# On crossed products of locally m-convex *-algebras

Maria Joița

University of Bucharest

EU-NCG 4, Bucharest 2011

# Locally m-convex *-algebras

## Definition

Let $A$ be a $*$-algebra. A submultiplicative $*$-seminorm on $A$ is a seminorm $p$ on $A$ which verifies:

1. $p(ab) \leq p(a)p(b), \forall a, b \in A;$
2. $p(a^*) = p(a), \forall a \in A.$

## Definition

A locally *m*-convex $*$-algebra (lmc $*$-algebra) is a Hausdorff topological complex $*$-algebra $A$ whose topology is determined by a directed family of submultiplicative $*$-seminorms.

# Locally m-convex *-algebras (continued)

## Examples

- Banach $*$-algebras
- Inverse limits of Banach $*$-algebras
- products of Banach $*$-algebras with the product topology

# Locally m-convex *-algebras(continued)

Let $A$ be an lmc $*$-algebra.

- $S(A) = \{p : A \to [0, \infty); p$ is a continuous submultiplicative $*$-seminorm$\}$
  - $S(A)$ is directed with the partial order $q \leq p$ if $q(a) \leq p(a)\ \forall a \in A$.
  - For $p \in S(A)$, $A/\ker p$ is a normed $*$-algebra in the $*$-norm $\|\cdot\|_p$ induced by $p$ (that is, $\|a\|_p = p(a) \forall a \in A$).
    Let $A_p$ be the completion of the normed $*$-algebra $A/\ker p$.
  - For $p, q \in S(A)$ with $q \leq p$ there is a continuous $*$-morphism with dense range $\pi_{pq}^A : A_p \to A_q$ such that
$$\pi_{pq}^A (a + \ker p) = a + \ker q.$$
- $\{A_p; \pi_{pq}^A\}_{p,q \in S(A)}$ is an inverse system of Banach $*$-algebras.

**Theorem**

*Let $A$ be a complete lmc $*$-algebra. Then*
$$A \equiv \lim_{\substack{\leftarrow \\ p}} A_p \quad \textit{up to a *-isomorphism.}$$

# Locally m-convex *-algebras(continued)

The terms used for locally m-convex $*$-algebras:

- m*- convex algebras
- Arens -Michael *- algebras ( A. Ya. Helemskii)

R. Arens and E. A. Michael (1952) studied independently lmc $*$-algebras as inverse limit of Banach $*$-algebras

# Pro-C *-algebras

## Definition

A pro-$C^*$-algebra is a complete lmc $^*$-algebra $A$ whose topology is determined by a directed family of $C^*$-seminorms (a $C^*$-seminorm on $A$ is a sumultipliative $*$-seminorm $p$ on $A$ with "C*-property"

$$p\left(a^* a\right) = p(a)^2, \forall a, b \in A).$$

The terms used for pro-$C^*$-algebras:

- locally $C^*$-algebras (A. Mallios, A. Inoue, M. Fragoulopoulou etc.)
- $LMC^*$ -algebras (G. Lassner, K. Schmüdgen)
- $b$ $*$-algebras (C. Apostol)

The term 'pro-$C^*$-algebra' was first used by D. Voiculescu and W. Arveson.

# Pro-C*-algebras (continued)

Let $A$ be a pro-$C^*$-algebra.

- $S(A) = \{p : A \to [0,\infty); p$ is a continuous $C^*$-seminorm$\}$
  - $S(A)$ is directed with the partial order $q \leq p$ if $q(a) \leq p(a)\, \forall a \in A$.
  - For $p \in S(A)$, $A/\ker p$ is a $C^*$-algebra in the $C^*$-norm $\|\cdot\|_p$ induced by $p$ ( that is, $\|a\|_p = p(a) \forall a \in A$).
    Let $A_p = A/\ker p$.
  - For $p, q \in S(A)$ with $q \leq p$ there is a surjective $C^*$-morphism $\pi^A_{pq} : A_p \to A_q$ such that

$$\pi^A_{pq}(a + \ker p) = a + \ker q.$$

- $\{A_p; \pi^A_{pq}\}_{p,q \in S(A)}$ is an inverse system of $C^*$-algebras.

