Graph C^*-algebras and crossed products by endomorphisms.

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Overview

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Section Crossed products

Crossed products by automorphisms

Let (A, α) be a *classical C^*-dynamical system*.

Let A be a unital C^*-algebra

and let $\alpha : A \longrightarrow A$ be a $*$-automorphism.

The *crossed product* $A \times_\alpha \mathbb{Z}$ is the C^*-algebra generated by the universal covariant representation of (A, α).

$A \times_\alpha \mathbb{Z}$ is generated by A and a unitary U such that $UaU^* = \alpha(a)$

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Crossed product by endomorphism

Let (A, α) be a *non-classical C^*-dynamical system*.

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The idea of the construction comes originally from Cuntz.

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} \cdots \longrightarrow & A_\infty \\
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Crossed product by endomorphism

So construct the full corner sub-C^*-algebra

$$p(A_\infty \times_{\alpha_\infty} \mathbb{Z})p$$

where $p = \iota_{1,\infty}(1)$.

Also observe that $pU_\infty p$ is an isometry in $p(A_\infty \times_{\alpha_\infty} \mathbb{Z})p$, that together with $\iota_{1,\infty}(A)$ generates all $p(A_\infty \times_{\alpha_\infty} \mathbb{Z})p$.

Later Paschke gave a generalization of Cuntz's construction and described and studied C^*-algebras generated by a unital algebra and an isometry.

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Crossed product by endomorphism

Stacey gave an elegant description of the crossed products by endomorphisms, actually in the most general context of crossed products by semigroups.

A *covariant representation* of (A, α) is a pair (π, V) such that:

1. $\pi : A \longrightarrow B(\mathcal{H})$ is a non-degenerate representation
2. $V \in B(\mathcal{H})$ an isometry such that $V\pi(a)V^* = \pi(\alpha(a))$.

He proved that if $A_\infty \neq 0$ there exists (ι, V) a universal covariant representation of (A, α), and denote by $A \times_\alpha \mathbb{N}$ the C^*-algebra generated by $\iota(A)$ and $\iota(A)V$.

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We have that

$$A \times_\alpha \mathbb{N} \cong p(A_\infty \times_{\alpha_\infty} \mathbb{Z})p\,.$$

Though $A_\infty \times_{\alpha_\infty} \mathbb{Z}$ and $A \times_\alpha \mathbb{N}$ are Morita equivalent, unluckily we cannot always recover the structure of α_∞ from α, and vice-versa.

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What is known?

There are result about the simplicity of $A \times_\alpha \mathbb{N}$ from Paschke, Adji-Laca-Nielsen-Raeburn and Olesen-Pedersen,

But the most satisfactory is the following from Schweizer.

Theorem 1 (Schweizer)

Let A be a unital C^*-algebra and let α be an injective $*$-endomorphism. T.F.A.E:

1. *α^n is outer for every $n \geq 1$ and there are no non-trivial ideals I of A such that $\alpha(I) \subseteq I$.*
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Rørdam gave sufficient conditions for the crossed product being simple and purely infinite.

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Section Graph C^*-algebras

Definition of Graph C^*-algebras.

Let $E = (E^0, E^1, r, s)$ be a directed graph.

We say that E is *locally finite* if $0 \leq |s^{-1}(v)|, |s^{-1}(v)| < \infty$.

A vertex $v \in E^0$ is a *sink* if $|s^{-1}(v)| = 0$, and it is a *source* if $|r^{-1}(v)| = 0$.

A *path* of length n is a sequence of edges $\alpha = \alpha_n \cdots \alpha_1$ with $r(\alpha_i) = s(\alpha_{i+1})$

A *cycle* is a path $\alpha = \alpha_n \cdots \alpha_1$ with $\alpha_i \neq \alpha_j$ if $i \neq j$ and with $r(\alpha_n) = s(\alpha_1)$

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The graph C^*-algebra $C^*(E)$ is the universal algebra generated by a family of mutually orthogonal projections $\{p_v\}_{v\in E^0}$ and isometries $\{S_e\}_{e\in E^1}$ satisfying:

- (CK1) $S_e^* S_f = \delta_{e,f} p_{s(e)}$
- (CK2) $p_v = \sum_{r(e)=v} S_e S_e^*$ if $0 < |r^{-1}(v)| < \infty$

We have that

$$C^*(E) = \overline{\text{span}}\{S_\eta S_\nu^* : \eta, \nu \in E^* \text{ with } s(\eta) = s(\nu)\}$$

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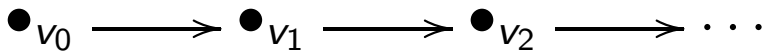

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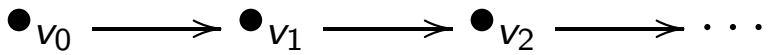
1. 

