$N = 2$ superconformal field theory and noncommutative geometry

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Operator algebraic approach to conformal field theory

→ Relations between subfactor theory and noncommutative geometry

(with S. Carpi, R. Hillier, R. Longo and F. Xu)

Outline of the talk:

1. Conformal symmetry and the Virasoro algebras
2. Analogy between conformal field theory and differential geometry
3. Supersymmetry and the Dirac operator
4. $N = 2$ supersymmetry, the Doplicher-Haag-Roberts theory and the Jaffe-Lesniewski-Osterwalder cocycle

Our spacetime is S^1 and the spacetime symmetry group is the the infinite dimensional Lie group $\mathrm{Diff}(S^1)$. It gives a Lie algebra generated by $L_n = -z^{n+1}\dfrac{\partial}{\partial z}$.

The Virasoro algebra is a central extension of its complexification. It is an infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a central element c with the following relations.

$$[L_m, L_n] \ = \ (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c.$$

We have a good understanding of its irreducible unitary highest weight representations, where the central charge c is mapped to a positive scalar.

Fix a nice representation π of the Virasoro algebra, called a vacuum representation, and simply write L_n for $\pi(L_n)$.

Consider $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, called the stress-energy tensor, for $z \in \mathbb{C}$ with $|z| = 1$. Regard it as a Fourier expansion of an operator-valued distribution on S^1. This is a typical example of a quantum field.

Fix an interval I and take a C^∞-function f with $\mathrm{supp}\, f \subset I$. We have an (unbounded) operator $\langle L, f\rangle$ as an application of an operator-valued distribution.

Let $A(I)$ be the von Neumann algebra of bounded linear operators generated by these operators with various f. The family $\{A(I)\}$ gives an example of a conformal field theory.

Operator algebraic axioms: (conformal field theory)

Motivation: Operator-valued distributions $\{T\}$ on S^1.

Fix an interval $I \subset S^1$, consider $\langle T, f\rangle$ with $\operatorname{supp} f \subset I$.

$A(I)$: the von Neumann algebra generated by these (possibly unbounded) operators

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. **(a commutator)**
3. $\mathrm{Diff}(S^1)$**-covariance (conformal covariance)**
4. **Positive energy**
5. **Vacuum vector**

Such a family $\{A(I)\}$ is called a local conformal net.

Geometric aspects of local conformal nets

Classical geometry: Consider the Laplacian Δ on an n-dimensional compact oriented Riemannian manifold. The classical Weyl formula gives an asymptotic expansion

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots),$$

as $t \to 0+$, where a_0 is the volume of the manifold, and if $n = 2$, then a_1 is (constant times) the Euler characteristic of the manifold.

So the coefficients in the asymptotic expansion have a geometric meaning. We look for their analogues in the setting of local conformal nets.

The conformal Hamiltonian L_0 of a local conformal net is the generator of the rotation group of S^1.

For a nice local conformal net, we have an expansion

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots),$$

where a_0, a_1, a_2 are explicitly given. (K-Longo)

This gives an analogy of the Laplacian Δ of a manifold and the conformal Hamiltonian L_0 of a local conformal net.

A "square root" of the Laplacian gives a classical Dirac operator. The Connes approach in noncommutative geometry uses its abstract axiomatization.

Noncommutative geometry:

Noncommutative operator algebras are regarded as function algebras on noncommutative spaces.
In geometry, we need manifolds rather than compact Hausdorff spaces or measure spaces.

The Connes axiomatization of a noncommutative compact Riemannian spin manifold: a spectral triple $(\mathcal{A}, H, D)$.

1. $\mathcal{A}$: $*$-subalgebra of $B(H)$, the smooth algebra $C^\infty(M)$.
2. H: a Hilbert space, the space of L^2-spinors.
3. D: an (unbounded) self-adjoint operator with compact resolvents, the Dirac operator.
4. We require $[D, x] \in B(H)$ for all $x \in \mathcal{A}$.

$N = 1$ super Virasoro algebras: (Adding a square root of L_0)

The infinite dimensional super Lie algebras generated by central element c, even elements L_n, $n \in \mathbb{Z}$, and odd elements G_r, $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$, with the following relations:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
[G_r, G_s] &= 2L_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}c.
\end{aligned}$$

Ramond algebra, if $r \in \mathbb{Z}$.

Neveu-Schwarz algebra, if $r \in \mathbb{Z} + 1/2$.

We again consider a unitary representation of (one of) the $N = 1$ super Virasoro algebras. Consider $L(z)$ as before and $G(z) = \sum_r G_r z^{-r-3/2}$ as operator-valued distributions on S^1.

Using test functions supported in an interval I, they produce a family $\{A(I)\}$ of von Neumann algebras parametrized by $I \subset S^1$. This gives a superconformal net, for which now the bracket in the axioms means a graded commutator.

To make a further study in connection to noncommutative geometry, we work on $N = 2$ super Virasoro algebra and its unitary representations. Instead of one series $\{G_r\}$, we next have two series $\{G_r^{\pm}\}$ for the $N = 2$ case.