**Theorem**

*Let $A$ be a pro-$C^*$-algebra. Then*

$$A \equiv \lim_{\substack{\leftarrow \\ p}} A_p \quad \text{up to an isomorphism of pro-}C^*\text{-algebras.}$$

# Pro-C*-algebras (continued)

## Examples

1. Any inverse limits of $C^*$-algebras is a pro-$C^*$-algebra.
2. If $X$ is a Hausdorff countably compactly generated topological space (that is, $\exists$ a countable family of compact spaces $K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n \subseteq \ldots$ such that $X = \lim\limits_{n\to} K_n$), then the $*$ -algebra $C(X)$ of all continuous complex valued functions on $X$ equipped with the topology defined by the family of $C^*$-seminorms $\{p_{K_n}\}_n$, where

$$p_{K_n}(f) = \sup\{|f(x)|, x \in K_n\},$$

   is a metrizable unital commutative pro-$C^*$-algebra.
3. $C_{cc}([0,1])$ the $*$-algebra of all complex valued continuous functions on $[0,1]$ with the topology of uniform convergence on the countable compact subsets of $[0,1]$ is a pro-$C^*$-algebra which is not topologically isomorphic to any $C^*$-algebra.
4. A product of $C^*$-algebras with the product topology is a pro-$C^*$-algebra.
5. The multiplier algebra of the Pedersen ideal of a $C^*$-algebra is a pro-$C^*$-algebra.

# The enveloping pro-C*-algebra of an lmc *-algebra

- R. M. Brooks (1971) constructed the enveloping pro-$C^*$-algebra of a unital metrizable lmc $^*$-algebra.
- A. Inoue (1971) constructed the enveloping pro-$C^*$-algebra of an lmc $^*$-algebra with bounded approximate unit.

# The enveloping pro-C*-algebra of an lmc *-algebra (continued)

Let $A$ be an lmc $^*$-algebra with bounded approximate unit.

**Definition**

A $*$-representation of $A$ on a Hilbert space $H$ is a continuous $*$-morphism $\varphi : A \to L(H)$.

- $\mathcal{R}(A) = \{\varphi; \varphi$ is a $*$-representation of $A\}$
- $\mathcal{R}_p(A) = \{\varphi; \varphi \in \mathcal{R}(A)$ with $\|\varphi(a)\| \leq p(a)\ \forall a \in A\}$
- $\mathcal{R}(A) = \cup_{p \in S(A)} \mathcal{R}_p(A)$
- $I = \{a \in A; \varphi(a) = 0 \forall \varphi \in \mathcal{R}(A)\}$ is a closed bilateral $*$-ideal of $A$.
    - $A/I$ is an algebra with involution.
    - For $p \in S(A), \widehat{p} : A/I \to [0, \infty)$, $\widehat{p}(a + I) = \sup\{\|\varphi(a)\|; \varphi \in \mathcal{R}_p(A)\}$ is a $C^*$-seminorm on $A/I$.

# The enveloping pro-C*-algebra of an lmc *-algebra (continued)

**Definition**

The pro-$C^*$-algebra $\mathcal{E}(A) = \overline{\left(A/I, \{\widehat{p}\}_{p\in S(A)}\right)}$ is called the enveloping pro-$C^*$-algebra of $A$.
$\delta_A : A \longrightarrow \mathcal{E}(A)$ denotes the canonical map.

**Proposition**

*Let $A$ be an lmc $^*$-algebra with bounded approximate unit. Then*

$$\mathcal{E}(A) \equiv \lim_{\substack{\longleftarrow \\ p}} \mathcal{E}(A_p).$$

# Locally m-convex *-algebras with enveloping C*-algebras

## Definition

We say that an lmc $^*$-algebra $A$ with bounded approximate unit has an enveloping $C^*$-algebra if the enveloping pro-$C^*$-algebra $\mathcal{E}(A)$ is isomorphic with a $C^*$-algebra.

