Deformations of Quantum Field Theories on de Sitter Spacetime

Eric Morfa-Morales

ESI Vienna

EU-NCG 4th Annual Meeting
Bucharest, 27.04.2011
Outline

• Motivation: QFT in terms of wedge triples
• de Sitter spacetime and wedges
• Wedge triples with global gauge symmetry
• Deformation by warped convolutions
• Conclusions and outlook
Motivation: QFT in terms of wedge triples

Minkowski space \((\mathbb{R}^4, \eta)\) with isometry group \(\mathcal{P} = \text{SO}(1, 3)_0 \rtimes \mathbb{R}^4\)

A wedge triple \((\mathcal{A}_0, \mathcal{A}, \alpha)\) consists of \(C^*\)-algebras \(\mathcal{A}_0 \subset \mathcal{A}\) and a strongly continuous action \(\alpha : \mathcal{P} \to \text{Aut}(\mathcal{A})\) satisfying

\[
gW_0 \subset W_0 \Rightarrow \alpha_g(\mathcal{A}_0) \subset \mathcal{A}_0, \quad \alpha_{jW_0}(\mathcal{A}_0) \subset \mathcal{A}_0' \cap \mathcal{A}.
\]

\[W_0 = \{x \in \mathbb{R}^4 : x^1 > |x^0|\}\]
Proposition (Reconstruction of QFT from wedge triples)

Let \((A_0, A, \alpha)\) be a wedge triple. Then

\[
W := gW_0 \mapsto \alpha_g(A_0) =: A(W)
\]

is a QFT in the sense of Haag and Kastler.

Conversely, every QFT (over wedges) gives rise to a wedge triple.

QFT over compact regions \(\mathcal{O} = \bigcap_{W \supset \mathcal{O}} W:\)

\[
A(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} A(W)
\]

is a QFT in the sense of Haag and Kastler.
Motivation: Deformation of wedge triples

Consider \((\mathcal{A}_0, \mathcal{A}, \alpha)\) in a covariant representation on a Hilbert space \(\mathcal{H}\).

**Warped Convolution:**

\[
AQ := \frac{1}{(2\pi)^4} \int dv \, dv' \, e^{-ivv'} U(Qv) \mathcal{A}U(-Qv)U(v'), \quad A \in \mathcal{A}_0 \text{ 'smooth'}
\]

where \(U\) is a representation of \(\mathbb{R}^4 \subset \mathcal{P}\) with spectrum condition and

\[
Q = \begin{pmatrix}
0 & \kappa & 0 & 0 \\
\kappa & 0 & 0 & 0 \\
0 & 0 & 0 & \kappa' \\
0 & 0 & -\kappa' & 0
\end{pmatrix}, \quad \kappa \geq 0, \kappa' \in \mathbb{R}.
\]

(close connection to Rieffel’s deformation of \(C^*\)-algebras)

**Theorem** [Buchholz, Lechner, Summers:2010]

Let \((\mathcal{A}_0)_Q = \{AQ : A \in \mathcal{A}_0\}''\). Then \(((\mathcal{A}_0)_Q, \mathcal{A}, \alpha)\) is a wedge triple.

Associated QFT: non-trivial 2-particle scattering!
Questions:

• Are translations special?
• Do other Abelian subgroups of $\mathcal{P}$ yield 'interesting' deformations?
• What about gauge groups?
• Is this procedure also feasible on a curved spacetime?

Natural setting: QFT with gauge symmetry in de Sitter spacetime
de Sitter spacetime

Solution \((M, g)\) of Einstein equation with positive cosmological constant

\[
M = \{ x \in \mathbb{R}^5 : (x^0)^2 - \sum_{j=1}^{4} (x^j)^2 = -1 \}
\]

\[
g = i^* \eta, \ i : M \hookrightarrow \mathbb{R}^5
\]

(maximally symmetric, globally hyperbolic)

- causal structure inherited from \((\mathbb{R}^5, \eta)\)
- isometry group:
  \[\text{O}(1,4) \supset \text{SO}(1,4)_0 =: \mathcal{L}_0\] (de Sitter group)
- covering:
  \[\text{Spin}(1,4) =: \tilde{\mathcal{L}}_0 \xrightarrow{\pi} \mathcal{L}_0\]

physically interesting: (inflationary) cosmology
de Sitter wedges

Causal closure of timelike geodesics (uniformly accelerated observers).

