On Dirac Operators and Spectral Geometry of compact Quantum Groups

Antti J. Harju

Cardiff University and Helsinki University

27.4.2011

Dirac Operator on a Lie Group

- Let G be a simple, simply connected and compact Lie group, $\mathfrak{g}$ its Lie algebra with orthonormal basis $\{x_i : 1 \leq i \leq n\}$.

Dirac Operator on a Lie Group

- Let G be a simple, simply connected and compact Lie group, $\mathfrak{g}$ its Lie algebra with orthonormal basis $\{x_i : 1 \leq i \leq n\}$.
- The symmetric,nondegenerate bilinear form of $\mathfrak{g}$ defines a Clifford algebra $\mathrm{cl}(\mathfrak{g})$. Denote by $\gamma : \mathfrak{g} \to \mathrm{cl}(\mathfrak{g})$ the canonical embedding and (Σ, s) an irreducible representation of $\mathrm{cl}(\mathfrak{g})$.

Dirac Operator on a Lie Group

- Let G be a simple, simply connected and compact Lie group, $\mathfrak{g}$ its Lie algebra with orthonormal basis $\{x_i : 1 \leq i \leq n\}$.
- The symmetric,nondegenerate bilinear form of $\mathfrak{g}$ defines a Clifford algebra $\mathrm{cl}(\mathfrak{g})$. Denote by $\gamma : \mathfrak{g} \to \mathrm{cl}(\mathfrak{g})$ the canonical embedding and (Σ, s) an irreducible representation of $\mathrm{cl}(\mathfrak{g})$.
- A homomorphism from $\mathfrak{g}$ to $\mathrm{cl}(\mathfrak{g})$ is given by

$$x \mapsto \widetilde{\mathrm{ad}}(x) := \frac{1}{4} \sum_k \gamma(x_k)\gamma([x, x_k]) \in \mathrm{cl}(\mathfrak{g}).$$

For all $x, y \in \mathfrak{g}$:

$$\gamma([x, y]) = [\widetilde{\mathrm{ad}}(x), \gamma(y)].$$

Dirac Operator on a Lie Group

- Let G be a simple, simply connected and compact Lie group, $\mathfrak{g}$ its Lie algebra with orthonormal basis $\{x_i : 1 \leq i \leq n\}$.
- The symmetric,nondegenerate bilinear form of $\mathfrak{g}$ defines a Clifford algebra $\mathrm{cl}(\mathfrak{g})$. Denote by $\gamma : \mathfrak{g} \to \mathrm{cl}(\mathfrak{g})$ the canonical embedding and (Σ, s) an irreducible representation of $\mathrm{cl}(\mathfrak{g})$.
- A homomorphism from $\mathfrak{g}$ to $\mathrm{cl}(\mathfrak{g})$ is given by

$$x \mapsto \widetilde{\mathrm{ad}}(x) := \frac{1}{4} \sum_k \gamma(x_k)\gamma([x, x_k]) \in \mathrm{cl}(\mathfrak{g}).$$

  For all $x, y \in \mathfrak{g}$:

$$\gamma([x, y]) = [\widetilde{\mathrm{ad}}(x), \gamma(y)].$$

- The classical Dirac operator $\mathcal{D} \in U(\mathfrak{g}) \otimes \mathrm{cl}(\mathfrak{g})$ is defined by

$$\mathcal{D} = \sum_k (x_k \otimes \gamma(x_k) + N \otimes \gamma(x_k)\widetilde{\mathrm{ad}}(x_k)) \in U(\mathfrak{g}) \otimes \mathrm{cl}(\mathfrak{g}).$$

  ($\sum_k x_k \otimes x_k$ is invariant under the adjoint action of $\mathfrak{g}$.), $N \in \mathbb{R}$.

1. $\mathcal{D}$ commutes with the algebra homomorphism

$$x \mapsto (\mathrm{id} \otimes \widetilde{\mathrm{ad}})\triangle(x) = x' \otimes \widetilde{\mathrm{ad}}(x'')$$

The square of $\mathcal{D}$ is a sum of Casimir elements in $U(\mathfrak{g}) \otimes \mathrm{cl}(\mathfrak{g})$.

1. $\mathcal{D}$ commutes with the algebra homomorphism

$$x \mapsto (\mathrm{id} \otimes \widetilde{\mathrm{ad}})\triangle(x) = x' \otimes \widetilde{\mathrm{ad}}(x'')$$

   The square of $\mathcal{D}$ is a sum of Casimir elements in $U(\mathfrak{g}) \otimes \mathrm{cl}(\mathfrak{g})$.
2. Denote by D the Dirac opeator acting on $\mathbf{H} = L^2(G) \otimes \Sigma$. The spectral triple $(C^\infty(G), D, \mathbf{H})$ recovers the structure of Riemannian manifold G. The spectrum of D behaves as

$$|D|^{-n} \in L_{1+}(\mathbf{H}), \qquad n = \dim(G).$$

Quantum Group Preliminaries

- The quantum group $U_q(\mathfrak{g})$ is the unital associative algebra with generators k_i, k_i^{-1}, e_i, f_i $(1 \leq i \leq n)$ subject to

$$[k_i, k_j] = 0, \quad k_i k_i^{-1} = 1 \quad k_i e_j k_i^{-1} = q_i^{a_{ij}/2} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}/2} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q_i - q_i^{-1}}, \quad q_i = q^{d_i}$$

and the quantum Serre relations. (a_{ij} is the cartan matrix of $\mathfrak{g}$ and $\{d_i : 1 \leq i \leq n\}$ coprime positive integers such that $(d_i a_{ij})_{ij}$ is a symmetric matrix.) Choose $q \in (0, 1)$.

Quantum Group Preliminaries

- The quantum group $U_q(\mathfrak{g})$ is the unital associative algebra with generators k_i, k_i^{-1}, e_i, f_i $(1 \leq i \leq n)$ subject to

$$[k_i, k_j] = 0, \quad k_i k_i^{-1} = 1 \quad k_i e_j k_i^{-1} = q_i^{a_{ij}/2} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}/2} f_j,$$
$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q_i - q_i^{-1}}, \quad q_i = q^{d_i}$$

  and the quantum Serre relations. (a_{ij} is the cartan matrix of $\mathfrak{g}$ and $\{d_i : 1 \leq i \leq n\}$ coprime positive integers such that $(d_i a_{ij})_{ij}$ is a symmetric matrix.) Choose $q \in (0, 1)$.
- The structure of Hopf $*$-algebra can be chosen by

$$\triangle_q(k_i) = k_i \otimes k_i \ \triangle_q(e_i) = e_i \otimes k_i + k_i^{-1} \otimes e_i, \ \triangle_q(f_i) = f_i \otimes k_i + k_i^{-1} \otimes f_i$$
$$S_q(e_i) = -q e_i, \quad S_q(f_i) = -q^{-1} f_i, \quad S_q(k_i) = k_i^{-1},$$
$$\epsilon_q(k_i) = 1, \quad \epsilon_q(e_i) = \epsilon_q(f_i) = 0, \quad e_i^* = f_i, \quad f_i^* = e_i, \quad k_i^* = k_i.$$