S.J. Bhatt and D.J. Karia (1993) proved a necessary and sufficient condition for an lmc $^*$-algebra admits an enveloping $C^*$-algebra.

## Theorem

*An lmc $^*$-algebra with bounded approximate unit admits an enveloping $C^*$-algebra if and only if $A$ has a maximal continuous $C^*$-seminorm.*

# Locally m-convex *-algebras(continued)

## Examples

- $\mathcal{AC}^{\omega}[0,1] = \bigcap\limits_{n\geq 1} \mathcal{AC}^{n}[0,1]$, where

$$\mathcal{AC}^{n}[0,1] = \{f \in \mathcal{C}[0,1]; f' \text{ exists and } f' \in L^{n}[0,1]\}$$

is a **complete lmc $^*$-algebra** with pointwise operations, involution

$$f \to f^*, f^*(t) = \overline{f(t)}$$

and the topology given by the family of submultiplicative $*$-seminorms $\{p_n\}_{n\geq 1}$

$$p_n(f) = \|f\|_{\infty} + \left(\int\limits_0^1 |f'(t)|^n \, dt\right)^{\frac{1}{n}}.$$

$\mathcal{E}(\mathcal{AC}^{\omega}[0,1]) \sim \mathcal{C}[0,1]$ (S.J. Bhatt and D.J. Karia)

- Let $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$and $U = \{z \in \mathbb{C}; |z| < 1\}$.
  $\mathcal{A}^{\omega}(\mathbb{D}) = \bigcap\limits_{n \geq 0} \mathcal{A}^{n}(\mathbb{D})$ , where
  $\mathcal{A}^{n}(\mathbb{D}) = \{f : U \to \mathbb{C}; f$ is analytic and $f^{(k)}$ has continuous extension on $\mathbb{D}, \forall k$ with $0 \leq k \leq n\}$
  is a **complete lmc** $^*$**-algebra** with pointwise operations, involution

  $$f \to f^{*}, f^{*}(z) = \overline{f(\overline{z})}$$

  and the topology given by the family of submultiplicative $*$-seminorms $\{p_n\}_{n \geq 1}$,

  $$p_n(f) = \sum_{k=0}^{n} \frac{1}{k!} \sup\{\left|f^{(k)}(z)\right|; z \in \mathbb{D}\}.$$

  $\mathcal{E}(\mathcal{A}^{\omega}(\mathbb{D})) \sim \mathcal{C}[-1, 1]$ (S.J. Bhatt and D.J. Karia)

- Let $A = \{f \in C^{\infty}(\mathbb{R}), f^{(n)} \in L^1(\mathbb{R})\ \forall n \in \mathbb{N}\}$.
  $A$ with the topology determined by the family of submultiplicative $*$-seminorms $\{p_n\}_n$,

$$p_n(f) = \|f\|_1 + \left\|f^{(n)}\right\|_1$$

  is a complete lmc $*$-algebra.
  $\mathcal{E}(A) \sim \mathcal{E}\left(L^1(\mathbb{R})\right)$ (S.J. Bhatt and D.J. Karia)

# Group actions on lmc *-algebras (pro-C*-algebras)

Let $A$ be an lmc $^*$- algebra (pro-$C^*$-algebra).
$Aut(A) = \{\sigma : A \to A; \sigma$ is a $*$-isomorphism$\}$

## Definition

A **continuous action** of a locally compact group $G$ on an lmc $^*$-convex algebra (pro-$C^*$-algebra) $A$ is a group morphism $g \mapsto \alpha_g$ from $G$ to $Aut(A)$ such that the map

$$g \in G \to \alpha_g (a) \in A$$

is continuous for each $a \in A$.
We say that the **continuous action** $\alpha$ **is** $G$**-invariant** if for each $p \in S(A)$, there is $M_p > 0$ such that

$$p (\alpha_g (a)) \leq M_p p(a), \forall g \in G \text{ and } \forall a \in A.$$

# Group actions on lmc *-algebras (pro-C*-algebras) (continued)

Suppose that $\alpha$ is a $G$-invariant continuos action of $G$ on $A$.

