A New Algorithm for Finding the Multivariable Alexander Polynomial of a Link

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University of California, Santa Barbara and Cardiff University

EU - NCG 4th Annual Meeting in Bucharest, Romania

Outline

- Alexander (1923/28) defined Alexander polynomial, $P(q)$
  “Topological Invariants of Knots and Links"

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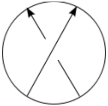
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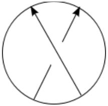
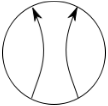

$= (q - q^{-1})$


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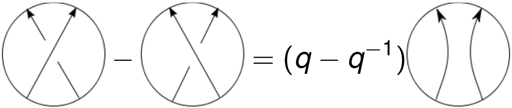

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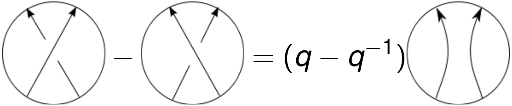

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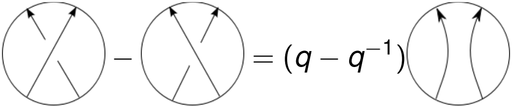
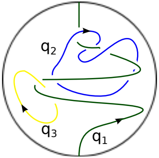


- Conway (1970) Multivariable Alexander Polynomial (MVAP) (or Conway Potential Function)
- Murakami (1993) List of axioms for the MVAP
- Bigelow (Spring 2010) diagrammatic algorithm for the single variable Alexander polynomial using planar algebras

Setup

Take knots and links to be oriented tangles with 1 unclosed strand with endpoints on the boundary of a disk

- Each component is assigned a “color" (or variable)



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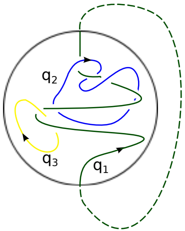

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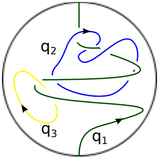

My Steps to Getting the MVAP

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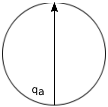
$$(\text{MVP}) \times$$ 

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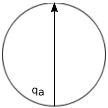
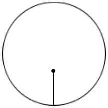
(MVP) $\times$ 

4. MVAP=(MVP)(a normalizing coefficient)

Let $\mathcal{P}'$ be the planar algebra generated by a single element in P_1:



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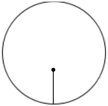
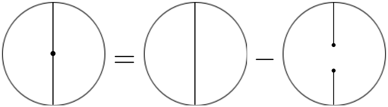


To ease notation, define the dotted strand in P_2 as



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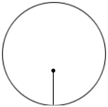

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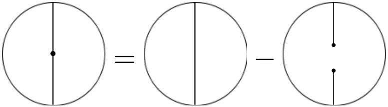
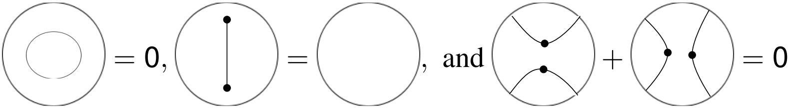


. The planar algebra $\mathcal{P}'$ has relations in P_0 and P_4 respectively



1. Resolve Crossings

Replace the crossings in the link with a linear combination of diagrams in P_4.

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And you get

$$\Sigma_{k=1}^{5^n} P_k(colors) D_k$$

where $n =$crossings and each D_k is a diagram in P_2

2. Applying Relations to get the MVAP

Apply relations in the planar algebra $\mathcal{P}'$ and the definition of the dotted strand.

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This will give a multivariable polynomial times a single strand, and this gives the MVAP up to a normalizing coefficient for R1 and the Murakami relations.

Theorem (Bigelow, K.)

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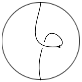
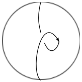
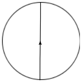
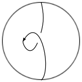
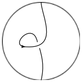
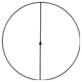
 $=$  $= -q^{-1}$ 

 $=$  $= -q$ 

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This algorithm defines the MVAP.

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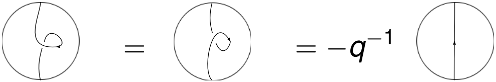
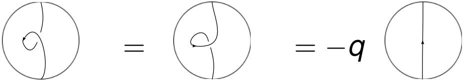
- R2, R3 exactly
- The following version or R1:





- and the Murakami Relations

Checking R2

There are four versions of R2 that we must check. Two of them must be checked by hand, and the other two follow from the other Reidemeister moves.

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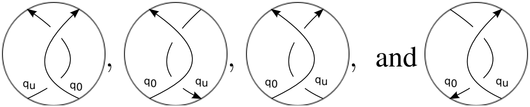
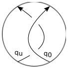


Manually Checking One Version of R2

 $=$

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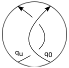
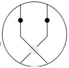
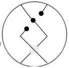


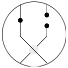
$= q_o$



$+ q_o$



$+ (q_u - q_u^{-1})$



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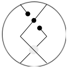

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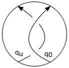
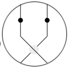
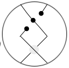
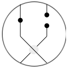
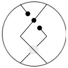
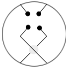
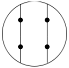
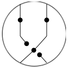
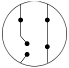
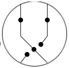
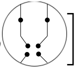


$= q_o$  $+ q_o$  $+ (q_u - q_u^{-1})$  $+ q_o^{-1}$  $- q_o^{-1}$ 

$= q_o \Big[q_o^{-1}$  $+ q_o^{-1}$  $- (q_u - q_u^{-1})$  $+ q_o$  $- q_o$  $\Big]$