Quantum Group Preliminaries

- The quantum group $U_q(\mathfrak{g})$ is the unital associative algebra with generators k_i, k_i^{-1}, e_i, f_i $(1 \leq i \leq n)$ subject to

$$[k_i, k_j] = 0, \quad k_i k_i^{-1} = 1 \quad k_i e_j k_i^{-1} = q_i^{a_{ij}/2} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}/2} f_j,$$
$$[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q_i - q_i^{-1}}, \quad q_i = q^{d_i}$$

  and the quantum Serre relations. (a_{ij} is the cartan matrix of $\mathfrak{g}$ and $\{d_i : 1 \leq i \leq n\}$ coprime positive integers such that $(d_i a_{ij})_{ij}$ is a symmetric matrix.) Choose $q \in (0, 1)$.
- The structure of Hopf $*$-algebra can be chosen by

$$\triangle_q(k_i) = k_i \otimes k_i \ \triangle_q(e_i) = e_i \otimes k_i + k_i^{-1} \otimes e_i, \ \triangle_q(f_i) = f_i \otimes k_i + k_i^{-1} \otimes f_i$$
$$S_q(e_i) = -q e_i, \quad S_q(f_i) = -q^{-1} f_i, \quad S_q(k_i) = k_i^{-1},$$
$$\epsilon_q(k_i) = 1, \quad \epsilon_q(e_i) = \epsilon_q(f_i) = 0, \quad e_i^* = f_i, \quad f_i^* = e_i, \quad k_i^* = k_i.$$

- The comultiplication is noncocommutative but there exists $R \in \overline{U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})}$ so that

$$x'' \otimes x' = \triangle_q^{\mathrm{op}}(x) = R \triangle_q(x) R^{-1},$$

  Equipped with R, the Hopf algebra $U_q(\mathfrak{g})$ is quasitriangular.

Algebraic Dirac Operator: Harju 2010

- Let $(V, \overset{\mathrm{ad}}{\triangleright})$ denote the adjoint representation of $U_q(\mathfrak{g})$ with a basis $\{|n\rangle : n \in I\}$ and let V^* be its dual with an orthonormal dual basis $\{\langle n| : n \in I\}$.

Algebraic Dirac Operator: Harju 2010

- Let $(V, \overset{\mathrm{ad}}{\triangleright})$ denote the adjoint representation of $U_q(\mathfrak{g})$ with a basis $\{|n\rangle : n \in I\}$ and let V^* be its dual with an orthonormal dual basis $\{\langle n| : n \in I\}$.
- Put $\Omega = \sum_{n \in I} |n\rangle \otimes \langle n|$. Ω spans the singlet of $V \otimes V^*$:

$$(x' \otimes x'') \overset{\mathrm{ad}}{\triangleright} \Omega = \epsilon(x)\Omega,$$

for all $x \in U_q(\mathfrak{g})$.

Algebraic Dirac Operator: Harju 2010

- Let $(V, \overset{\mathrm{ad}}{\triangleright})$ denote the adjoint representation of $U_q(\mathfrak{g})$ with a basis $\{|n\rangle : n \in I\}$ and let V^* be its dual with an orthonormal dual basis $\{\langle n| : n \in I\}$.
- Put $\Omega = \sum_{n \in I} |n\rangle \otimes \langle n|$. Ω spans the singlet of $V \otimes V^*$:

$$(x' \otimes x'') \overset{\mathrm{ad}}{\triangleright} \Omega = \epsilon(x)\Omega,$$

  for all $x \in U_q(\mathfrak{g})$.
- We would like to define

$$\mathcal{D}'_q = (\theta \otimes \bar{\gamma}_q)\Omega \in U_q(\mathfrak{g}) \otimes \mathrm{cl}_q(\mathfrak{g})$$

  where $\mathrm{cl}_q(\mathfrak{g})$ is a deformation of $\mathrm{cl}(\mathfrak{g})$ and a $U_q(\mathfrak{g})$-module algebra, $\bar{\gamma}_q : V^* \to \mathrm{cl}_q(\mathfrak{g})$ an embedding and $\theta : V \to \mathfrak{L}_q(\mathfrak{g}) \subset U_q(\mathfrak{g})$ are module isomorphisms.

Clifford Algebra $\mathrm{cl}_q(\mathfrak{g})$

- Denote by B_q the nondegenerate bilinear form $V \otimes V \to \mathbb{C}$ which is invariant

$$B_q(\triangle_q(x) \overset{\mathrm{ad}}{\triangleright} (u \otimes v)) = \epsilon_q(x) B_q(u \otimes v),$$

for all $x \in U_q(\mathfrak{g})$. Since $\triangle_q$ is noncocommutative B_q is nonsymmetric.

Clifford Algebra $\mathrm{cl}_q(\mathfrak{g})$

- Denote by B_q the nondegenerate bilinear form $V \otimes V \to \mathbb{C}$ which is invariant

  $$B_q(\triangle_q(x) \overset{\mathrm{ad}}{\triangleright} (u \otimes v)) = \epsilon_q(x) B_q(u \otimes v),$$

  for all $x \in U_q(\mathfrak{g})$. Since $\triangle_q$ is noncocommutative B_q is nonsymmetric.
- The braid operator $\check{R} = \sigma R$ is an automorphism of $V \otimes V$ and commutes with the representation. σ is the flip automorphism.

Clifford Algebra $\mathrm{cl}_q(\mathfrak{g})$

- Denote by B_q the nondegenerate bilinear form $V \otimes V \to \mathbb{C}$ which is invariant

  $$B_q(\triangle_q(x) \overset{\mathrm{ad}}{\triangleright} (u \otimes v)) = \epsilon_q(x) B_q(u \otimes v),$$

  for all $x \in U_q(\mathfrak{g})$. Since $\triangle_q$ is noncocommutative B_q is nonsymmetric.
- The braid operator $\check{R} = \sigma R$ is an automorphism of $V \otimes V$ and commutes with the representation. σ is the flip automorphism.
- Each irreducible component of $V \otimes V$ is an eigenspace of $\check{R}$. The eigenvalues are real because $\check{R}$ is self adjoint and do not reach zero because $\check{R}$ is automorphism.

Clifford Algebra $\mathrm{cl}_q(\mathfrak{g})$

- Denote by $T(V)$ the tensor algebra of V.

