Centers of C^*-algebras rich in modular ideals

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Clearly every primitive ideal of a C^*-algebra A that does not contain the center of A is modular. It is also obvious that the set of all these ideals is open in $\mathrm{Prim}(A)$. Thus, if the center of A is nonzero, the set of its modular primitive ideals has a nonempty interior in $\mathrm{Prim}(A)$.

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The main purpose of a 1968 paper of C. Anantharaman-Delaroche is an investigation of the converse: does the existence of a nonempty open set of modular primitive ideals imply a nonzero center? Among other results, an affirmative answer is obtained for liminal C^*-algebras.

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However, two examples of postliminal C^*-algebras with zero center are given there: one separable which has a nonempty open set of modular primitive ideals and another one that is nonseparable but whose all primitive ideals are modular.

I discuss conditions which ensure that a C^*-algebra with 'many' modular ideals has a nonzero center. In particular, I shall treat the case of a postliminal algebra. Further I shall give an example of a postliminal AF algebra with zero center whose all primitive ideals are modular. This answers a question of Anantharaman-Delaroche in that 1968 paper.

By the term ideal we shall mean everywhere a two sided closed ideal. $\mathrm{Id}(A)$ will denote the collection of all the ideals of the C^*-algebra A. For $I \in \mathrm{Id}(A)$ we shall let $\theta_I : A \to A/I$ be the quotient map.

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On $\mathrm{Id}(A)$ I shall consider a compact Hausdorff topology; a net $\{I_\alpha\}$ converges to I in this topology if and only if $\|\theta_{I_\alpha}(a)\| \to \|\theta_I(a)\|$ for every $a \in A$. This topology was discussed on some length by R. Archbold in a paper on primal ideals. If it is not mentioned otherwise, $\mathrm{Id}(A)$ and its subsets will be endowed with this topology.

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However, on the primitive ideal space of A, denoted $\mathrm{Prim}(A)$, I shall always work with the usual Jacobson topology.

A primal ideal I of a C^*-algebra A is defined by the following property: whenever $I_1, \ldots I_n$, $n \geq 2$, are ideals of A such that $I_1 \cdot I_2 \cdots I_n = \{0\}$ then $I_k \subseteq I$ for some k. Every prime (in particular every primitive) ideal is primal and by using Zorn's lemma one sees that every primal ideal must contain a minimal primal ideal. The collection of all the minimal primal ideals of A is denoted by $\mathrm{Min\text{-}Primal}(A)$.

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Two primitive ideals P, Q of the C^*-algebra A are said to be equivalent if $f(P) = f(Q)$ for every continuous $f : \mathrm{Prim}(A) \to \mathbb{C}$. Each equivalence class is the hull of an ideal called a Glimm ideal of A; the collection of these ideals is denoted $\mathrm{Glimm}(A)$ and the quotient map $\phi_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ is called the complete regularization map; it was introduced in the well known Memoir of Dauns-Hofmann and discussed further in a 1990 paper of R. Archbold and D. Somerset. $\mathrm{Glimm}(A)$ will be considered with its quotient topology induced by this map.

The Dauns-Hofmann theorem implies that if $a \in A$ and $f : \mathrm{Glimm}(A) \to \mathbb{C}$ is a bounded continuous function then there is a unique $b \in A$ such that $\theta_G(b) = f(G)\theta_G(a)$ for every $G \in \mathrm{Glimm}(A)$.

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An ideal I is called semi-Glimm if it contains a Glimm ideal; this Glimm ideal is necessarily unique since its hull must contain the hull of I. Obviously every Glimm ideal is semi-Glimm and every proper primal ideal is semi-Glimm by a result of Archbold and Somerset. Set $S - \mathrm{Glimm}(A)$ for the family of all the semi-Glimm ideals of A and let ψ_A be the map that takes each $I \in S - \mathrm{Glimm}(A)$ to the Glimm ideal it contains. It is easily seen that ψ_A is continuous.

Positive results

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A family $\mathcal{F}$ of ideals of the C^*-algebra A is called sufficiently large if $\cup\{\mathrm{Prim}(A/I) \mid I \in \mathcal{F}\}$ is dense in $\mathrm{Prim}(A)$.

Theorem

Let A be a C^*-algebra that has a countable approximate identity and suppose there exists a sufficiently large Baire subspace $\mathcal{S}$ of $\underline{S - Glimm}(A)$ consisting of modular ideals. Suppose, moreover, that every non-void (relatively) open subset of $\mathcal{S}$ contains the preimage by $\psi_A^{\mathcal{S}} := \psi_A|\mathcal{S}$ of a non-void relatively open subset of $\psi_A^{\mathcal{S}}(\mathcal{S}) \subseteq \underline{Glimm}(A)$. Then A has a non-zero center.

