Radial Multipliers on Reduced Free Products

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Freenes

Definition

For $(A_i) \subseteq (A, \phi)$ unital C*-subalgebras we say (A_i) are free in A if

$$\forall n \in \mathbb{N} \quad \forall a_j \in A_{i_j}, 1 \leq j \leq n, i_j \neq i_{j+1}, \phi(a_j) = 0$$

we have

$$\phi(a_1 a_2 \cdots a_n) = 0.$$

Reduced free product

Given unital C*-algebras (A_i, ϕ_i) construct algebra (A,ϕ) such that

- (A_i) free in (A, ϕ)
- $\phi|_{A_i} = \phi_i$

Notation

$$(A, \phi) = *_i(A_i, \phi_i)$$

Properties of reduced free product

$$\forall i : A_i \subseteq B(H_i) \Rightarrow A \subseteq B(*_i H_i)$$

$$*_i C_r^*(G_i) = C_r^*(*_i G_i)$$

$$A = \overline{\mathbb{C}1 \oplus \bigoplus_n \bigoplus_{i_1,\ldots i_n, i_j \neq i_{j+1}} \mathring{\mathcal{A}}_{i_1} \cdots \mathring{\mathcal{A}}_{i_n}}$$

where $\mathring{\mathcal{A}}_i = \ker \phi_i$.

Problem

Let $\phi : \mathbb{N} \to \mathbb{C}$ and $\mathcal{A} = *_i \mathcal{A}_i$ and define

$$M_\phi(a_1 \ldots a_n) = \phi(n) a_1 \ldots a_n. \quad (1)$$

- Is M_ϕ welldefined on $\mathcal{A}$?
- When is M_ϕ completely bounded?
- For which $\mathcal{A}_i$?
- For which ϕ?
- $\|M_\phi\|_{cb} =?$

Class $\mathcal{C}$

Definition

Let $\mathcal{C}$ denote the set of functions $\phi : \mathbb{N}_0 \to \mathbb{C}$ for which the Hankel matrices

$$h = (\phi(i+j) - \phi(i+j+1))_{i,j\in\mathbb{N}_0}$$
$$k = (\phi(i+j+1) - \phi(i+j+2))_{i,j\in\mathbb{N}_0}$$

are of trace class and $c = \lim_{n\to\infty} \phi(n)$ exists.
For $\phi \in \mathcal{C}$ put

$$\|\phi\|_{\mathcal{C}} = \|h\|_1 + \|k\|_1 + |c|.$$

Known results

Theorem (Wysoczanski 1995)

Let $G = *_{i \in I} G_i$ and $\phi \in \mathcal{C}$ then $M_\phi : C_r^*(G) \to C_r^*(G)$ is welldefined and

$$\|M_\phi\|_{cb} \leq \|\phi\|_{\mathcal{C}}.$$

Theorem (Ricard-Xu 2006)

Let $\mathcal{A} = *_i \mathcal{A}_i$ and $\phi(n) = s^n$, $s \in]0, 1[$ then $M_\phi : \mathcal{A} \to \mathcal{A}$ is welldefined and

$$\|M_\phi\|_{cb} \leq 1.$$

Main result

Theorem (Haagerup-M)

Let $\mathcal{A} = *_{i\in I}(\mathcal{A}_i, \omega_i)$ *be the reduced free product of unital C*-algebras* $(\mathcal{A}_i)_{i\in I}$ *with respect to states* $(\omega_i)_{i\in I}$ *for which the GNS-representation* π_{ω_i} *is faithful for all* $i \in I$.
If $\phi : \mathbb{N}_0 \to \mathbb{C}$ *belongs to* $\mathcal{C}$, *then there is an unique linear completely bounded map*

$$M_\phi : \mathcal{A} \to \mathcal{A}$$

such that $M_\phi(1) = \phi(0)1$ *and*

$$M_\phi(a_1 a_2 \dots a_n) = \phi(n) a_1 a_2 \dots a_n$$

whenever $a_j \in \mathring{\mathcal{A}}_{i_j} = ker(\omega_{i_j})$ *and* $i_1 \neq i_2 \neq \dots \neq i_n$.
Moreover $\|M_\phi\|_{cb} \leq \|\phi\|_{\mathcal{C}}$.

Example

Let $\mathbb{D} = \{s \in \mathbb{C} | |s| < 1\}$. For every $s \in \mathbb{D}$

$$\phi_s(n) = s^n \tag{2}$$

defines a radial multiplier M_ϕ on $\mathcal{A} = *_{i \in I}(\mathcal{A}_i, \omega_i)$ with

$$\|M_{\phi_s}\|_{cb} \leq \frac{|1-s|}{1-|s|}. \tag{3}$$

Strategy

- Uniqueness of M_ϕ
- Reduce to $A_i = B(H_i)$
- Equivalent description of M_ϕ
- Construct $\Phi^i_{x,y}$
- Construct T_1, T_2, T
- Show T is M_ϕ
- Estimate $\|T\|_{cb}$

Notation

$$H = \mathbb{C}\Omega \oplus \bigoplus_{n=0}^{\infty} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathring{H}_{i_1} \otimes \cdots \otimes \mathring{H}_{i_n}. \quad (4)$$

and denote basis by

$$\Lambda = \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{\gamma_1 \otimes \cdots \otimes \gamma_n | \gamma_j \in \mathring{\Gamma}_{i_j}, i_1 \neq \cdots \neq i_n\}. \quad (5)$$

For $\gamma \in H$, define $L_\gamma \in B(H)$ as

$$L_\gamma(\chi) = \begin{cases} \gamma \otimes \chi & \text{if } i \neq i_1 \\ 0 & \text{if } i = i_1 \end{cases}$$

For $\eta, \xi \in H$ let *case 2* if $\eta_{|\eta|}, \xi_{|\xi|} \in H_i$ and *case 1* otherwise.