Clifford Algebra $\mathrm{cl}_q(\mathfrak{g})$

- Denote by $T(V)$ the tensor algebra of V.
- Denote by B_q^i and $\check{R}_i$ the bilinear form and braid operator acting on i'th and $(i+1)$'th tensor component and $\{b_{i,k} : k \in J\}$ the positive eigenvalues of $\check{R}_i$. Define an ideal $\mathfrak{I}$ of $T(V)$ by

$$\mathfrak{I} = \{(\mathrm{id} - B_q^i)t : t \in \mathrm{Ker}(\check{R}_i - b_{i,k}) \quad \text{for some} \quad i \in \mathbb{N}, k \in J\}.$$

Clifford Algebra $\mathrm{cl}_q(\mathfrak{g})$

- Denote by $T(V)$ the tensor algebra of V.
- Denote by B_q^i and $\check{R}_i$ the bilinear form and braid operator acting on i'th and $(i+1)$'th tensor component and $\{b_{i,k} : k \in J\}$ the positive eigenvalues of $\check{R}_i$. Define an ideal $\mathfrak{I}$ of $T(V)$ by

$$\mathfrak{I} = \{(\mathrm{id} - B_q^i)t : t \in \mathrm{Ker}(\check{R}_i - b_{i,k}) \quad \text{for some} \quad i \in \mathbb{N}, k \in J\}.$$

- Define

$$\mathrm{cl}_q(\mathfrak{g}) = T(V)/\mathfrak{I}.$$

Clifford Algebra $\mathrm{cl}_q(\mathfrak{g})$

- Denote by $T(V)$ the tensor algebra of V.
- Denote by B_q^i and $\check{R}_i$ the bilinear form and braid operator acting on i'th and $(i+1)$'th tensor component and $\{b_{i,k} : k \in J\}$ the positive eigenvalues of $\check{R}_i$. Define an ideal $\mathfrak{I}$ of $T(V)$ by

$$\mathfrak{I} = \{(\mathrm{id} - B_q^i)t : t \in \mathrm{Ker}(\check{R}_i - b_{i,k}) \quad \text{for some} \quad i \in \mathbb{N}, k \in J\}.$$

- Define

$$\mathrm{cl}_q(\mathfrak{g}) = T(V)/\mathfrak{I}.$$

- $\mathrm{cl}_q(\mathfrak{g})$ is $U_q(\mathfrak{g})$-module algebra.

Clifford Algebra $\mathrm{cl}_q(\mathfrak{g})$

- Denote by $T(V)$ the tensor algebra of V.
- Denote by B_q^i and $\check{R}_i$ the bilinear form and braid operator acting on i'th and $(i+1)$'th tensor component and $\{b_{i,k} : k \in J\}$ the positive eigenvalues of $\check{R}_i$. Define an ideal $\mathfrak{I}$ of $T(V)$ by

$$\mathfrak{I} = \{(\mathrm{id} - B_q^i)t : t \in \mathrm{Ker}(\check{R}_i - b_{i,k}) \quad \text{for some} \quad i \in \mathbb{N}, k \in J\}.$$

- Define

$$\mathrm{cl}_q(\mathfrak{g}) = T(V)/\mathfrak{I}.$$

- $\mathrm{cl}_q(\mathfrak{g})$ is $U_q(\mathfrak{g})$-module algebra.
- Denote by $\gamma_q : V \to \mathrm{cl}_q(\mathfrak{g})$ the canonical embedding.

Spinor module

- There exists a homomorphism $\widetilde{\mathrm{ad}}_q : U_q(\mathfrak{g}) \to \mathrm{cl}_q(\mathfrak{g})$ so that

$$\gamma_q(x \overset{\mathrm{ad}}{\triangleright} \psi) = \widetilde{\mathrm{ad}}_q(x')\gamma_q(\psi)\widetilde{\mathrm{ad}}_q(S_q(x'')),$$

for all $x \in U_q(\mathfrak{g})$.

Spinor module

- There exists a homomorphism $\widetilde{\mathrm{ad}}_q : U_q(\mathfrak{g}) \to \mathrm{cl}_q(\mathfrak{g})$ so that

$$\gamma_q(x \overset{\mathrm{ad}}{\triangleright} \psi) = \widetilde{\mathrm{ad}}_q(x')\gamma_q(\psi)\widetilde{\mathrm{ad}}_q(S_q(x'')),$$

  for all $x \in U_q(\mathfrak{g})$.
- Denote by (Σ, s_q) an irreducible representations of $\mathrm{cl}_q(\mathfrak{g})$.

Quantum Lie algebra: Delius, Gould (1997)

- Define the opposite adjoint action of $U_q(\mathfrak{g})$ on itself by

$$x \overset{\mathrm{ad}}{\blacktriangleright} y = x'' y S_q^{\mathrm{op}}(x').$$

Quantum Lie algebra: Delius, Gould (1997)

- Define the opposite adjoint action of $U_q(\mathfrak{g})$ on itself by

$$x \overset{\mathrm{ad}}{\blacktriangleright} y = x'' y S_q^{\mathrm{op}}(x').$$

- Let (U, π) denote the defining reprsentation of $U_q(\mathfrak{g})$ and

$$Z = H^{-1}((R^t)^{\mathrm{op}} R^{\mathrm{op}} - 1) \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}),$$
$$Z_{lk} = (\pi_{lk} \otimes \mathrm{id}) Z \in \mathbb{C} \otimes U_q(\mathfrak{g}) \simeq U_q(\mathfrak{g}).$$

Quantum Lie algebra: Delius, Gould (1997)

- Define the opposite adjoint action of $U_q(\mathfrak{g})$ on itself by

$$x \overset{\text{ad}}{\blacktriangleright} y = x'' y S_q^{\text{op}}(x').$$

- Let (U, π) denote the defining reprsentation of $U_q(\mathfrak{g})$ and

$$Z = H^{-1}((R^t)^{\text{op}} R^{\text{op}} - 1) \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}),$$
$$Z_{lk} = (\pi_{lk} \otimes \text{id}) Z \in \mathbb{C} \otimes U_q(\mathfrak{g}) \simeq U_q(\mathfrak{g}).$$

- The vectors Z_{lk} transform covariantly under the adjoint action

$$x \overset{\text{ad}}{\blacktriangleright} Z_{lk} = Z_{ij} \pi^*_{il}(x') \pi_{jk}(x''), \qquad \text{for all} \qquad x \in U_q(\mathfrak{g}).$$

Quantum Lie algebra: Delius, Gould (1997)

- Define the opposite adjoint action of $U_q(\mathfrak{g})$ on itself by

$$x \overset{\mathrm{ad}}{\blacktriangleright} y = x'' y S_q^{\mathrm{op}}(x').$$

- Let (U, π) denote the defining reprsentation of $U_q(\mathfrak{g})$ and

$$Z = H^{-1}((R^t)^{\mathrm{op}} R^{\mathrm{op}} - 1) \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}),$$
$$Z_{lk} = (\pi_{lk} \otimes \mathrm{id}) Z \in \mathbb{C} \otimes U_q(\mathfrak{g}) \simeq U_q(\mathfrak{g}).$$

- The vectors Z_{lk} transform covariantly under the adjoint action

$$x \overset{\mathrm{ad}}{\blacktriangleright} Z_{lk} = Z_{ij} \pi^*_{il}(x') \pi_{jk}(x''), \qquad \text{for all} \qquad x \in U_q(\mathfrak{g}).$$

- Pick the C-G coefficients of the module homomorphism $V \to U^* \otimes U$. Define

$$Z_a = C_a^{ij} (\pi_{ij} \otimes \mathrm{id}) Z.$$

Z_a's span a quantum Lie algebra $\mathfrak{L}_q(\mathfrak{g}) \subset U_q(\mathfrak{g})$ which is a deformation of $\mathfrak{g}$ and isomorphic to the adjoint representation of $U_q(\mathfrak{g})$.

