C^*-algebras associated to product systems of C^*-correspondences

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Outline

- The Toeplitz algebra of a C^*-correspondence.
- Product system X of C^*-correspondences over a semigroup P.
- When (G, P) is quasi-lattice ordered: look for compactly aligned X.
- C^*-algebras of product systems: Fowler's Toeplitz algebra, Toeplitz covariant algebra and Cuntz-Pimsner algebra, and Sims and Yeend's Cuntz-Nica-Pimsner algebra.
- A universal and a co-universal C^*-algebra. A gauge-invariant uniqueness result.
- Examples.

The Toeplitz algebra of a C^*-correspondence

X is a C^*-correspondence over A if X is a right Hilbert A-module with a homomorphism $\phi : A \to \mathcal{L}(X)$ (also say X is a right-Hilbert A–A-bimodule).

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- Toeplitz representation: a linear map $\psi : X \to B(H)$ and a homomorphism $\pi : A \to B(H)$ compatible with module actions and s.t. $\psi(\xi)^*\psi(\eta) = \pi(\langle \xi, \eta \rangle_A)$ for $\xi, \eta \in X$.

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- There is a universal algebra $\mathcal{T}_X$ for Toeplitz representations, and is generated by $i = (\psi_0, \pi_0)$: any (ψ, π) gives rise to a repr. $\psi \times \pi$ of $\mathcal{T}_X$ on H s.t. $(\psi \times \pi) \circ i$ restricts to (ψ, π).

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- A *concrete* algebra $\mathcal{T}_X^{F(X)}$ on the Fock space $F(X)$. Fact: $\mathcal{T}_X \cong \mathcal{T}_X^{F(X)}$ (Pimsner, 1994).

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X can be thought of as a generalised endomorphism of A and $\mathcal{T}_X$ as a kind of crossed product of A by $\mathbb{N}$.

X: product system over P is a semigroup with a homomorphism $d\colon X \to P$ s.t. $X_p := d^{-1}(p)$ is a C^*-correspondence over A for $p \in P$ and $X_e = {}_AA_A$,

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and the right and left actions of X_e (Arveson, Dinh, Fowler).

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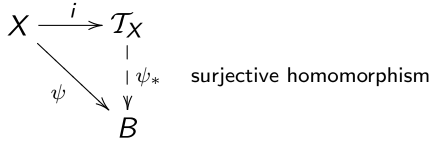

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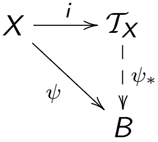

surjective homomorphism

$\mathcal{T}_X$: a sort of crossed product of A by generalised action of P.

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Facts: $\mathfrak{l}$ is a Toeplitz representation of X (take $\mathfrak{l}_e = \oplus_s \phi_s$). It is *isometric* (i.e. $\mathfrak{l}_e$ is injective) because ϕ_e is.

$(\mathfrak{l}_s, \mathfrak{l}_e)$ is a Toeplitz repr. of X_s for $s \in P$. By Pimsner, there is a homomorphism $\mathfrak{l}^{(s)} : \mathcal{K}(X_s) \to \mathcal{L}(F(X))$ with $\mathfrak{l}^{(s)}(\theta_{\xi,\eta}) = \mathfrak{l}_s(\xi)\mathfrak{l}_s(\eta)^*$.

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For $q \in sP$ let $i_s^q : \mathcal{L}(X_s) \to \mathcal{L}(X_q)$ be

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What can be said of $K_{s,r} := \mathfrak{l}^{(s)}(\theta_{\xi,\eta})\mathfrak{l}^{(r)}(\theta_{z,w})$ in $\mathcal{L}(F(X))$?

Quasi-lattice ordered groups (A. Nica 1992). G a discrete group, P a subsemigroup with $P \cap P^{-1} = \{e\}$. Partial order on G: $g \leq h \iff g^{-1}h \in P$.

(G, P) is quasi-lattice ordered (q.l.o.) if every pair $p, q \in G$ with a common upper bound in G has a l.u.b. $p \vee q$. If so, write $p \vee q < \infty$, or else $p \vee q = \infty$.

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Def. (Fowler, 2002). X is compactly aligned if

$$i_s^{s\vee r}(S)i_r^{s\vee r}(R) \in \mathcal{K}(X_{s\vee r}),$$

whenever $S \in \mathcal{K}(X_s)$, $R \in \mathcal{K}(X_r)$, $s \vee r < \infty$.