Equivalently:

- $W_0 = \{ x \in \mathbb{R}^5 : x^1 > |x^0| \} \cap M$
- $\mathcal{W} = \{ gW_0 : g \in \tilde{L}_0 \}$, $gW_0 := \pi(g)W_0$

Properties:

- $W' \in \mathcal{W}$ and $W'' = (W')' = W$
- $\forall W \in \mathcal{W}$: $\exists \Gamma_W = \{ \Lambda_W(t) \in \tilde{L}_0 : t \in \mathbb{R} \}$:
  \[ \Lambda_W(t)W = W, \quad \Lambda_{gW}(t) = g\Lambda_W(t)g^{-1} \]
  
  associated with $\Gamma_W$ is a timelike Killing VF $\xi_W$

- $\forall W \in \mathcal{W}$: $\exists j_W \in \tilde{L}_0$:
  \[ j_W W = W', \quad j_{gW} = gj_Wg^{-1} \]
Lemma 1

Let $W \in \mathcal{W}$ and $\Lambda(t) \in \Gamma_W$, $j_W$ be as before. Then

$$g\Lambda_W(t)g^{-1} = \Lambda_W(t)$$

for all $g \in \tilde{\mathcal{L}}_0(W) = \{g \in \tilde{\mathcal{L}}_0 : gW = W\}$ and

$$\Lambda_{W'}(t) = j_W\Lambda_W(t)j_W = \Lambda_W(-t)$$

Lemma 2

Let $W_1, W_2 \in \mathcal{W}$ and $W_1 \subset W_2$. Then $W_1 = W_2$. 
Wedge triples for Bose/Fermi fields on de Sitter

$C^*$-algebras $\mathcal{F}_0 \subset \mathcal{F}$ and $\alpha : \tilde{\mathcal{L}}_0 \rightarrow \text{Aut}(\mathcal{F})$ strongly continuous action s.t.

$$\alpha_g(\mathcal{F}_0) = \mathcal{F}_0, \ \forall g \in \tilde{\mathcal{L}}_0(W_0), \ \alpha_{jW_0}(\mathcal{F}_0) \subset \mathcal{F}_0^{t'} \cap \mathcal{F}$$

with $\mathcal{F}_0^{t'} = 'twisted' \ commutant \ of \ \mathcal{F}_0$

$\rightarrow$ (anti)commutation relations for Bose/Fermi fields

Gauge symmetry: $G$ compact group, $\sigma : G \rightarrow \text{Aut}(\mathcal{F})$ strongly cont.:

- $\sigma_h \circ \alpha_g = \alpha_g \circ \sigma_h, \ \forall h \in G, g \in \tilde{\mathcal{L}}_0$
- $\sigma_h(\mathcal{F}_0) = \mathcal{F}_0, \ \forall h \in G$

**Proposition (Reconstruction of QFT)**

The map $gW_0 \mapsto \alpha_g(\mathcal{F}_0)$ is a QFT with global gauge symmetry (field net) in the sense of Doplicher, Haag and Roberts.

Example: free charged Dirac field ($G = U(1)$)
Deformation of wedge triples

Let \((\mathcal{F}_0 \subset \mathcal{F}, \alpha, \sigma)\) be a wedge triple with \(U(1)\) gauge symmetry in a covariant representation on a Hilbert space \(\mathcal{H}\).

Use the following \(\mathbb{R}^2\)-action (boosts + gauge symmetry)

\[
(t, s) \mapsto \alpha_{\Lambda_{W_0}(t)} \circ \sigma_{e^{is}} = \tau_{t,s}^\xi : \mathcal{F} \to \mathcal{F}
\]

for warped convolution.

Notation:

- \(\xi = \xi_{W_0} = \) Killing vector field associated with \(W_0\)
- \(\Lambda_\xi(t) = \Lambda_{W_0}(t)\)
- \(\text{Ad } U_\xi(t, s) = \tau_{t,s}^\xi\)
Deformation of wedge triples

**Definition**

\[ F_{\xi,\kappa} := \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dv \, dv' \, e^{-i\nu \nu'} \tau_Q^{\xi}(F) U_{\xi}(\nu'), \quad Q = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix}, \quad \kappa \in \mathbb{R}. \]

This integral exists for 'smooth' elements \( F \in \mathcal{F}_0 \) in an oscillatory sense.