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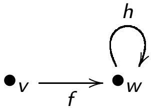
2. 

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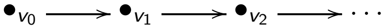

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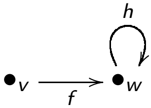

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What is known about graph C^*-algebras?.

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Properties like simplicity, ideal structure and purely infiniteness are completely determined by properties of the graph.

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The core of $C^*(E)$.

We can define a group homomorphism $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ where for every $z \in \mathbb{T}$ we define the automorphism $\gamma_z : C^*(E) \to C^*(E)$ given by

$$\gamma_z(p_v) = p_v \qquad \text{and} \qquad \gamma_z(S_e) = zS_e$$

for every $v \in E^0$ and $e \in E^1$.

We define the core

$$C^*(E)^\gamma := \{x \in C^*(E) : \gamma_z(x) = x \text{ for all } z \in \mathbb{T}\} .$$

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The core of $C^*(E)$.

Given $v \in E^0$ and $n \geq 0$ we define

$$\mathcal{F}_n(v) = \overline{\text{span}}\{S_\eta S_\nu^* : |\eta| = |\nu| = n\} \cong M_{k_{n,v}}(\mathbb{C})\,.$$

So let $\mathcal{F}_n := \oplus_{v \in E^0} \mathcal{F}_n(v)$ and $C_n = \mathcal{F}_0 + \cdots + \mathcal{F}_n$.

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Graph C^*-algebras as crossed products.

Let E be a locally finite graph without sinks,

then

$$T = \sum_{e \in E^1} |s^{-1}(s(e))|^{-1/2} S_e$$

is an isometry in $M(C^*(E))$.

Then define the endomorphism

$$\beta_E : C^*(E)^\gamma \longrightarrow C^*(E)^\gamma$$
$$x \longmapsto TxT^*$$

Theorem 3 (an Huef-Raeburn)

Let E be a locally finite graph without sinks, then $C^*(E) \cong C^*(E)^\gamma \times_{\beta_E} \mathbb{N}$.

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Section Gauge invariant ideals

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We can naturally define a gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(A \times_\alpha \mathbb{N})$.

An ideal $I \lhd A \times_\alpha \mathbb{N}$ is called gauge invariant if $\gamma_z(I) = I$ for every $z \in \mathbb{T}$.

Definition 4

Let $\alpha : A \to A$ be a $*$-homomorphism, we say that $I \lhd A$ is:

1. weakly α-invariant if $\alpha(I) \subseteq I$.
2. α-invariant if $\overline{\alpha(A) I \alpha(A)} = \alpha(I)$.
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We define a bijection between the hereditary and saturated sets of E^0 and the β_E-invariant ideals of $C^*(E)^\gamma$ as

$$H \longmapsto \overline{\sum_{v\in H, n\geq 0} \mathcal{F}_n(v)}$$

and

$$I \longmapsto \{v \in E^0 : p_v \in I\}.$$

Proposition 5 (Bates-Pask-Raeburn-Szymański, Katsura, Ortega)

Let E be a locally finite graph without sinks. There is a bijection between the following sets:

1. *The β_E-invariant ideals of $C^*(E)^\gamma$.*
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The Cuntz Krieger uniqueness theorem.

We say that $A \times_\alpha \mathbb{N}$ satisfies the Cuntz-Krieger uniqueness theorem if given any $*$-homomorphism

$$\Phi : A \times_\alpha \mathbb{N} \longrightarrow B$$

with $\Phi_{|A}$ injective , then Φ is injective

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Theorem 7 (Ortega)

Let E be a locally finite graph without sinks. T.F.A.E:

1. *E satisfies condition (L).*
2. *β_E is a strongly outer endomorphism.*

Theorem 8 (Bates-Pask-Raeburn-Szymański)

Let E be a locally finite graph without sinks. T.F.A.E:

1. *$C^*(E) \cong C^*(E)^\gamma \times_{\beta_E} \mathbb{N}$ is simple.*
2. *$C^*(E)^\gamma$ has no non-trivial β_E-invariant ideals and β_E is strongly outer.*

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Logo

Simple Purely Infinite graph C^*-algebra.

Theorem 9 (Hong-Szymański, Rørdam, Ortega)

Let E be a locally finite graph without sinks. T.F.A.E:

1. $C^*(E)^\gamma$ *is unital,* $\beta_E(1) \neq 1$ *and* $C^*(E)^\gamma$ *has no non-trivial* β_E*-invariant ideals.*
2. $C^*(E)^\gamma \times_{\beta_E} \mathbb{N} \cong C^*(E)$ *is a unital simple purely infinite* C^**-algebra.*