Definition

- Let $\sigma : V^* \to V$ and $\theta : V \to \mathfrak{L}_q(\mathfrak{g})$ be module isomorphisms

Definition

- Let $\sigma : V^* \to V$ and $\theta : V \to \mathfrak{L}_q(\mathfrak{g})$ be module isomorphisms
- Let N' be a constant. Define

$$\mathcal{D}'_q = (\theta \otimes \gamma_q \circ \sigma)\Omega + N' \otimes \sum_n \gamma_q(|n\rangle)\widetilde{\mathrm{ad}}(\sigma(\langle n|) \in U_q(\mathfrak{g}) \otimes \mathrm{cl}_q(\mathfrak{g}).$$

Definition

- Let $\sigma : V^* \to V$ and $\theta : V \to \mathfrak{L}_q(\mathfrak{g})$ be module isomorphisms
- Let N' be a constant. Define

$$\mathcal{D}'_q = (\theta \otimes \gamma_q \circ \sigma)\Omega + N' \otimes \sum_n \gamma_q(|n\rangle)\widetilde{\mathrm{ad}}(\sigma(\langle n|) \in U_q(\mathfrak{g}) \otimes \mathrm{cl}_q(\mathfrak{g}).$$

- $\mathcal{D}'_q$ commutes with the image of $x \mapsto (\mathrm{id} \otimes \widetilde{\mathrm{ad}}_q)\triangle_q(x)$:

Definition

- Let $\sigma : V^* \to V$ and $\theta : V \to \mathfrak{L}_q(\mathfrak{g})$ be module isomorphisms
- Let N' be a constant. Define

$$\mathcal{D}'_q = (\theta \otimes \gamma_q \circ \sigma)\Omega + N' \otimes \sum_n \gamma_q(|n\rangle)\widetilde{\mathrm{ad}}(\sigma(\langle n|) \in U_q(\mathfrak{g}) \otimes \mathrm{cl}_q(\mathfrak{g}).$$

- $\mathcal{D}'_q$ commutes with the image of $x \mapsto (\mathrm{id} \otimes \widetilde{\mathrm{ad}}_q)\triangle_q(x)$:
-

$$\begin{aligned}
&(x' \otimes \widetilde{\mathrm{ad}}_q(x''))(\theta \otimes \gamma_q \circ \sigma)(\Omega) \\
&= \sum_n x'''\theta(|n\rangle)S_q^{\mathrm{op}}(x'')x' \otimes \widetilde{\mathrm{ad}}_q(x^{(4)})(\gamma_q \circ \sigma(\langle n|))\widetilde{\mathrm{ad}}_q(S_q(x^{(5)})x^{(6)}) \\
&= \sum_n (\theta \otimes \gamma_q \circ \sigma)((x'' \otimes x''') \overset{\mathrm{ad}}{\triangleright} \Omega)(x' \otimes \widetilde{\mathrm{ad}}_q(x'''')) \\
&= (\theta \otimes \gamma_q \circ \sigma)(\Omega)(x' \otimes \widetilde{\mathrm{ad}}_q(x'')),
\end{aligned}$$

for all $x \in U_q(\mathfrak{g})$. Above we used $x = x'\epsilon_q(x'') = x''\epsilon_q(x')$ and $\epsilon_q(x) = S_q(x')x'' = S_q^{\mathrm{op}}(x'')x'$.

Geometric Dirac Operator: Neshveyev, Tuset (2010)

- Denote by $W^*(G)$ the Hopf von Neumann algebra of G generated by the operators π_λ of (fixed) irreducible representations of G. ($W^*(G)$ is the l^∞ sum of $B(V_\lambda)$.)

Geometric Dirac Operator: Neshveyev, Tuset (2010)

- Denote by $W^*(G)$ the Hopf von Neumann algebra of G generated by the operators π_λ of (fixed) irreducible representations of G. ($W^*(G)$ is the l^∞ sum of $B(V_\lambda)$.)
- Define $U(G) = \prod_\lambda B(V_\lambda)$: densely defined unbounded operators affiliated with $W^*(G)$,

Geometric Dirac Operator: Neshveyev, Tuset (2010)

- Denote by $W^*(G)$ the Hopf von Neumann algebra of G generated by the operators π_λ of (fixed) irreducible representations of G. ($W^*(G)$ is the l^∞ sum of $B(V_\lambda)$.)
- Define $U(G) = \prod_\lambda B(V_\lambda)$: densely defined unbounded operators affiliated with $W^*(G)$,
- The representations of $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are in one to one correspondence: There exists an isomorphism of algebras

$$\phi : W^*(G_q) \to W^*(G),$$

The algebra $U_q(\mathfrak{g})$ is a subalgebra in $U(G_q)$. ϕ extends to an isomorphism $U_q(G) \to U(G_q)$.

Geometric Dirac Operator: Neshveyev, Tuset (2010)

- Denote by $W^*(G)$ the Hopf von Neumann algebra of G generated by the operators π_λ of (fixed) irreducible representations of G. ($W^*(G)$ is the l^∞ sum of $B(V_\lambda)$.)
- Define $U(G) = \prod_\lambda B(V_\lambda)$: densely defined unbounded operators affiliated with $W^*(G)$,
- The representations of $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are in one to one correspondence: There exists an isomorphism of algebras

  $$\phi : W^*(G_q) \to W^*(G),$$

  The algebra $U_q(\mathfrak{g})$ is a subalgebra in $U(G_q)$. ϕ extends to an isomorphism $U_q(G) \to U(G_q)$.
- The coproducts are related by a unitary $F \in U(G \otimes G)$, $x \in U(G)$:

  $$(\phi \otimes \phi)\triangle_q(x) \circ \phi^{-1} = F\triangle(x)F^{-1}$$

Geometric Dirac Operator: Neshveyev, Tuset (2010)

- Denote by $W^*(G)$ the Hopf von Neumann algebra of G generated by the operators π_λ of (fixed) irreducible representations of G. ($W^*(G)$ is the l^∞ sum of $B(V_\lambda)$.)
- Define $U(G) = \prod_\lambda B(V_\lambda)$: densely defined unbounded operators affiliated with $W^*(G)$,
- The representations of $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are in one to one correspondence: There exists an isomorphism of algebras

  $$\phi : W^*(G_q) \to W^*(G),$$

  The algebra $U_q(\mathfrak{g})$ is a subalgebra in $U(G_q)$. ϕ extends to an isomorphism $U_q(G) \to U(G_q)$.
- The coproducts are related by a unitary $F \in U(G \otimes G)$, $x \in U(G)$:

  $$(\phi \otimes \phi)\triangle_q(x) \circ \phi^{-1} = F\triangle(x)F^{-1}$$

- Let $\mathcal{D}$ denote the classical operator. Define

  $$\mathcal{D}_q = (\phi^{-1} \otimes \mathrm{id})\Big((\mathrm{id} \otimes \widetilde{\mathrm{ad}})(F)\mathcal{D}(\mathrm{id} \otimes \widetilde{\mathrm{ad}})(F^*)\Big) \in U_q(\mathfrak{g}) \otimes \mathrm{cl}(\mathfrak{g}).$$

