

A New Algorithm for Finding the Multivariable Alexander Polynomial of a Link

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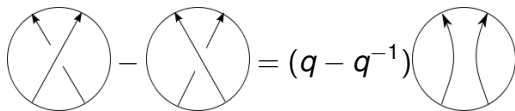
EU - NCG 4th Annual Meeting in Bucharest, Romania

Outline

- 1 Background
- 2 The Main Idea
- 3 The Planar Algebra \mathcal{P}'
- 4 The Algorithm
- 5 Theorem and Partial Proof
- 6 Conclusion

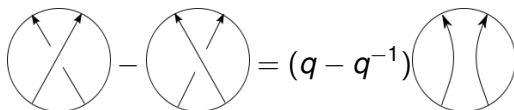
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The diagram illustrates the skein relation for the Alexander polynomial. It shows three circular diagrams arranged in a sequence, separated by a minus sign and an equals sign. The first diagram is a circle with two strands crossing each other, with arrows pointing upwards from the crossing. The second diagram is a circle with two strands crossing each other, with arrows pointing upwards from the crossing. The third diagram is a circle with two parallel strands, with arrows pointing upwards from the strands. The equation is: $\text{Crossing} - \text{Crossing} = (q - q^{-1}) \text{Parallel}$

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The diagram illustrates the skein relation for the Alexander polynomial. It shows three circular diagrams arranged horizontally, separated by a minus sign and an equals sign. The first diagram on the left has two strands crossing each other, with arrows pointing upwards. The second diagram in the middle has the same two strands, but they are parallel and do not cross. The third diagram on the right has two parallel strands, each with an arrow pointing upwards. The equation is: $(\text{crossing}) - (\text{parallel}) = (q - q^{-1})(\text{parallel})$.

- Conway (1970) Multivariable Alexander Polynomial (MVAP)
(or Conway Potential Function)

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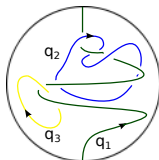
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- Murakami (1993) List of axioms for the MVAP
- Bigelow (Spring 2010) diagrammatic algorithm for the
single variable Alexander polynomial using planar algebras

Setup

Take knots and links to be oriented tangles with 1 unclosed strand with endpoints on the boundary of a disk

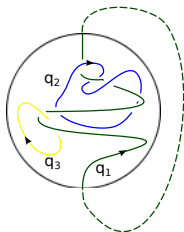
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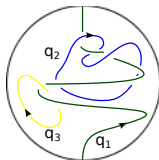
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My Steps to Getting the MVAP

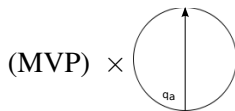
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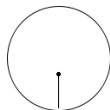
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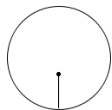
$$(\text{MVP}) \times \left(\begin{array}{c} \circlearrowleft \\ \uparrow \\ q_a \end{array} \right)$$

- 4 $\text{MVAP} = (\text{MVP})(\text{a normalizing coefficient})$

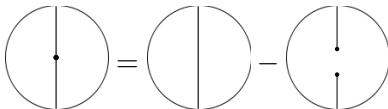
Let \mathcal{P}' be the planar algebra generated by a single element in \mathcal{P}_1 :



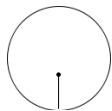
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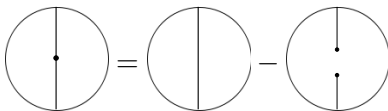
To ease notation, define the dotted strand in \mathcal{P}_2 as



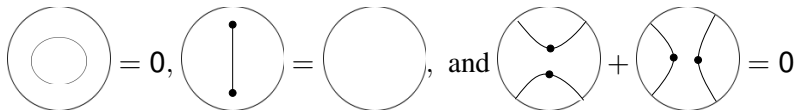
Let \mathcal{P}' be the planar algebra generated by a single element in P_1 :



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. The planar algebra \mathcal{P}' has relations in P_0 and P_4 respectively



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Replace the crossings in the link with a linear combination of diagrams in P_4 .

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And you get

$$\sum_{k=1}^{5^n} P_k(\text{colors}) D_k$$

where $n = \text{crossings}$ and each D_k is a diagram in P_2

2. Applying Relations to get the MVAP

Apply relations in the planar algebra \mathcal{P}' and the definition of the dotted strand.

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Apply relations in the planar algebra \mathcal{P}' and the definition of the dotted strand.