Equivalent description of M_ϕ

Lemma

Let $T : B(H) \to B(H)$ *be a bounded linear normal map, and let* $\phi : \mathbb{N}_0 \to \mathbb{C}$ *be a function on* $\mathbb{N}_0$*. TFAE*

(a) $T(1) = \phi(0)1$ *and*

$$T(a_1 a_2 \dots a_n) = \phi(n) a_1 a_2 \dots a_n \tag{6}$$

whenever $a_j \in B(\mathring{H}_{i_j}) = ker(\omega_{i_j})$ *and* $i_1 \neq i_2 \neq \dots \neq i_n$*.*

(b) *For all* $k, l \in \mathbb{N}_0$ *and* $\xi \in \Lambda(k), \eta \in \Lambda(l)$ *we have*

$$T(L_\xi L_\eta^*) = \begin{cases} \phi(k+l) L_\xi L_\eta^* & \text{in case 1} \\ \phi(k+l-1) L_\xi L_\eta^* & \text{in case 2.} \end{cases} \tag{7}$$

Construction of maps

For $x, y \in l^2(\mathbb{N}_0)$ and $a \in B(H)$ put

$$\Phi_{x,y}^{(1)}(a) = \sum_{n=0}^{\infty} D_{(S^*)^n x} a D_{(S^*)^n y}^* + \sum_{n=1}^{\infty} D_{S^n x} \rho^n(a) D_{S^n y}^*$$

$$\Phi_{x,y}^{(2)}(a) = \sum_{n=0}^{\infty} D_{(S^*)^n x} a D_{(S^*)^n y}^* + \sum_{n=1}^{\infty} D_{S^n x} \rho^{n-1}(\epsilon(a)) D_{S^n y}^*$$

$$T_1 = \sum_{i=1}^{\infty} \Phi_{x_i,y_i}^{(1)} \qquad \text{for } h = \sum_{i=1}^{\infty} x_i \odot y_i$$

$$T_2 = \sum_{i=1}^{\infty} \Phi_{z_i,w_i}^{(2)} \qquad \text{for } k = \sum_{i=1}^{\infty} z_i \odot w_i$$

$$T = T_1 + T_2 + cI.$$

Properties of $\Phi^{(i)}_{x,y}$

Lemma

If $\xi \in \Lambda(k), \eta \in \Lambda(l)$ *then*

$$\Phi^{(1)}_{x,y}(L_\xi L_\eta^*) = \left(\sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)} \right) L_\xi L_\eta^*$$

and

$$\Phi^{(2)}_{x,y}(L_\xi L_\eta^*) = \begin{cases} \sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)} L_\xi L_\eta^* & \text{in case 1} \\ \sum_{t=0}^{\infty} x(k+t-1)\overline{y(l+t-1)} L_\xi L_\eta^* & \text{in case 2.} \end{cases}$$

Properties of $\phi \in \mathcal{C}$

Lemma

With

$$\phi(n) = \psi_1(n) + \psi_2(n) + c$$

$$\psi_1(k+l) = \sum_{i=1}^{\infty}\sum_{t=0}^{\infty} x_i(k+t)\overline{y_i(l+t)} \tag{8}$$

$$\psi_2(k+l) = \sum_{i=1}^{\infty}\sum_{t=0}^{\infty} z_i(k+t)\overline{w_i(l+t)}.$$

$\cdots$ Hence T has the right behavior

Estimate $\|T\|_{cb}$

$$\|\Phi^{(1)}_{x_i,y_i}\|_{cb} \leq \|x_i\|_2\|y_i\|_2$$

$$\|\Phi^{(2)}_{z_i,w_i}\|_{cb} \leq \|z_i\|_2\|w_i\|_2$$

$$\|T_1\|_{cb} \leq \sum_{i=1}^{\infty} \|\Phi^{(1)}_{x_i,y_i}\|_{cb} \leq \|h\|_1$$

$$\|T_2\|_{cb} \leq \sum_{i=1}^{\infty} \|\Phi^{(2)}_{z_i,w_i}\|_{cb} \leq \|k\|_1$$

$$\|T\|_{cb} \leq \|T_1\|_{cb} + \|T_2\|_{cb} + \|cId\|_{cb} \leq \|\phi\|_{\mathcal{C}}$$

Open questions

- For which $(A_i, \omega_i)_{i \in I}$ holds $\|M_\phi\|_{cb} = \|\phi\|_{\mathcal{C}}$ for all $\phi \in \mathcal{C}$?
- Find other examples for $\phi \in \mathcal{C}$ with calculable $\|\phi\|_{\mathcal{C}}$
- Find $\phi \in \mathcal{C}$ with finite support
- Find $\phi_n \in \mathcal{C}$ where $\phi_n \to 1$ pointwise