Geometric Dirac Operator: Neshveyev, Tuset (2010)

- Denote by $W^*(G)$ the Hopf von Neumann algebra of G generated by the operators π_λ of (fixed) irreducible representations of G. ($W^*(G)$ is the l^∞ sum of $B(V_\lambda)$.)
- Define $U(G) = \prod_\lambda B(V_\lambda)$: densely defined unbounded operators affiliated with $W^*(G)$,
- The representations of $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ are in one to one correspondence: There exists an isomorphism of algebras

$$\phi : W^*(G_q) \to W^*(G),$$

  The algebra $U_q(\mathfrak{g})$ is a subalgebra in $U(G_q)$. ϕ extends to an isomorphism $U_q(G) \to U(G_q)$.
- The coproducts are related by a unitary $F \in U(G \otimes G)$, $x \in U(G)$:

$$(\phi \otimes \phi)\triangle_q(x) \circ \phi^{-1} = F\triangle(x)F^{-1}$$

- Let $\mathcal{D}$ denote the classical operator. Define

$$\mathcal{D}_q = (\phi^{-1} \otimes \mathrm{id})\Big((\mathrm{id} \otimes \widetilde{\mathrm{ad}})(F)\mathcal{D}(\mathrm{id} \otimes \widetilde{\mathrm{ad}})(F^*)\Big) \in U_q(\mathfrak{g}) \otimes \mathrm{cl}(\mathfrak{g}).$$

- $\mathcal{D}_q$ commutes with the image of the homomorphism $x \mapsto (\mathrm{id} \otimes (\widetilde{\mathrm{ad}} \circ \phi))\triangle_q(x)$ in $U_q(\mathfrak{g}) \otimes \mathrm{cl}(\mathfrak{g})$ which is a consequence of the corresponding property of $\mathcal{D}$ and the definition of F.

Geometry of G_q

- Define $\mathbb{C}[G_q]$ the Hopf-algebra of representative functions on G_q: It is spanned by the matrix elements of irreducible finite dimensional representations of $U_q(\mathfrak{g})$, and the product is determined from C-G coefficients. We can idetify

$$\mathbb{C}[G_q] = \bigoplus_{\lambda \in P_+} V_\lambda \otimes V_\lambda^*.$$

Geometry of G_q

- Define $\mathbb{C}[G_q]$ the Hopf-algebra of representative functions on G_q: It is spanned by the matrix elements of irreducible finite dimensional representations of $U_q(\mathfrak{g})$, and the product is determined from C-G coefficients. We can idetify

$$\mathbb{C}[G_q] = \bigoplus_{\lambda \in P_+} V_\lambda \otimes V_\lambda^*.$$

- $\mathbb{C}[G_q]$ accepts a Haar state. Denote by $L^2(G_q)$ the completion and by ρ the representation of $\mathbb{C}[G_q]$ on $L^2(G_q)$.

Geometry of G_q

- Define $\mathbb{C}[G_q]$ the Hopf-algebra of representative functions on G_q: It is spanned by the matrix elements of irreducible finite dimensional representations of $U_q(\mathfrak{g})$, and the product is determined from C-G coefficients. We can idetify

$$\mathbb{C}[G_q] = \bigoplus_{\lambda \in P_+} V_\lambda \otimes V_\lambda^*.$$

- $\mathbb{C}[G_q]$ accepts a Haar state. Denote by $L^2(G_q)$ the completion and by ρ the representation of $\mathbb{C}[G_q]$ on $L^2(G_q)$.
- $U_q(\mathfrak{g})$ acts on the orthonormal basis of $L^2(G_q)$ by

$$\partial(x)|\lambda, n\rangle \otimes \langle\lambda, m| = (\pi_{\lambda,q}(x)|\lambda, n\rangle) \otimes \langle\lambda, m|.$$

Geometry of G_q

- Define $\mathbb{C}[G_q]$ the Hopf-algebra of representative functions on G_q: It is spanned by the matrix elements of irreducible finite dimensional representations of $U_q(\mathfrak{g})$, and the product is determined from C-G coefficients. We can idetify

$$\mathbb{C}[G_q] = \bigoplus_{\lambda \in P_+} V_\lambda \otimes V_\lambda^*.$$

- $\mathbb{C}[G_q]$ accepts a Haar state. Denote by $L^2(G_q)$ the completion and by ρ the representation of $\mathbb{C}[G_q]$ on $L^2(G_q)$.
- $U_q(\mathfrak{g})$ acts on the orthonormal basis of $L^2(G_q)$ by

$$\partial(x)|\lambda, n\rangle \otimes \langle \lambda, m| = (\pi_{\lambda,q}(x)|\lambda, n\rangle) \otimes \langle \lambda, m|.$$

- Theorem (Nesyenev, Tuset): The triple

$$(\mathbb{C}[G_q], D_q, \mathbf{H})$$

is a spectral triple; $D_q = (\partial \otimes s)\mathcal{D}_q$ and $\mathbf{H} = L^2(G_q) \otimes \Sigma$.

Example: $SU_q(2)$

- Choose the generators $\{j_\pm, j_0\}$ of $\mathfrak{su}_2$ so that

$$[j_0, j_\pm] = \pm j_\pm, \qquad [j_+, j_-] = 2j_0$$

Example: $SU_q(2)$

- Choose the generators $\{j_\pm, j_0\}$ of $\mathfrak{su}_2$ so that

$$[j_0, j_\pm] = \pm j_\pm, \qquad [j_+, j_-] = 2j_0$$

- The irreducible representations $\{(V_l, \pi_l) : l \in \frac{1}{2}\mathbb{N}_0\}$ are

$$\pi_l(j_\pm)|l, m\rangle = \sqrt{l(l+1) - m(m \pm 1)}|l, m \pm 1\rangle,$$
$$\pi_l(j_0)|l, m\rangle = m|l, m\rangle.$$

where the basis is chosen by $\{|l, m\rangle : -l \leq m \leq l\}$ for each V_l.

Example: $SU_q(2)$

- Choose the generators $\{j_\pm, j_0\}$ of $\mathfrak{su}_2$ so that

$$[j_0, j_\pm] = \pm j_\pm, \qquad [j_+, j_-] = 2j_0$$

- The irreducible representations $\{(V_l, \pi_l) : l \in \frac{1}{2}\mathbb{N}_0\}$ are

$$\pi_l(j_\pm)|l,m\rangle = \sqrt{l(l+1) - m(m\pm 1)}|l, m\pm 1\rangle,$$
$$\pi_l(j_0)|l,m\rangle = m|l,m\rangle.$$

  where the basis is chosen by $\{|l,m\rangle : -l \leq m \leq l\}$ for each V_l.