**Basic properties:**

- \((F_{\xi,\kappa})^* = (F^*)_{\xi,\kappa}\)
- \(F_{\xi,\kappa} G_{\xi,\kappa} = (F \times_{\xi,\kappa} G)_{\xi,\kappa}\) with Rieffel product \( \times_{\xi,\kappa}\)
- If \([\tau_{\nu}^{\xi}(F), G] = 0, \forall \nu \in \mathbb{R}^2\), then \([F_{\xi,\kappa}, G_{\xi,-\kappa}] = 0\)

**Transformation properties:**

- \(\alpha_g(F_{\xi,\kappa}) = \alpha_g(F')_{g^*\xi,\kappa}\)
- \(\sigma_h(F_{\xi,\kappa}) = \sigma_h(F')_{\xi,\kappa}\)
Deformation of wedge triples

Theorem

Let \((\mathcal{F}_0)_{\xi,\kappa} := \{F_{\xi,\kappa} : F \in \mathcal{F}_0 \text{ smooth}\}\). Then \(((\mathcal{F}_0)_{\xi,\kappa} \subset \mathcal{F}, \alpha, \sigma)\) is a wedge-triple, i.e,

a) \(\alpha_g((\mathcal{F}_0)_{\xi,\kappa}) = (\mathcal{F}_0)_{\xi,\kappa}, \quad \forall g \in \tilde{\mathcal{L}}_0(W_0)\)

b) \(\alpha_{jW_0}((\mathcal{F}_0)_{\xi,\kappa}) \subset ((\mathcal{F}_0)_{\xi,\kappa})^{t'}\)

c) \(\sigma_h((\mathcal{F}_0)_{\xi,\kappa}) = (\mathcal{F}_0)_{\xi,\kappa}, \quad \forall h \in U(1)\).

proof: locality property b) (Bose case): let \(F, G \in \mathcal{F}_0\)

\[
\alpha_{jW_0} (G_{\xi,\kappa}) = \alpha_{jW_0} (G)_{jW_0 \ast \xi,\kappa} = \alpha_{jW_0} (G)_{\xi,-\kappa}
\]

As \([\tau^\xi_v (F), \alpha_{jW_0} (G)] = 0, \quad \forall v \in \mathbb{R}^2\) there follows

\[
[F_{\xi,\kappa}, \alpha_{jW_0} (G_{\xi,\kappa})] = [F_{\xi,\kappa}, \alpha_{jW_0} (G)_{\xi,-\kappa}] = 0.
\]
Deformation of wedge triples

Example of wedge-triples with gauge symmetry: selfdual CAR-algebras

\[ \mathcal{F}_0 = \text{CAR}(\mathcal{H}_0, C) \subset \text{CAR}(\mathcal{H}, C) = \mathcal{F} \]

coming from \((\mathcal{H}_0 \subset \mathcal{H}, U, V)\).

Gauge transformations generated by charge operator (grading).

Results:

• deformed operators can be computed: \( F : \mathcal{H}_n \rightarrow \mathcal{H}_{n+m} \)

\[ F_{\xi, \kappa} = \sum_{n \in \mathbb{Z}} U_\xi(\kappa n) FU_\xi(-\kappa(n + m)) E(n) \]

\( E(n) \) = projector on \( \mathcal{H}_n \) (charge \( n \) subspace of \( \mathcal{H} \))

• deformation fix-points for observables: \( \mathbb{C} \cdot 1 \)

• unitary inequivalence of the associated nets for a variety of models
Conclusion and outlook

Used boosts and $U(1)$ symmetry to define a warped convolution

- obtained deformed wedge triple
- inequivalence in a variety of models (e.g. free charged Dirac field)

Deformations with other Abelian groups

- pure gauge symmetry: trivial deformation
- subgroups of $SO(1, 4)_0$: deformation does not yield a wedge triple
  $\rightarrow$ modification of warping formula required

Outlook:

- applications in cosmology
- connection with [Dappiaggi, Lechner, M:2010]
  $\rightarrow$ deformations with non-Abelian groups, e.g. $SO(3)$
- chiral conformal theories: affine group [Beliavsky:2007]
- general deformation theory for wedge triples