The first consequence one can draw from the above result has also been obtained by Archbold and Somerset in a paper to appear in Münster J. Math.

Proposition

Let A be a C^*-algebra with a countable approximate identity. Suppose that $\phi_A : \underline{Prim}(A) \to \underline{Glimm}(A)$ is open and each Glimm ideal is modular. Then A has a non-zero center.

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From the openness of the the complete regularization map one infers that $G \to \|\theta(a)\|$ is continuous on $(\mathrm{Glimm}(A), \tau_q)$ for every $a \in A$, hence the identity map from $(\mathrm{Glimm}(A), \tau)$ to $(\mathrm{Glimm}(A), \tau_q)$, which is the restriction of ψ_A to $(\mathrm{Glimm}(A), \tau)$, is a homeomorphism. From the fact that ϕ_A is open one also infers that $(\mathrm{Glimm}(A), \tau_q)$ is a locally compact Hausdorff space hence a Baire space. The Theorem yields the conclusion.

A topological space X is called quasi-completely regular if for every non-void open subset U of X there is a non-zero real valued continuous function on X that is identically 0 on $X \setminus U$. Such spaces were called "quasi-uniformisable" by Anantharaman-Delaroche but this term is nowadays used in another sense in topology.

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The restriction φ_A of ψ_A to $\mathrm{Min\text{-}Primal}(A)$ maps this space onto $\mathrm{Glimm}(A)$, again since every primitive ideal contains a minimal primal ideal. One must show that every non-void open subset of $\mathrm{Min\text{-}Primal}(A)$ contains the preimage by φ_A of an open subset of $\mathrm{Glimm}(A)$. So let $\mathcal{U}$ be a non-void open subset of $\mathrm{Min\text{-}Primal}(A)$. By using the fact, proved by Archbold, that τ on $\mathrm{Min\text{-}Primal}(A)$ is equal to the restriction of a weaker topology on $\mathrm{Id}(A)$ one gets an ideal J of A such that $\{I \in \mathrm{Min\text{-}Primal}(A) \mid J \not\subseteq I\} \subseteq \mathcal{U}$. $\mathrm{Prim}(A)$ is quasi-completely regular so there exists a non-zerocontinuous function $f : \mathrm{Prim}(A) \to \mathbb{R}$ that vanishes off $\mathrm{Prim}(J)$. Let $g : \mathrm{Glimm}(A) \to \mathbb{R}$ be such that $f = g \circ \phi_A$. Then $\{G \in \mathrm{Glimm}(A) \mid g(G) > 0\}$ is open and its preimage by φ_A is contained in $\mathcal{V}$. The other conditions of the Theorem are easily verified.

The C^*-algebra obtained by adjoining a unit to the ideal of compact operators on an infinite-dimensional Hilbert space is an example that satisfies the conditions of the first Proposition but not those of the second Proposition. There is an example of a C^*-algebra in the situation described by the second Proposition for which the complete regularization map is not open.

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A C^*-algebra A was called by Anantharaman-Delaroche generalized quasi-central if for every ideal I of A, $I \neq A$, the center of A/I is non-zero.

Corollary

Let A be a C^*-algebra that has a countable approximate identity. Suppose that every minimal primal ideal of A is modular and every closed subset of $\mathrm{Prim}(A)$ is a quasi-completely regular space with its relative topology. Then A is generalized quasi-central

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Obviously every quasi-completely regular space has the property that every non-empty open subset contains a closed subset with non-empty interior. In certain topological spaces this easily verifiable property implies that the space is quasi-completely regular. Namely, this is true for locally compact spaces that contain an open dense Hausdorff subset.

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Proposition

Let A be a postliminal C^*-algebra with a countable approximate identity. Suppose that every minimal primal ideal of A is modular and $\mathrm{Prim}(A)$ has the property that every non-void open subset of $\mathrm{Prim}(A)$ contains a closed subset with non-empty interior. Then A has a non-zero center.