- For $p \in S(A)$ and $g \in G \Longrightarrow \exists\ M_p > 0$ such that

$$p\left(\alpha_g\left(a\right)\right) \leq M_p p(a) \forall a \in A$$

$\Longrightarrow \exists\ \alpha_g^p : A_p \to A_p$ such that

$$\alpha_g^p \circ \pi_p^A = \pi_p^A \circ \alpha_g$$

Moreover, $\alpha_g^p \circ \alpha_{g^{-1}}^p = \alpha_{g^{-1}}^p \circ \alpha_g^p = \mathrm{id}_{A_p}$.

$\Longrightarrow \alpha_g^p \in \mathrm{Aut}(A_p) \Longrightarrow g \to \alpha_g^p$ is a continuous action of $G$ on $A_p$

- $\left(\alpha_g^p\right)_p$ is an inverse system of $*$-isomorphisms
- $\alpha_g = \lim\limits_{\leftarrow p} \alpha_g^p$

# Group actions on lmc *-algebras (pro-C*-algebras) (continued)

**Lemma**

*Let $\alpha$ be a $G$-invariant continuos action of a locally compact group $G$ on an lmc $^*$-algebra (pro-$C^*$-algebra) $A$. Then there are continuous actions $\alpha^p$ of $G$ on $A_p$, $p \in S(A)$ such that*

$$\alpha_g = \lim_{\leftarrow p} \alpha_g^p$$

*for each $g \in G$.*

# Group actions on lmc *-algebras

Suppose that $\alpha$ is a $G$-invariant continuous action of $G$ on a complete lmc *-algebra $A$ with bounded approximate unit.

- For each $g \in G$, $\alpha_g = \lim_{\leftarrow p} \alpha_g^p$, where $g \mapsto \alpha_g^p$ is a continuous action of $G$ on $A_p$, $p \in S(A)$.
- For $p \in S(A)$, $g \in G \Rightarrow \alpha_g^p \in \mathrm{Aut}(A_p) \Rightarrow \exists\ \widetilde{\alpha_g^p} \in \mathrm{Aut}(\mathcal{E}(A_p))$ such that
$$\widetilde{\alpha_g^p} \circ \delta_{A_p} = \delta_{A_p} \circ \alpha_g^p.$$
- For $p \in S(A) \Rightarrow g \mapsto \widetilde{\alpha_g^p}$ is a continuous action of $G$ on $\mathcal{E}(A_p)$.
- For each $g \in G$, $\left(\widetilde{\alpha_g^p}\right)_p$ is an inverse system of $C^*$-isomorphisms. Let $\widetilde{\alpha_g} = \lim_{\leftarrow p} \widetilde{\alpha_g^p}$.
- $g \mapsto \widetilde{\alpha_g}$ is a $G$-invariant continuous action of $G$ on $\mathcal{E}(A)$.

# Group actions on lmc *-algebras

## Proposition

*Any $G$-invariant continuous action $\alpha$ of a locally compact group $G$ on a complete lmc *-algebra $A$ with bounded approximate unit induces a $G$-invariant continuous action $\widetilde{\alpha}$ of $G$ on $\mathcal{E}(A)$.*

# Group actions on lmc *-algebras (pro-C*-algebras) (continued)

Examples

## Example

Let $(G, X)$ be a transformation group with $X$ a Hausdorff countably compactly generated topological space and $G$ a compact group. Then the map $\alpha_g : C(X) \to C(X)$ defined by

$$\alpha_g(f)(x) = f\left(g^{-1} \cdot x\right)$$

is an isomorphism of pro-$C^*$-algebras for each $g \in G$, and the map

$$g \to \alpha_g(f) \text{ from } G \text{ to } C(X)$$

is a $G$-invariant continuous action of $G$ on $C(X)$. Moreover, if $\alpha$ is a$G$-invariant action of $G$ on a metrizable unital commutative pro-$C^*$-algebra $A$, then there is a transformation group $(G, X)$, $X =$Sp$(A)$ which induces the action $\alpha$.