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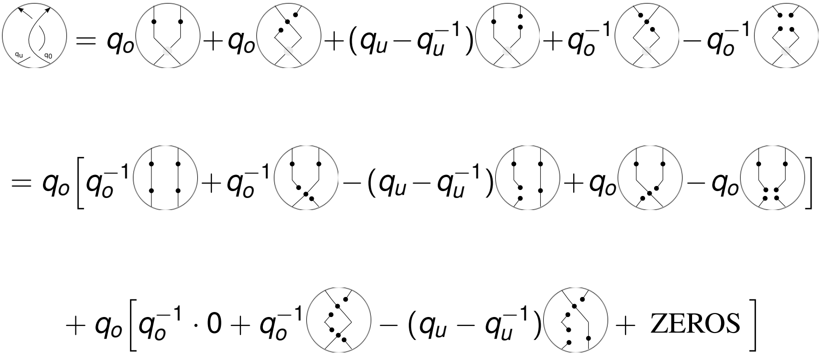

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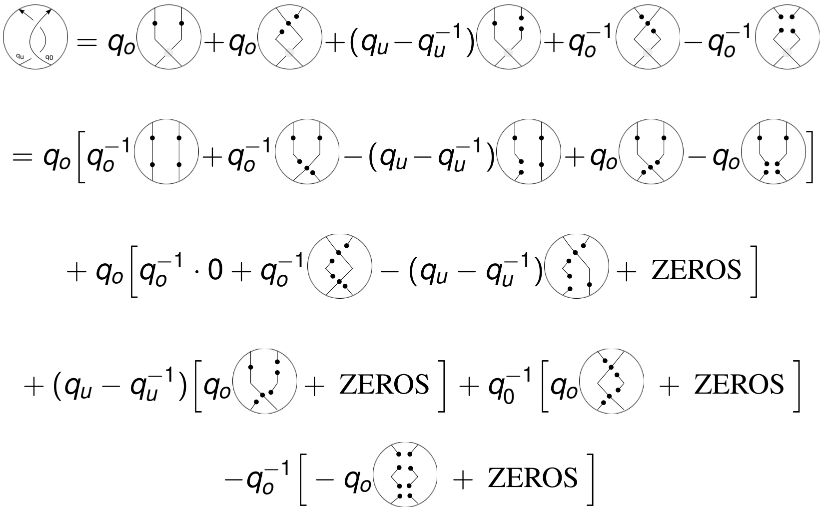

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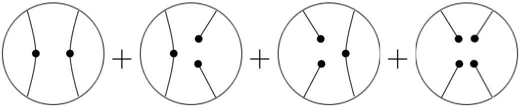

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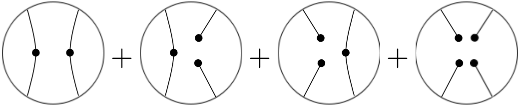

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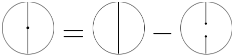
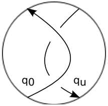
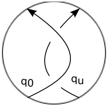
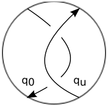
Two iterations of  will finish the verification.

Other versions of R2

- 

- 

- 

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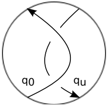
- 

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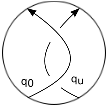
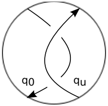
- 

- 

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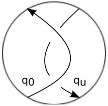
- 

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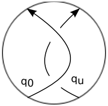
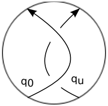
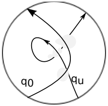
- 

Use R1, R2, R3, and 

$= (-q_u)^{-1}$ 

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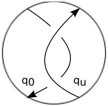
- 

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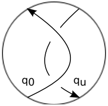
- 

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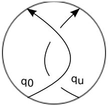
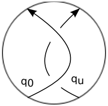
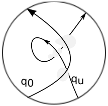
- 

Use R1, R2, R3, and 

$= (-q_u)^{-1}$ 

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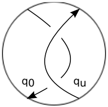
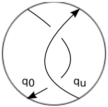
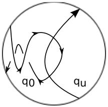
- 

Use R1, R2, R3, and 

$=$ 

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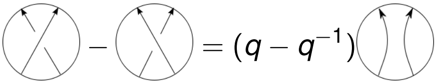
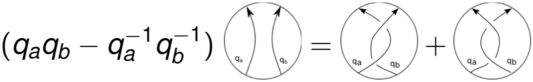
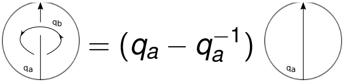
Murakami's Axioms

1. 

2. 

3. Gets its own slide.

4. If L is the trivial knot with color q_a, then $\Delta(L) = \frac{1}{q_a - q_a^{-1}}$.

5. 

6. If L is the split union of a link and trivial knot, then $\Delta(L)$ is zero.

Murakami's Third Axiom

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- From left to right the colors of the strands in B_3 are q_1, q_2, and q_3.

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$$f_+(q_1)f_-(q_2)\Delta([\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}]) - f_-(q_2)f_+(q_3)\Delta([\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}]) -$$

$$f_-(q_1^{-1}q_3)[\Delta([\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}]) + \Delta([\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}])] + f_-(q_1^{-1}q_2q_3)\Delta([\sigma_2^{-1}\sigma_2^{-1}])$$

$$-f_+(q_1)f_-(q_1q_2q_3^{-1})\Delta([\sigma_1^{-1}\sigma_1^{-1}]) + f_-(q_1^{-2}q_3^2)\Delta([e])$$

$$= \text{ZERO}$$

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Why we like this:
This invariant generalizes easily to tangle invariants up to R1.

- What does the image in $\mathcal{P}'$ of the tangles look like?
- Can this be done for the HOMFLY polynomial?
- Is it possible to find new knot invariants with this method?