An example

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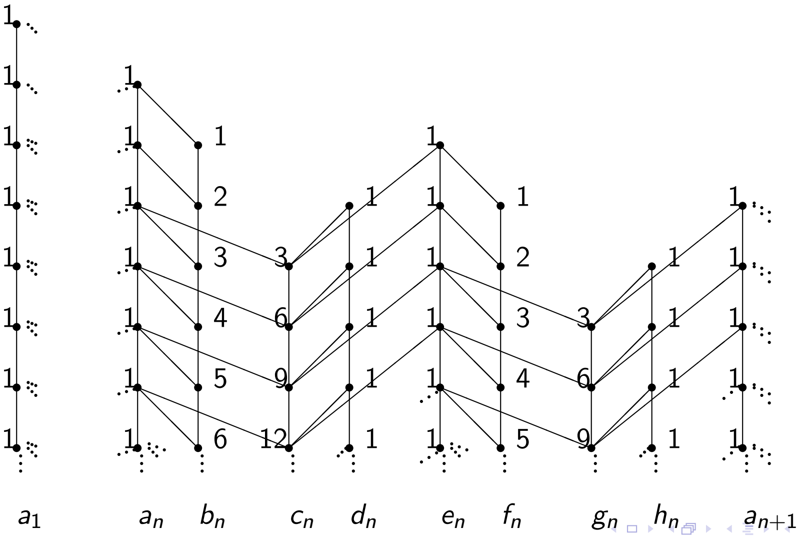
As promised I present now a postliminal (separable) AF algebra whose all primitive ideals are modular but with center reduced to $\{0\}$. As a matter of fact, all the minimal primal ideals of this algebra are modular so the hypothesis made on the primitive ideal space in the last Proposition cannot be eliminated.


1 1 1 1 1 1 1 1
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1 1 1 1 1
a_1 a_n b_n c_n d_n e_n f_n g_n h_n a_{n+1}

This is a Bratteli diagram of an AF algebra, A say. In this diagram the first vertex of the connected sequence a_1 should be thought at the level 1 while the first vertex of the connected sequence a_n should be imagined at the level $1 + 2(n - 1)$.

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Recall that a subdiagram E of a diagram D of an AF algebra A is the diagram of an ideal I of A if and only if E has the following two properties: the descendants of every vertex of E belong to E and if every descendant of a vertex belongs to E then that vertex itself belongs to E. If this is the case then $D \setminus E$ is a diagram of A/I. The ideal I is primitive if and only if every two vertices in $D \setminus E$ have a common descendant in $D \setminus E$.

It can be immediately checked that the diagram presented has the property that for every connected sequence $\{x_m\}_{m=1}^{\infty}$ in it, the vertex x_{m+1} is a descendant of x_m with multiplicity one. Hence, by a result of AJL and D. Taylor (1980), A is a postliminal algebra.

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By direct examination one finds that the primitive quotients of A have one of the following diagrams: $\{a_n\}$, $\{a_n, b_n\}$, $\{a_n, c_n, d_n, e_n\}$, $\{d_n\}$, $\{e_n\}$, $\{e_n, f_n\}$, $\{e_n, g_n, h_n, a_{n+1}\}$, $\{h_n\}$, $n = 1, 2, \ldots$. Denote the primitive ideals determined by the complementary diagrams by P_n, Q_n, R_n, S_n, T_n, U_n, V_n, W_n, $n = 1, 2, \ldots$, respectively.

All the quotients of A by its minimal primal ideals are modular. Indeed, by a result of F. Beckhoff, an ideal I of an AF algebra is primal if and only if its associated diagram D_I has the property that every finite set of vertices not in D_I has a common descendant in the diagram of the algebra. It is then easily seen that all the diagrams of the quotients of A by the minimal primal ideals are $\{a_n, b_n\}$, $\{a_n, c_n, d_n, e_n\}$, $\{e_n, f_n\}$, $\{e_n, g_n, h_n, a_{n+1}\}$, $n = 1, 2, \ldots$ and all these are diagrams of unital AF algebras.

Now I am going to show that there are no nonzero elements in the center of A. To this end I prove that every real valued continuous function on $\mathrm{Prim}(A)$ is constant. First remark that by the definition of the hull-kernel topology of the primitive ideal space we have:

$$\overline{\{P_n\}} = \{P_n\}, \overline{\{Q_n\}} = \{Q_n, P_n\}, \overline{\{R_n\}} = \{R_n, P_n, S_n, T_n\}, \overline{\{S_n\}} = \{S_n\},$$
$$\overline{\{T_n\}} = \{T_n\}, \overline{\{U_n\}} = \{U_n, T_n\}, \overline{\{V_n\}} = \{V_n, T_n, W_n, P_{n+1}\}, \overline{\{W_n\}} = \{W_n\}.$$

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If $f : \mathrm{Prim}(A) \to \mathbb{R}$ is a continuous function with $f(P_1) = \cdots = f((P_n) = \alpha$, then by the above equalities we must have $\alpha = f(P_n) = f(Q_n) = f(R_n) = f(S_n) = f(T_n) = f(U_n) = f(V_n) = f(W_n) = f(P_{n+1})$ and we conclude that f is a constant function.

One gathers from the Dauns-Hofmann theorem that the center of the multiplier algebra of A consists only of the scalar multiples of the unit. On the other hand, A has no unit since vertices without ancestors appear at infinitely many levels and we are done.

Thank you for your attention