# Group actions on lmc *-algebras (pro-C*-algebras) (continued)

## Example

Let $A$ be a complete lmc $^*$-algebra with bounded approximate unit (pro-$C^*$-algebra) and $\varphi \in Aut(A)$.
The map $n \to \alpha_n$, where

$$\alpha_n = \varphi^n,$$

is continuous action of $\mathbb{Z}$ on $A$.
Moreover, if $\varphi$ is an inverse limit of $*$-isomorphisms, then $\alpha$ is $\mathbb{Z}$-invariant.

## Example

The map $t \to \alpha_t$ from $\mathbb{Z}_2$ to $\mathrm{Aut}(\mathcal{A}^\omega(\mathbb{D}))$, where

$$\alpha_0(f) = f \text{ and } \alpha_1(f)(z) = f(-z), \forall f \in \mathcal{A}^\omega(\mathbb{D}), \forall z \in \mathbb{D},$$

is a continuous action of $\mathbb{Z}_2$ on $\mathcal{A}^\omega(\mathbb{D})$.

# The covariance algebra

Let $\alpha$ be a $G$-invariant action of $G$ on a complete lmc $^*$-algebra $A$ with bounded approximate unit.

- $C_c(G, A) = \{f : G \to A; f$ is continuous with compact support$\}$ is a topological $*$-algebra with:
  - $(f * h)(t) = \int_G f(g)\alpha_g\left(h(g^{-1}t)\right) dg$
  - $f^{\#}(t) = \Delta\left(t^{-1}\right)\alpha_t\left(f\left(t^{-1}\right)\right)^*$, where $\Delta$ is the modular function on $G$
  - $\{N_p\}_{p \in S(A)}, N_p(f) = \int_G p(f(g))\, dg$.

**Definition**

The lmc $^*$-algebra $L^1(G, \alpha, A)$ obtained by the Hausdorff completion of $C_c(G, A)$ is called the covariance algebra associated to $\alpha$.

# The covariance algebra (continued)

## Proposition

*Let $\alpha$ be a $G$-invariant action of $G$ on a complete lmc $^*$-algebra $A$ with bounded approximate unit.*

1. $L^1(G, \alpha, A)$ *is a complete lmc* $^*$*-algebra with bounded approximate unit.*
2. $\{L^1(G, \alpha^p, A_p), \chi_{pq}\}_{p,q \in S(A), p \geq q}$, *where*

$$\chi_{pq} : \ L^1(G, \alpha^p, A_p) \to L^1(G, \alpha^q, A_q), \ \chi_{pq}(f) = \pi_p^A \circ f,$$

*is an inverse system of Banach $*$-algebras and moreover,*

$$L^1(G, \alpha, A) \equiv \lim_{\underset{p}{\leftarrow}} L^1(G, \alpha^p, A_p) \text{ up to an } * \text{-isomorphism.}$$

# Question

Let $\alpha$ be a $G$-invariant action of $G$ on a complete lmc $^*$-algebra $A$ with bounded approximate unit.

## Question

*When is $\mathcal{E}\left(L^1(G,\alpha,A)\right)$ isomorphic to a $C^*$-algebra?*

## Definition

The enveloping pro-$C^*$-algebra of $L^1(G,\alpha,A)$ is called the crossed product of $A$ by $\alpha$ and is denoted by $G\times_\alpha A$.