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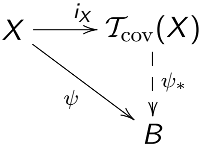
The Toeplitz covariant algebra is $\mathcal{T}_{\mathrm{cov}}(X) := \mathcal{T}_X/\mathcal{I}$ and is generated by $i_X = q_{\mathcal{I}} \circ i$ which is Nica covariant (Fowler, Carlsen-L-Sims-Vittadello). Universal property



surjective homomorphism

Example 1. (Fowler 2002) Y is a right-Hilbert A–A-bimodule. Put $Y_0 = A$ and $Y_n = Y^{\otimes n}$ for $n \geq 1$. Then $Y^{\otimes} = \sqcup_n Y_n$ is a product system over $\mathbb{N}$ with $\mathcal{T}_{Y^{\otimes}} \cong \mathcal{T}_Y$. Now $\mathbb{N}$ is totally ordered, so $i_s^{s\vee r}(S)$ or $i_r^{s\vee r}(R)$ is in $\mathcal{K}(X_{s\vee r})$. $Y^{\otimes}$ is compactly aligned and $\mathcal{T}_{\mathrm{cov}}(Y^{\otimes}) \cong \mathcal{T}_Y$.

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Pimsner: Y Hilbert bimodule over A with algebra $(\mathcal{T}_Y, i)$.
$\mathcal{O}_Y$ is the quotient of $\mathcal{T}_Y$ by the ideal $\mathcal{I}_0$ generated by $i^{(1)}(\phi(a)) - i|_A(a)$ for all a with $\phi(a) \in \mathcal{K}(Y)$.

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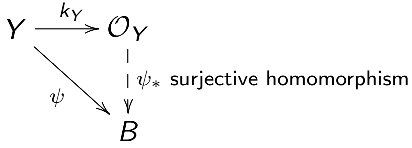

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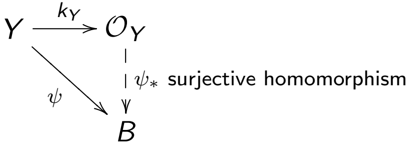

Katsura: uses largest $\mathcal{I}_Y$ on which ϕ is injective into $\mathcal{K}(Y)$. Theorem (Katsura 2004). The representation k_Y is injective and $\mathcal{O}_Y$ has the gauge-invariant uniqueness property: ψ_* is injective iff ψ is an injective repr. and B admits an action of $\mathbb{T}$ compatible with the gauge-action on $\mathcal{O}_Y$.

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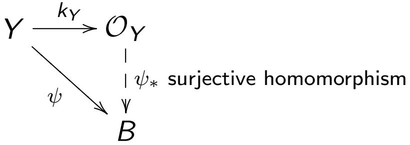

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Earlier: an Huef-Raeburn 1997, Fowler-Muhly-Raeburn 2003.

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When $\phi_m \in \mathcal{K}(X_m)$ for all $m \in P$ and $m \vee n < \infty$ for all $m, n \in P$ (e.g. for $(\mathbb{Z}^k, \mathbb{N}^k)$), the algebra $\mathcal{NO}_X$ is Fowler's $\mathcal{O}_X$.

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A coaction $\delta : A \to A \otimes C^*(G)$ is an injective nondegenerate homom. satisfying

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where $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ comes from $s \mapsto s \otimes s$. It is normal if $(\mathrm{id} \otimes \lambda_G) \circ \delta$ from $A \to A \otimes C_r^*(G)$ is injective.