- The Killing form is normalized so that the vectors

$$x_1 = j_+ + j_-, \qquad x_2 = -i(j_+ - j_-), \qquad x_3 = 2j_0$$

  form an orthonormal basis of $\mathfrak{g}$.

- The representations of the algebras $\mathrm{cl}(\mathfrak{su}_2)$ and $\mathfrak{su}_2$ on $\Sigma = V_{1/2}$ are

$$s : \gamma(x_i) \mapsto \pi_{1/2}(x_i), \qquad \widetilde{\mathrm{ad}}(x_i) \mapsto \pi_{1/2}(x_i).$$

- The representations of the algebras $\mathrm{cl}(\mathfrak{su}_2)$ and $\mathfrak{su}_2$ on $\Sigma = V_{1/2}$ are

$$s : \gamma(x_i) \mapsto \pi_{1/2}(x_i), \qquad \widetilde{\mathrm{ad}}(x_i) \mapsto \pi_{1/2}(x_i).$$

- The classical Dirac operator is

$$D = (\partial \otimes s)\mathcal{D} = 2\partial \begin{pmatrix} j_0 & j_- \\ j_+ & -j_0 \end{pmatrix} + 3N\mathbf{1}.$$

- The representations of the algebras $\mathrm{cl}(\mathfrak{su}_2)$ and $\mathfrak{su}_2$ on $\Sigma = V_{1/2}$ are

$$s : \gamma(x_i) \mapsto \pi_{1/2}(x_i), \qquad \widetilde{\mathrm{ad}}(x_i) \mapsto \pi_{1/2}(x_i).$$

- The classical Dirac operator is

$$D = (\partial \otimes s)\mathcal{D} = 2\partial \begin{pmatrix} j_0 & j_- \\ j_+ & -j_0 \end{pmatrix} + 3N\mathbf{1}.$$

- Eigenvalues of D on irreducible components of $V_l \otimes \Sigma \simeq V_{l-1/2} \oplus V_{l+1/2}$

$$D|l + \frac{1}{2}, m\rangle_0 = (2l + 3N)|l + \frac{1}{2}, m\rangle_0,$$
$$D|l - \frac{1}{2}, n\rangle_0 = (-(2l + 2) + 3N)|l - \frac{1}{2}, n\rangle_0.$$

for each m, n.

- The irreducible representations $(V_l, \pi_{l,q})$ of $U_q(\mathfrak{su}_2)$ are $(l \in \frac{1}{2}\mathbb{N}_0)$:

$$\pi_{l,q}(k)|l, m\rangle = q^m|l, m\rangle$$
$$\pi_{l,q}(e)|l, m\rangle = \sqrt{[l-m]_q[l+m+1]_q}|l, m+1\rangle$$
$$\pi_{l,q}(f)|l, m\rangle = \sqrt{[l-m+1]_q[l+m]_q}|l, m-1\rangle$$

- The irreducible representations $(V_l, \pi_{l,q})$ of $U_q(\mathfrak{su}_2)$ are $(l \in \frac{1}{2}\mathbb{N}_0)$:

$$\pi_{l,q}(k)|l,m\rangle = q^m|l,m\rangle$$
$$\pi_{l,q}(e)|l,m\rangle = \sqrt{[l-m]_q[l+m+1]_q}|l,m+1\rangle$$
$$\pi_{l,q}(f)|l,m\rangle = \sqrt{[l-m+1]_q[l+m]_q}|l,m-1\rangle$$

- D_q acts on $V_l \otimes \Sigma \simeq V_{l-1/2} \oplus V_{l+1/2}$ by

$$D_q|l+\frac{1}{2},m\rangle = (2j+3N)|l+\frac{1}{2},m\rangle$$
$$D_q|l-\frac{1}{2},n\rangle = (-(2j+2)+3N)|l-\frac{1}{2},n\rangle, .$$

where the tensor product is reduced to $U_q(\mathfrak{g})$ invariant components.

- How is $\mathcal{D}_q$ defined in $U_q(\mathfrak{g}) \otimes \mathrm{cl}(\mathfrak{g})$?. In the following let us put $N = 0$. The following relation holds:

$$(\phi \otimes \phi)(R^t R) = Fq^{\sum_i x_i \otimes x_i} F^*$$

Therefore, (recall $\gamma = \widetilde{\mathrm{ad}}$ now)

$$\begin{aligned} q^{D_q} &= (\partial \circ \phi^{-1} \otimes s \circ \gamma)(Fq^{\sum_i x_i \otimes x_i} F^*) = (\partial \otimes \pi_{1/2,q})(R^t R) \\ &= \partial\Big[\begin{pmatrix} t^2 & 0 \\ 0 & -t^2 \end{pmatrix} + (q - q^{-1}) \begin{pmatrix} (1 - q^{-2})fe & q^{-1/2} ft^{-1} \\ q^{-1/2} t^{-1} f & 0 \end{pmatrix} \Big]. \end{aligned}$$

Algebraic Operator on $SU_q(2)$

- The adjoint module of $U_q(\mathfrak{su}_2)$ is $(V_1, \pi_{1,q})$. Then $V_1 \otimes V_1 = V_2 \oplus V_1 \oplus V_0$ where V_2 and V_0 are q-symmetric modules. Then

$$
\begin{aligned}
&\psi_1\psi_1 = \psi_{-1}\psi_{-1} = 0 \\
&q^{-1}\psi_1\psi_0 + q\psi_0\psi_1 = 0 \\
&q^{-2}\psi_1\psi_{-1} + [2]_q\psi_0\psi_0 + q^2\psi_{-1}\psi_1 = 0 \\
&\psi_0\psi_{-1} + q^2\psi_{-1}\psi_0 = 0 \\
&\psi_1\psi_{-1} + \psi_{-1}\psi_1 = b,
\end{aligned}
$$

where $\psi_i = \gamma_q(|1, i\rangle)$ and b is some constant fixed from the normalization of the form B_q.

