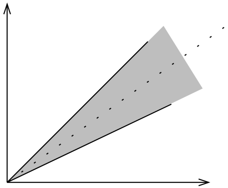
$$X(0) = 0 \,, \qquad a(t_0) = a_0 = \lambda t_0, \qquad \tau(t_0) = \tau_0$$

a and τ are functional of X.
Banach space $\mathcal{B} := \{f \in C^1(0, t_0) \,, \; f(0) = 0\}$. Norm

$$\|f\|_{\mathcal{B}} := \sup_{(0,t_0)} \left|\dot{f}\right| \,.$$

Closed ball $B_c \subset \mathcal{B}$ where $\frac{2}{3} < c < 1$. Radius $1/2 - c/2$. Center

$$f_0 := \frac{1+c}{2}\, t \,.$$



Properties of the elements of B_c

- Implements the initial condition $X(0) = 0$
- $t = 0$ corresponds to $\Im^-$ $\Longrightarrow$ **NBB** Power law inflationary scenario
- For every smooth X in B_c, $\omega_{1,0}$ is well posed and Hadamard

Analysis of φ^2

back

$$\langle\varphi^2(x)\rangle_{\omega_{1,0}} = \lim_{y\to x}\left[\omega_{1,0}(x,y) - \mathcal{H}(x,y)\right] + \alpha R + \beta m^2$$

Prescription for fixing the renormalization freedom:

- Minkowski spacetime on Minkowski vacuum, fixes β.
- α changes the value of H_c or $X_c \implies H_c$ **is a ren. constant**

We regularize on Minkowski spacetime the problem $-\Box_{\mathbb{M}}\tilde{\varphi} + (ma)^2\tilde{\varphi} = 0$

$$\lim_{y\to x}\mathcal{H}(y,x) - \frac{1}{a(\tau_x)a(\tau_y)}\mathcal{H}_{\mathbb{M}}(y,x) = \frac{m^2}{8\pi^2}\log a + \alpha' R\ .$$

Other reg. scheme

Point splitting at fixed time, then it is enough to subtract

$$\mathcal{H}^0_{\mathbb{M}}(y,x) := \frac{1}{(4\pi)^2}\left(\frac{2}{\sigma_\epsilon} + m^2 a(\tau_x)^2 \log\left(\frac{\sigma_\epsilon}{\lambda^2}\right)\right)$$

Comparison with the first order adiabatic approximation

$$\mathcal{H}^0_{\mathbb{M}}(y,x) - \frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{k}(\mathbf{y}-\mathbf{x})}}{2\sqrt{\mathbf{k}^2 + m^2 a(\tau)^2}}\ d^3\mathbf{k}$$

is a continuous function

$$\langle\varphi^2\rangle_{\omega_{1,0}} :=$$

$$\frac{1}{2\pi^2 a^2}\int_0^\infty k^2 dk \left[\overline{\chi}_k \chi_k - \Theta(k - ma)\left(\frac{1}{2k} - \frac{m^2 a^2}{4k^3}\right)\right] - \frac{m^2}{8\pi^2} + \alpha R,$$

Construction of the χ

$$\chi_k'' + (k^2 + m^2 a^2)\chi = 0$$

Perturbative const. over the massless solution $\chi_k^0(a,\tau)(t) = \frac{e^{-ik\tau(t)}}{\sqrt{2k}}$

$$\chi_k = \sum_{n=0}^{\infty} \chi_k^n$$

$$\chi_k^n(t) = -\int_0^t \frac{\sin(k(\tau - \tau'))}{k} a(t') m^2 \chi_k^{n-1}(t')\, dt' \ ,$$

Proposition

The series converges absolutely on $[0, t_0]$*, and*

$$|\chi_k| \leq \frac{1}{\sqrt{2k}} \exp\left(\frac{m^2 a(t) t}{k}\right) , \qquad |\chi_k| \leq \frac{1}{\sqrt{2k}} \exp\left(m^2 t^2\right) .$$

Every χ_k^n is $O(m^{2n})$ back

Analysis of the fluctuations

The solution is meaningful provided the variance of ${T_\mu}^\mu$ is small

- The anomaly is a $C-$number
- The variance of $\langle\varphi^2\rangle$

$$\Delta_\omega(\varphi^2) := \omega(\varphi^2\varphi^2) - \omega(\varphi^2)\omega(\varphi^2)$$

diverges: it is proportional to $\omega_2 \cdot \omega_2(x, x)$

When smeared the situation is better, consider the family centered in x_τ

$$f_{n_1,n_2}(\tau', \mathbf{x}) = \frac{n_1}{n_2^3} f\left(n_1(\tau' - \tau) + \tau, \frac{\mathbf{x}}{n_2}\right)$$

where

$$f(x_\tau) = 1\ , \qquad \int_M f\ d\mu(g) = 1\ , \qquad f \geq 0$$

We study the limit

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty}\left[R(f_{n_1,n_2}) + 8\pi\langle T\rangle_\omega(f_{n_1,n_2})\right] = R(x_\tau) + 8\pi\langle T\rangle_\omega(x_\tau)$$

Theorem

We have

$$\lim_{n_2\to\infty}\Delta_{\omega_{1,0}}(\varphi^2(f_{n_1,n_2})) = 0\ .$$

- In a **weaker** sense, the solution we have found is meaningful also when H is very large.

back

Comparison with the ΛCDM model

- The late time behavior is **not** under control $\Longrightarrow$ some assumptions

Before "local vacuum": $\langle \varphi^2 \rangle_\omega \sim 0$ with certain α and β

Now "local thermal state": $\langle \varphi^2 \rangle_\omega \sim \frac{T^3}{a^3} + O\left(\frac{1}{a^5}\right)$

(A minimal model with two fields a massive scalar field a massless one)

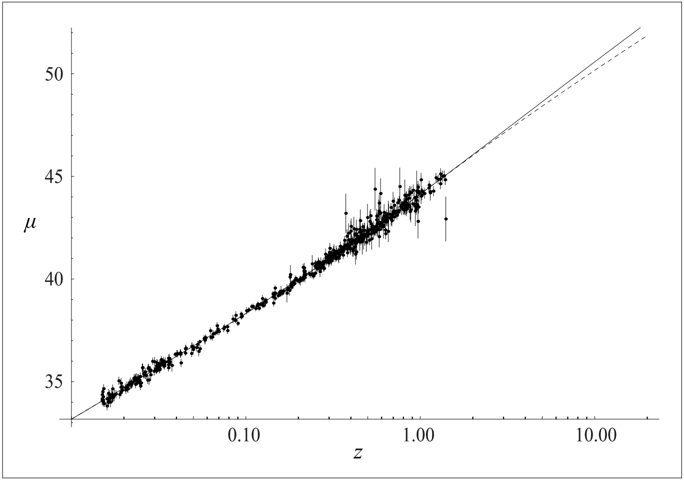
$$H^2 = H_*^2 \pm \sqrt{H_*^4 - \frac{C_1}{a^4} - C_2 - C_3 \frac{T^3}{a^3}}$$

  - lower branch if H_*^4 is very large we get ΛCDM plus quantum correction
  - upper branch looks crazy *(the energies appear with negative sign)*

- Phenomenological law for the luminosity distance μ *(spatial distance)* w.r.t. red-shift $z = \frac{1}{a} - 1$ *(temporal distance)* for the SN1a explosions.

$$\mu(z) = 5 \log \left((1+z) \int_0^z \frac{1}{H(z')} dz' \right) + K$$

- Compare it with observations: best fit is obtained by minimizing χ^2.



Union2 supernova compilation *[Amanullah et al. 2010]*