This will give a multivariable polynomial times a single strand, and this gives the MVAP up to a normalizing coefficient for R1 and the Murakami relations.

Theorem (Bigelow, K.)

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- R2, R3 exactly
- The following version or R1:

$$\begin{array}{c} \circlearrowleft \end{array} = \begin{array}{c} \circlearrowright \end{array} = -q^{-1} \begin{array}{c} | \\ \circlearrowleft \end{array}$$

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Theorem (Bigelow, K.)

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We need to show that these resolutions and relations satisfy

- R2, R3 exactly
- The following version of R1:

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} = -q^{-1} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array}$$

The first row shows a resolution of a crossing in a circle. The left diagram has a crossing where the strand from the top-left goes over the strand from the bottom-left. The middle diagram has the strand from the top-right going over the strand from the bottom-right. The right diagram is a circle with a vertical line through its center.

$$\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} = \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} = -q \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array}$$

The second row shows a resolution of a crossing in a circle. The left diagram has a crossing where the strand from the top-left goes under the strand from the bottom-left. The middle diagram has the strand from the top-right going under the strand from the bottom-right. The right diagram is a circle with a vertical line through its center.

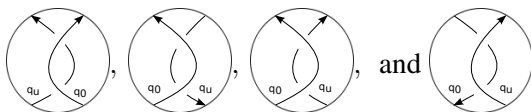
- and the Murakami Relations

Checking R2

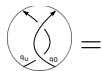
There are four versions of R2 that we must check. Two of them must be checked by hand, and the other two follow from the other Reidemeister moves.

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Manually Checking One Version of R2



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$$\begin{array}{c} \circlearrowleft \\ \text{qu} \quad \text{qo} \end{array} = q_o \begin{array}{c} \circlearrowleft \\ \text{qu} \quad \text{qo} \end{array} + q_o \begin{array}{c} \circlearrowleft \\ \text{qu} \quad \text{qo} \end{array} + (q_u - q_u^{-1}) \begin{array}{c} \circlearrowleft \\ \text{qu} \quad \text{qo} \end{array} + q_o^{-1} \begin{array}{c} \circlearrowleft \\ \text{qu} \quad \text{qo} \end{array} - q_o^{-1} \begin{array}{c} \circlearrowleft \\ \text{qu} \quad \text{qo} \end{array}$$

Manually Checking One Version of R2

$$\begin{aligned}
 \text{Diagram 1} &= q_o \text{Diagram 2} + q_o \text{Diagram 3} + (q_u - q_u^{-1}) \text{Diagram 4} + q_o^{-1} \text{Diagram 5} - q_o^{-1} \text{Diagram 6} \\
 &= q_o \left[q_o^{-1} \text{Diagram 7} + q_o^{-1} \text{Diagram 8} - (q_u - q_u^{-1}) \text{Diagram 9} + q_o \text{Diagram 10} - q_o \text{Diagram 11} \right]
 \end{aligned}$$

The diagrams are planar algebra generators:

- Diagram 1: A circle with two strands forming a loop, labeled q_u and q_o .
- Diagram 2: A circle with two strands crossing, labeled q_o .
- Diagram 3: A circle with two strands crossing, labeled q_o .
- Diagram 4: A circle with two strands crossing, labeled $(q_u - q_u^{-1})$.
- Diagram 5: A circle with two strands crossing, labeled q_o^{-1} .
- Diagram 6: A circle with two strands crossing, labeled q_o^{-1} .
- Diagram 7: A circle with two vertical strands, labeled q_o^{-1} .
- Diagram 8: A circle with two vertical strands, labeled q_o^{-1} .
- Diagram 9: A circle with two vertical strands, labeled $(q_u - q_u^{-1})$.
- Diagram 10: A circle with two vertical strands, labeled q_o .
- Diagram 11: A circle with two vertical strands, labeled q_o .