Algebraic Operator on $SU_q(2)$

- The adjoint module of $U_q(\mathfrak{su}_2)$ is $(V_1, \pi_{1,q})$. Then $V_1 \otimes V_1 = V_2 \oplus V_1 \oplus V_0$ where V_2 and V_0 are q-symmetric modules. Then

$$\begin{aligned} &\psi_1\psi_1 = \psi_{-1}\psi_{-1} = 0 \\ &q^{-1}\psi_1\psi_0 + q\psi_0\psi_1 = 0 \\ &q^{-2}\psi_1\psi_{-1} + [2]_q\psi_0\psi_0 + q^2\psi_{-1}\psi_1 = 0 \\ &\psi_0\psi_{-1} + q^2\psi_{-1}\psi_0 = 0 \\ &\psi_1\psi_{-1} + \psi_{-1}\psi_1 = b, \end{aligned}$$

  where $\psi_i = \gamma_q(|1, i\rangle)$ and b is some constant fixed from the normalization of the form B_q.
- The irreducible representation on $(\Sigma = V_{1/2}, s_q)$ are

$$s_q(\psi_1) = \begin{pmatrix} 0 & \sqrt{q} \\ 0 & 0 \end{pmatrix}, \; s_q(\psi_0) = -\frac{1}{\sqrt{[2]_q}} \begin{pmatrix} q^{-1} & 0 \\ 0 & -q \end{pmatrix}$$

$$s_q(\psi_{-1}) = \begin{pmatrix} 0 & 0 \\ -\sqrt{q^{-1}} & 0 \end{pmatrix}, \; s_q(\widetilde{\mathrm{ad}}_q(x)) = \pi_{1/2,q}(x).$$

- The isomorphism $V \to \mathfrak{L}(\mathfrak{su}_2)$ is defined by

$$\theta(|1,1\rangle) = t^{-1}e, \qquad \theta(|1,0\rangle) = \frac{1}{\sqrt{[2]_q}}(q^{-1}fe - qef),$$
$$\theta(|1,-1\rangle) = -t^{-1}f.$$

- The isomorphism $V \to \mathfrak{L}(\mathfrak{su}_2)$ is defined by

$$\theta(|1,1\rangle) = t^{-1}e, \qquad \theta(|1,0\rangle) = \frac{1}{\sqrt{[2]_q}}(q^{-1}fe - qef),$$
$$\theta(|1,-1\rangle) = -t^{-1}f.$$

- For $N' = 0$ we have

$$D'_q = \partial \begin{pmatrix} ef - q^{-2}fe & q^{-1/2}[2]_q t^{-1} f \\ q^{1/2}[2]_q t^{-1} e & -q^2 ef + fe \end{pmatrix}.$$

- The isomorphism $V \to \mathfrak{L}(\mathfrak{su}_2)$ is defined by

$$\theta(|1,1\rangle) = t^{-1}e, \qquad \theta(|1,0\rangle) = \frac{1}{\sqrt{[2]_q}}(q^{-1}fe - qef),$$
$$\theta(|1,-1\rangle) = -t^{-1}f.$$

- For $N' = 0$ we have

$$D'_q = \partial \begin{pmatrix} ef - q^{-2}fe & q^{-1/2}[2]_q t^{-1} f \\ q^{1/2}[2]_q t^{-1} e & -q^2 ef + fe \end{pmatrix}.$$

- The relation to geometric approach:

$$D'_q = [D_q]_q = \frac{q^{D_q} - q^{-D_q}}{q - q^{-1}}.$$

which can be checked using the formula

$$q^{-D_q} = (\partial \otimes \pi_{1/2,q})(R^{-1}(R^t)^{-1}).$$

Geometry of $SU_q(2)$

- Denote by **H** the completion of the prehilbert space

$$(\bigoplus_{2l=0}^{\infty} V_l \otimes V_l^*) \otimes \Sigma \simeq (\bigoplus_{2l=0}^{\infty} V_l \otimes V_l) \otimes V_{1/2}$$

  and the representation of $U_q(\mathfrak{g})$ on **H** is defined by $x \mapsto (\partial \otimes \widetilde{\mathrm{ad}} \circ \phi)\triangle_q(x)$.

Geometry of $SU_q(2)$

- Denote by **H** the completion of the prehilbert space

$$(\bigoplus_{2l=0}^{\infty} V_l \otimes V_l^*) \otimes \Sigma \simeq (\bigoplus_{2l=0}^{\infty} V_l \otimes V_l) \otimes V_{1/2}$$

  and the representation of $U_q(\mathfrak{g})$ on **H** is defined by $x \mapsto (\partial \otimes \widetilde{\mathrm{ad}} \circ \phi)\triangle_q(x)$.

- The prehilbert space reduces into irreducible components under this action as

$$(\bigoplus_{2l=0}^{\infty} V_l \otimes V_l) \otimes \Sigma \simeq V_{1/2} \oplus \bigoplus_{2j=1}^{\infty}(V_{j+1/2} \otimes V_j) \oplus (V_{j-1/2} \otimes V_j)$$

$$= W_0^{\uparrow} \oplus \bigoplus_{2j=1}^{\infty} W_j^{\uparrow} \oplus W_j^{\downarrow}.$$

- The orthonormal basis of **H** is chosen by

$$\{|j\mu n\uparrow\rangle, |j'\mu' n\downarrow\rangle : j \in \frac{1}{2}\mathbb{N}_0,\ j' \in \frac{1}{2}\mathbb{N},\ |\mu| \leq j+1,\ |\mu'| \leq j-1,\ |n| \leq j\}$$

where $|j\mu n\uparrow\rangle \in W_j^{\uparrow}$ and $|j\mu n\downarrow\rangle \in W_j^{\downarrow}$.'

- The orthonormal basis of $\mathbf{H}$ is chosen by

$$\{|j\mu n\uparrow\rangle, |j'\mu' n\downarrow\rangle : j \in \frac{1}{2}\mathbb{N}_0,\ j' \in \frac{1}{2}\mathbb{N},\ |\mu| \leq j+1,\ |\mu'| \leq j-1,\ |n| \leq j\}$$

where $|j\mu n\uparrow\rangle \in W_j^{\uparrow}$ and $|j\mu n\downarrow\rangle \in W_j^{\downarrow}$.'

- The action of the Dirac opeator:

$$D_q|j\mu n\uparrow\rangle = (2j+3N)|j\mu n\uparrow\rangle$$
$$D_q|j\mu n\downarrow\rangle = (-(2j+2)+3N)|j\mu n\downarrow\rangle.$$

- The orthonormal basis of **H** is chosen by

$$\{|j\mu n\uparrow\rangle, |j'\mu' n\downarrow\rangle : j \in \frac{1}{2}\mathbb{N}_0,\ j' \in \frac{1}{2}\mathbb{N},\ |\mu| \leq j+1,\ |\mu'| \leq j-1,\ |n| \leq j\}$$

where $|j\mu n\uparrow\rangle \in W_j^{\uparrow}$ and $|j\mu n\downarrow\rangle \in W_j^{\downarrow}$.'

- The action of the Dirac opeator:

$$D_q|j\mu n\uparrow\rangle = (2j+3N)|j\mu n\uparrow\rangle$$
$$D_q|j\mu n\downarrow\rangle = (-(2j+2)+3N)|j\mu n\downarrow\rangle.$$

- The algbra $\mathbb{C}[SU_q(2)]$ has a faithful $*$-representation on **H**.