$N=2$ super Virasoro algebra: Generated by c, L_n, J_n and $G^{\pm}_{n\pm a}$, $n\in\mathbb{Z}$, with the following relations. (a: a parameter)

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c, \\
[J_m, J_n] &= \frac{m}{3}\delta_{m+n,0}c \\
[L_n, J_m] &= -mJ_{m+n}, \\
[G^+_{n+a}, G^+_{m+a}] &= [G^-_{n-a}, G^-_{m-a}] = 0, \\
[L_n, G^\pm_{m\pm a}] &= \left(\frac{n}{2}-(m\pm a)\right)G^\pm_{m+n\pm a}, \\
[J_n, G^\pm_{m\pm a}] &= \pm G^\pm_{m+n\pm a}, \\
[G^+_{n+a}, G^-_{m-a}] &= 2L_{m+n} + (n-m+2a)J_{n+m} + \\
&\qquad \frac{1}{3}\left((n+a)^2-\frac{1}{4}\right)\delta_{m+n,0}c.
\end{aligned}$$

The definition makes sense for a general $a \in \mathbb{R}$, but for all values a, the resulting super Lie algebras are mutually isomorphic. The isomorphism is given by the spectral flow.

We call the algebra the Ramond algebra for $a = 0$ and the Neveu-Schwarz algebra for $a = 1/2$ just as in the $N = 1$ super Virasoro algebra, but now the two super Lie algebras are isomorphic.

The spectral flow isomorphism can be also formulated in the framework of von Neumann algebras. The notions of Neveu-Schwarz and Ramond representations also make sense in the framework of Doplicher-Haag-Roberts representation theory.

It is known that an irreducible unitary representation maps c to a scalar in the set

$$\left\{ \frac{3m}{m+2} \,\middle|\, m = 1, 2, 3, \ldots \right\} \cup [3, \infty).$$

We consider only the case $c = 3m/(m+2)$ now.

We use $G^1_n = (G^+_n + G^-_n)/\sqrt{2}$ and $G^2_n = -i(G^+_n - G^-_n)/\sqrt{2}$. We fix a unitary representation and write L_n, G^1_n, G^2_n, J_n for their images in the representation. They are closed unbounded operators.

We then use the four operator-valued distributions $L(z) = \sum_n L_n z^{-n-2}$, $G^j(z) = \sum_n G^j_n z^{-n-3/2}$ $(j = 1, 2)$ and $J(z) = \sum_n J_n z^{-n-1}$, where $z \in \mathbb{C}$ with $|z| = 1$.

As before, using these four operator-valued distributions and test functions supported in $I \subset S^1$, we obtain a family of von Neumann algebras $\{A(I)\}$ parametrized by the intervals I.

We now would like to construct a family of spectral triples parameterized by the intervals I. We need the Dirac operator. We have two candidates G_0^1 and G_0^2 in the Ramond representation, and we can also mix them so that we have $(\cos t)G_0^1 + (\sin t)G_0^2$. They are unitarily equivalent, so we just choose G_0^1, and put $\delta(x) = [G_0^1, x]$ for a bounded linear operator x on the representation space. We put

$$\mathcal{A}(I) = A(I) \cap \bigcap_{n=1}^{\infty} \mathrm{dom}(\delta^n).$$

Each $\mathcal{A}(I)$ satisfies $\delta(\mathcal{A}(I)) \subset \mathcal{A}(I)$. That is, our spectral triple $(\mathcal{A}(I), H, G_0^1)$ gives a quantum algebra in the sense of Jaffe-Lesniewski-Osterwalder.

We now would like to make a study on some invariants in noncommutative geometry, and we deal with entire cyclic cohomology introduced by Connes.

Our Dirac operator $D = G_0^1$ satisfies the condition $\mathrm{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$. Such a spectral triple is called θ-summable and this condition gives a class of well-behaved "infinite dimensional noncommutative manifolds".

A JLO cocycle for a θ-summable spectral triple is defined and it gives an element in the entire cyclic cohomology.

The entire cyclic cohomology is a nice cohomology theory for dealing with "infinite dimensional" spectral triples, where the dimension in the sense of noncommutative geometry is defined in terms of the spectrum of D^2.

A cocycle here is given by a sequence of multilinear functionals on the $*$-algebra. A JLO cocycle is such a sequence of functionals defined in terms of traces and integrals involving e^{-tD^2} and $[D, \cdot\,]$.

We are interested in spectral triples arising from the Ramond representations with the lowest conformal weight $h = c/24$. There are different such representations and they are distinguished with $U(1)$-charge q.

For dealing with such representations, we introduce the universal von Neumann algebra for a local conformal net. This is related to a definition of Fredenhagen, and each representation gives a subfactor.

Within this von Neumann algebra, we define a $*$-subalgebra. It has different representations, corresponding to the $U(1)$-charge q, and each image gives a spectral triple with an appropriate Dirac operator in each representation. We thus have different JLO-cocycles for the same $*$-algebra.

The different representations give different projections in the $*$-algebra, called the characteristic projections. Each gives an element in the K_0-group of the $*$-subalgebra.

In general, we have the index pairing between the K_0-group and the entire cyclic cohomology, producing a number.

In the above, we have the K_0-elements depending on the Ramond representations and the JLO-cocycles in the entire cyclic cohomology depending on the same Ramond representations. Our result says that the pairing between them give the Kronecker δ of the representations. In this way, subfactor theory and noncommutative geometry are connected.

Carpi-Hillier have preceding results on such index pairing for examples arising from the loop group construction of superconformal nets.

Program: Operator Algebras and their Applications
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`http://www.ms.u-tokyo.ac.jp/~yasuyuki/rims2011.htm`

Kyoto is 500 km away from Fukushima.

See you in Kyoto!