Manually Checking One Version of R2

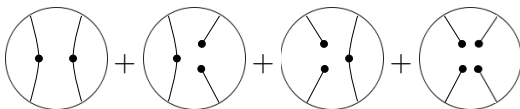
$$\begin{aligned}
 \text{Diagram 1} &= q_o \text{Diagram 2} + q_o \text{Diagram 3} + (q_u - q_u^{-1}) \text{Diagram 4} + q_o^{-1} \text{Diagram 5} - q_o^{-1} \text{Diagram 6} \\
 &= q_o \left[q_o^{-1} \text{Diagram 7} + q_o^{-1} \text{Diagram 8} - (q_u - q_u^{-1}) \text{Diagram 9} + q_o \text{Diagram 10} - q_o \text{Diagram 11} \right] \\
 &\quad + q_o \left[q_o^{-1} \cdot 0 + q_o^{-1} \text{Diagram 12} - (q_u - q_u^{-1}) \text{Diagram 13} + \text{ZEROS} \right]
 \end{aligned}$$

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 &\quad + q_o \left[q_o^{-1} \cdot 0 + q_o^{-1} \text{Diagram 12} - (q_u - q_u^{-1}) \text{Diagram 13} + \text{ZEROS} \right] \\
 &\quad + (q_u - q_u^{-1}) \left[q_o \text{Diagram 14} + \text{ZEROS} \right] + q_o^{-1} \left[q_o \text{Diagram 15} + \text{ZEROS} \right] \\
 &\quad - q_o^{-1} \left[-q_o \text{Diagram 16} + \text{ZEROS} \right]
 \end{aligned}$$

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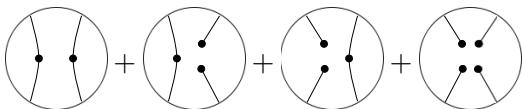
And the problem reduces to checking



evaluates to the same thing as two uncrossed strands.

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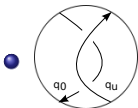
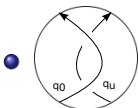
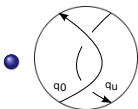
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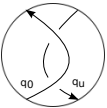
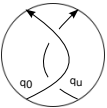
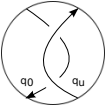
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Two iterations of $\begin{array}{c} \bigcirc \\ | \\ \bigcirc \end{array} = \begin{array}{c} \bigcirc \\ | \\ \bigcirc \end{array} - \begin{array}{c} \bigcirc \\ | \\ \bigcirc \end{array}$ will finish the verification.

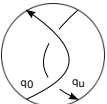
Other versions of R2

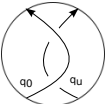


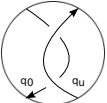
Other versions of R2

-  Just like the previous version of R2.
- 
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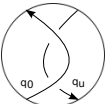
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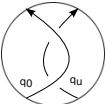
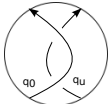
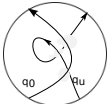
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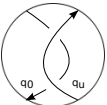
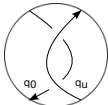

- 
 Use R1, R2, R3, and $\text{crossing} = (-q_u)^{-1} \text{crossing}$.

- 

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 $=$


Murakami's Axioms

1

$$\text{Diagram 1} - \text{Diagram 2} = (q - q^{-1}) \text{Diagram 3}$$

2

$$(q_a q_b - q_a^{-1} q_b^{-1}) \text{Diagram 4} = \text{Diagram 5} + \text{Diagram 6}$$

3 Gets its own slide.

4 If L is the trivial knot with color q_a , then $\Delta(L) = \frac{1}{q_a - q_a^{-1}}$.

5

$$\text{Diagram 7} = (q_a - q_a^{-1}) \text{Diagram 8}$$

6 If L is the split union of a link and trivial knot, then $\Delta(L)$ is zero.

Murakami's Third Axiom

- Defined in the braid group with three strands, B_3 .
- From left to right the colors of the strands in B_3 are q_1 , q_2 , and q_3 .

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Define $f_+(x) = x + x^{-1}$ and $f_-(x) = x - x^{-1}$.

$$\begin{aligned}
 & f_+(q_1)f_-(q_2)\Delta([\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}]) - f_-(q_2)f_+(q_3)\Delta([\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}]) - \\
 & f_-(q_1^{-1}q_3)[\Delta([\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}]) + \Delta([\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}])] + f_-(q_1^{-1}q_2q_3)\Delta([\sigma_2^{-1}\sigma_2^{-1}]) \\
 & - f_+(q_1)f_-(q_1q_2q_3^{-1})\Delta([\sigma_1^{-1}\sigma_1^{-1}]) + f_-(q_1^{-2}q_3^2)\Delta([e]) \\
 & = \text{ZERO}
 \end{aligned}$$

Conclusion and Future Work

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This invariant generalizes easily to tangle invariants up to R1.

- What does the image in \mathcal{P}' of the tangles look like?
- Can this be done for the HOMFLY polynomial?
- Is it possible to find new knot invariants with this method?