A universal and a co-universal algebra

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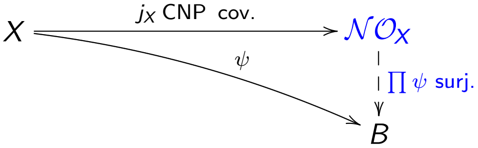

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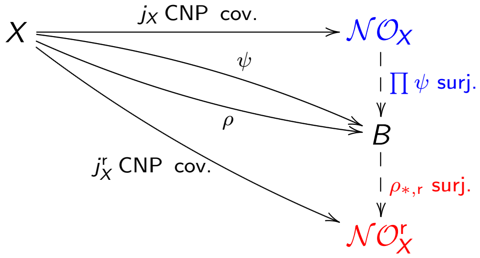

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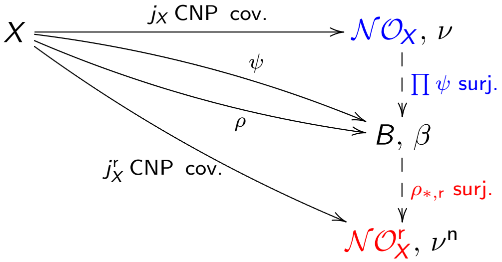

Theorem (Carlsen-L-Sims-Vittadello 2009). For X compactly aligned (with properties), $\mathcal{NO}_X^{\mathrm{r}}$ is co-universal for injective Nica covariant repr. ρ into B with a coaction β compatible with the normal coaction ν^{n}.

Back to example 2 where (G, P) is q.l.o and $\mathbb{C}^P$ has $X_p = \mathbb{C}$ for all $p \in P$. There is a *Nica spectrum* of (G, P) (Nica) and a boundary $\delta\Omega$ of Ω determined by elementary relations (Laca, Crisp-Laca).

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For (G, P) with either P directed (and so that $X \to \mathcal{NO}_X$ is an injective representation) or all left actions injective:

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by the co-universal property (Carlsen-L-Sims-Vittadello). As corollary $\mathcal{NO}_X \cong C(\delta\Omega) \times_\alpha G$ without having to check CNP covariance or the elementary relations.

The gauge-coactions

Let (G, P) q.l.o. and X compactly aligned. There is a coaction

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Theorem (Carlsen-L-Sims-Vittadello 2009). $\mathcal{NO}_X$ has the gauge-invariant uniqueness property precisely when it is isomorphic to $\mathcal{NO}_X^{\mathrm{r}}$. This is the case if, e.g., ν is also normal.

Another example (Carlsen-L-Sims-Vittadello): X_Λ product system over $\mathbb{N}^k$ for $k \geq 1$ from a topological higher-rank graph Λ of Yeend (Λ generalises the construction of topological graph of Katsura and of higher-rank graph of Kumjian-Pask).

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The gauge invariant uniqueness property for $\mathcal{NO}_X$ and maximal coactions; likewise (but differently), the gauge invariant uniqueness property of $\mathcal{NO}_X^r$ and normal coactions. (Kaliszewski-L-Quigg, work in progress). Same questions for $\mathcal{T}_{\mathrm{cov}}(X)$. Main point is to look at the Fell bundles.

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An aside: coactions and Fell bundles

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An aside: coactions and Fell bundles

A coaction $\delta : A \to A \otimes C^*(G)$ is a homom. s.t. $(\delta \otimes \mathrm{id}_{C^*(G)}) \circ \delta = (\mathrm{id}_A \otimes \delta_G) \circ \delta$ where $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ is the map $s \mapsto s \otimes s$. Let $A_g^\delta := \{ a \in A \mid \delta(a) = a \otimes i_G(g) \}$ for $g \in G$. The disjoint union $\mathcal{A} = \cup_g A_g^\delta \times \{g\}$ is a Fell bundle over G (Quigg 1996). Associated to a Fell bundle $\mathcal{A}$ there are a full cross sectional algebra $C^*(\mathcal{A})$ (Fell-Doran), and a reduced cross sectional algebra $C_r^*(\mathcal{A})$ – independently due to Exel (1997) and Quigg (1996) – and shown to be the same by Echterhoff and Quigg (1999). When $\mathcal{A}$ is the Fell bundle associated to a cosystem (A, G, δ), we let A^{r} be the reduced cross sectional algebra; there are a surjective homomorphism $\lambda_{\mathcal{A}} : A \to A^{\mathrm{r}}$ (Exel) and a normal coaction δ^{n} on A^{r} s.t. $\delta^{\mathrm{n}}(a_g) = a_g \otimes i_G(g)$ for $a_g \in A_g^\delta$ (Quigg). Normal means $(\mathrm{id} \otimes \lambda_G) \circ \delta^{\mathrm{n}}$ from $A^{\mathrm{r}} \to A^{\mathrm{r}} \otimes C_r^*(G)$ is injective.