## Proposition

*Then*

$$G\times_\alpha A \equiv \lim_{\overleftarrow{p}} G\times_{\alpha^p} A_p \text{ up to an } *\text{-isomorphism.}$$

# Covariant representations (recall)

Let $A$ be a Banach $*$-algebra with approximate unit and $\alpha$ a continuous action of $G$ on $A$.

**Definition**

A (non-degenerate) covariant $*$-representation is a triple $(\varphi, u, H)$, where $(\varphi, H)$ is a (non-degenerate) $*$-representation of $A$ and $(u, H)$ is a unitary representation of $G$, such that $\varphi(\alpha_g(a)) = u_g \varphi(a) u_g^*, \forall a \in A, \forall g \in G$.

$\mathrm{Rep}(G, \alpha, A) = \{(\varphi, u, H)\,;\, (\varphi, u, H)$ is a non-degenerate covariant $*$-representation$\}$

**Proposition**

*The map* $(\varphi, u, H) \in Rep(G, \alpha, A) \to (\varphi \times u, H) \in Rep\left(L^1(G, \alpha, A)\right)$, *where*

$$(\varphi \times u)(f) = \int_G \varphi(f(t))\, u_t dt,$$

*is a bijective correspondence.*

### Lemma

*Let* $\alpha$ *be a continuous action of* $G$ *on a Banach* $*$*-algebra* $A$ *with approximate unit. Then the map*

$$(\varphi, H) \mapsto \left(\varphi \circ \delta_{L^1(G,\widetilde{\alpha},\mathcal{E}(A))} \circ j, H\right)$$

*where*

$$j : L^1(G, \alpha, A) \to L^1(G,\widetilde{\alpha},\mathcal{E}(A)), \; j(f) = \delta_A \circ f$$

*is a bijective correspondence between* $Rep(G \times_{\widetilde{\alpha}} \mathcal{E}(A))$ *and* $Rep\left(L^1(G, \alpha, A)\right)$.

# Crossed products of complete lmc *-algebra

## Theorem

*Let $\alpha$ be a continuous action of $G$ on a Banach $*$-algebra $A$ with bounded approximate unit. Then there is a $C^*$-isomorphism $\Phi : G \times_{\widetilde{\alpha}} \mathcal{E}(A) \to G \times_{\alpha} A$ such that*

$$\Phi\left(\left(\delta_{L^1(G,\widetilde{\alpha},\mathcal{E}(A))} \circ j\right)(f)\right) = \delta_{L^1(G,\alpha,A)}(f)$$

*for all $f \in L^1(G,\alpha,A)$.*

## Corollary

*Let $\alpha$ be a $G$-invariant continuous action of $G$ on a complete lmc $*$-algebra $A$ with bounded approximate unit. Then*

$$G \times_{\alpha} A \sim G \times_{\widetilde{\alpha}} \mathcal{E}(A).$$

## Remark

*Suppose that $\alpha$ is a $G$-invariant continuous action of $G$ on a complete locally m-convex $*$-algebra $A$ with bounded approximate unit and $A$ has an enveloping $C^*$-algebra.*
*$\mathcal{E}(A)$ is isomorphic to a $C^*$-algebra $\Rightarrow G \times_{\widetilde{\alpha}} \mathcal{E}(A)$ is isomorphic to a $C^*$-algebra $\Rightarrow G \times_{\alpha} A$ is isomorphic to a $C^*$-algebra.*

# Multiplier algebra of a pro-C*-algebra

Let $A$ be a pro-$C^*$-algebra and

$M(A) = \{(l, r)\,; l, r : A \to A$ are left and right module morphisms such that $al(b) = r(a)b, \forall a, b \in A\}$

$M(A)$ equipped with the topology determined by the family of $C^*$-seminorms $\{p_{M(A)}\}_{p \in S(A)}$, where

$$p_{M(A)}\,(l, p) = \sup\{p\,(l(a))\,; p(a) \leq 1\},$$

is a pro-$C^*$-*algebra.

**Theorem**

*Let $A$ be a pro-$C^*$-algebra. Then*

$$M(A) \equiv \lim_{\substack{\leftarrow \\ p}} M(A_p) \text{ up to an isomorphism of } C^* \text{algebras.}$$

# Multiplier algebra and crossed products

Let $A$ be a $C^*$-algebra and $\alpha$ a continuous action of $G$ on $A$.
The map $\mathrm{i}_A : A \longrightarrow M(G \times_\alpha A)$ defined by

$$i_A(a)(f)(g) = (af(g), f(g)\alpha_g(a)) \text{ for all } f \in C_c(G, A), g \in G$$

is a faithful $C^*$-morphism.

## Corollary

*Let $A$ be a pro-$C^*$-algebra and $\alpha$ a $G$ -invariant continuous action of $G$ on $A$. Then $\left(i_{A_p}\right)_p$ is an inverse system of $C^*$-morphisms, and*

$$i_A = \lim_{\substack{\longleftarrow \\ p}} i_{A_p}$$

*is an embedding of $A$ into $M(G \times_\alpha A)$.*

## Remark

*Suppose that $\alpha$ is a $G$-invariant continuous action of $G$ on a complete locally m-convex $*$-algebra $A$ with bounded approximate unit and $G \times_{\alpha} A$ is isomorphic to a $C^*$-algebra.*

*$G \times_{\alpha} A$ is isomorphic to a $C^*$-algebra $\Rightarrow$ $G \times_{\widetilde{\alpha}} \mathcal{E}(A)$ is isomorphic to a $C^*$-algebra*

*$\Rightarrow M(G \times_{\widetilde{\alpha}} \mathcal{E}(A))$ is isomorphic to a $C^*$-algebra*

*Since $\mathcal{E}(A)$ can be identified with a pro-$C^*$-subalgebra of $M(G \times_{\widetilde{\alpha}} \mathcal{E}(A))$, $\mathcal{E}(A)$ is isomorphic to a $C^*$-algebra.*

# The main result

## Theorem

*Let $\alpha$ be a $G$-invariant continous action of $G$ on a complete lmc $^*$-algebra $A$ with bounded approximate unit. Then $G \times_\alpha A$ is isomorphic to a $C^*$-algebra if and only if $A$ has an enveloping $C^*$-algebra.*

## Corollary

*Let $G$ be a locally compact group and let $A$ be a complete lmc $^*$-algebra with bounded approximate unit. Then the projective tensor product $L^1(G) \widehat{\otimes}_\pi A$ of $L^1(G)$ and $A$ admits an enveloping $C^*$-algebra if and only if $A$ admits an enveloping $C^*$-algebra.*

$L^1(G)\ \widehat{\otimes}_\pi A \equiv L^1(G,\mathrm{id}, A)$, $\mathrm{id}_g = \mathrm{id}_A$ for all $g \in G$.

# References


- S. J. Bhatt, D. J. Karia, *Topological algebras with $C^*$-enveloping algebras*, Proc. Indian Acad. Sci (Math. Soc.) 102(1993), 201-215.
- S. J. Bhatt, *Topological algebras with $C^*$-enveloping algebras II*, Proc. Indian Acad. Sci (Math. Soc.) 111(2001), 65-94.
- M. Fragoulopoulou, *Topological algebras with involution,* North-Holland Mathematics Studies 200, 2005.
- M. Joița, *Crossed products of locally $C^*$-algebras,* Rocky Mountain J. Math. 37(2007), 5, 1623-1644.
- N. C. Phillips, *Representable K-theory for $\sigma$-$C^*$-algebras*, K-Theory, 3(1989), 5, 441–478.
- D. P. Williams, *Crossed products of $C^*$-algebras*, Mathematical Surveys and Monographs (134) 2007, American Mathematical Society.

Thank you for your attention!