- The orthonormal basis of **H** is chosen by

$$\{|j\mu n\uparrow\rangle, |j'\mu' n\downarrow\rangle : j \in \frac{1}{2}\mathbb{N}_0,\ j' \in \frac{1}{2}\mathbb{N},\ |\mu| \leq j+1,\ |\mu'| \leq j-1,\ |n| \leq j\}$$

  where $|j\mu n\uparrow\rangle \in W_j^{\uparrow}$ and $|j\mu n\downarrow\rangle \in W_j^{\downarrow}$.'
- The action of the Dirac opeator:

$$D_q|j\mu n\uparrow\rangle = (2j+3N)|j\mu n\uparrow\rangle$$
$$D_q|j\mu n\downarrow\rangle = (-(2j+2)+3N)|j\mu n\downarrow\rangle.$$

- The algbra $\mathbb{C}[SU_q(2)]$ has a faithful $*$-representation on **H**.
- For $N = 1/2$ the spectral triple $(\mathbb{C}[SU_q(2)], D_q, \mathbf{H})$ coincides with the isospectral deformation in "Dabrowski, Landi, Sitarz, van Suijlekom, Varilly (2005)". Thefore it is regular with dimension spectrum $\{1, 2, 3\}$.

- For $N' = 0$ the triple $(\mathbb{C}[SU_q(2)], F'_q, \mathbf{H})$ defines a Fredholm module, where $F'_q = D'_q(1 + (D'_q)^2)^{-1/2}$ and

$$D'_q|j\mu n\uparrow\rangle = [2j]_q|j\mu n\uparrow\rangle$$
$$D'_q|j\mu n\downarrow\rangle = -[2j+2]_q|j\mu n\downarrow\rangle.$$

- For $N' = 0$ the triple $(\mathbb{C}[SU_q(2)], F'_q, \mathbf{H})$ defines a Fredholm module, where $F'_q = D'_q(1 + (D'_q)^2)^{-1/2}$ and

$$D'_q|j\mu n\uparrow\rangle = [2j]_q|j\mu n\uparrow\rangle$$
$$D'_q|j\mu n\downarrow\rangle = -[2j+2]_q|j\mu n\downarrow\rangle.$$

- The Fredholm modules $(\mathbb{C}[SU_q(2)], F'_q, \mathbf{H})$ and $(\mathbb{C}[SU_q(2)], F_q, \mathbf{H})$ are homotopy equivalent:

$$(0,1] \to B(\mathbf{H}), \qquad t \mapsto \frac{[D_q]_{q^t}}{(1 + [D_q]^2_{q^t})^{1/2}}, \qquad f(0) = F_q = \frac{D_q}{(1 + (D_q)^2)^{1/2}},$$

where $[D_q]_{q^t} = (q^{tD_q} - q^{-tD_q})/(q^t - q^{-t})$.

- For $N' = 0$ the triple $(\mathbb{C}[SU_q(2)], F'_q, \mathbf{H})$ defines a Fredholm module, where $F'_q = D'_q(1+(D'_q)^2)^{-1/2}$ and

$$D'_q|j\mu n\uparrow\rangle = [2j]_q|j\mu n\uparrow\rangle$$
$$D'_q|j\mu n\downarrow\rangle = -[2j+2]_q|j\mu n\downarrow\rangle.$$

- The Fredholm modules $(\mathbb{C}[SU_q(2)], F'_q, \mathbf{H})$ and $(\mathbb{C}[SU_q(2)], F_q, \mathbf{H})$ are homotopy equivalent:

$$(0,1] \to B(\mathbf{H}), \qquad t \mapsto \frac{[D_q]_{q^t}}{(1+[D_q]_{q^t}^2)^{1/2}}, \qquad f(0) = F_q = \frac{D_q}{(1+(D_q)^2)^{1/2}},$$

  where $[D_q]_{q^t} = (q^{tD_q} - q^{-tD_q})/(q^t - q^{-t})$.
- Questinon: Is this true in general?

Example: $U_q(2)$

- $\mathfrak{u}_2$ is spanned by x_i $(0 \leq i \leq 3)$

$$[x_0, x_i] = 0, \qquad x_1, x_2, x_3 \text{ generate } \mathfrak{su}_2$$

Fix the normalization of the bilinear form so that these x_i form an orthonormal basis.

Example: $U_q(2)$

- $\mathfrak{u}_2$ is spanned by x_i $(0 \leq i \leq 3)$

$$[x_0, x_i] = 0, \qquad x_1, x_2, x_3 \text{ generate } \mathfrak{su}_2$$

  Fix the normalization of the bilinear form so that these x_i form an orthonormal basis.

- The irreducible representations are parametrized by the pairs (l, k), where $l \in \frac{1}{2}\mathbb{N}_0$ and $k \in l + \mathbb{Z}$ (l is the highest weight for $\mathfrak{su}_2$ and k fixes the action of the center.)

Example: $U_q(2)$

- $\mathfrak{u}_2$ is spanned by x_i $(0 \leq i \leq 3)$

$$[x_0, x_i] = 0, \qquad x_1, x_2, x_3 \text{ generate } \mathfrak{su}_2$$

  Fix the normalization of the bilinear form so that these x_i form an orthonormal basis.

- The irreducible representations are parametrized by the pairs (l, k), where $l \in \frac{1}{2}\mathbb{N}_0$ and $k \in l + \mathbb{Z}$ (l is the highest weight for $\mathfrak{su}_2$ and k fixes the action of the center.)
- The deformed algebra $U_q(\mathfrak{u}_2)$ is defined by adding the linearly independent generator $\xi = q^c$ to $U_q(\mathfrak{su}_2)$ which is central in $U_q(\mathfrak{g})$. The extension of the Hopf structure is defined by

$$\triangle_q(\xi) = \xi \otimes \xi, \qquad S_q(\xi) = \xi^{-1}, \qquad \epsilon(\xi) = 1.$$

Example: $U_q(2)$

- $\mathfrak{u}_2$ is spanned by x_i $(0 \leq i \leq 3)$

$$[x_0, x_i] = 0, \qquad x_1, x_2, x_3 \text{ generate } \mathfrak{su}_2$$

  Fix the normalization of the bilinear form so that these x_i form an orthonormal basis.

- The irreducible representations are parametrized by the pairs (l, k), where $l \in \frac{1}{2}\mathbb{N}_0$ and $k \in l + \mathbb{Z}$ (l is the highest weight for $\mathfrak{su}_2$ and k fixes the action of the center.)
- The deformed algebra $U_q(\mathfrak{u}_2)$ is defined by adding the linearly independent generator $\xi = q^c$ to $U_q(\mathfrak{su}_2)$ which is central in $U_q(\mathfrak{g})$. The extension of the Hopf structure is defined by

$$\triangle_q(\xi) = \xi \otimes \xi, \qquad S_q(\xi) = \xi^{-1}, \qquad \epsilon(\xi) = 1.$$

- $U_q(\mathfrak{u}_2)$ differs from $U_q(\mathfrak{su}_2)$ only by an element in the center, so the twist F and braiding $\check{R}$ are defined as for $U_q(\mathfrak{su}_2)$. The isomorphism ϕ is extended to an isomorphism $U_q(\mathfrak{u}_2) \to U(\mathfrak{u}_2)$ by setting $\phi(\xi) = q^{x_0}$.

- The Clifford algebra $\mathrm{cl}(\mathfrak{u}_2)$ has 4 dimensional irreducible representation $\hat{\Sigma}$ given by

$$s(\gamma(x_0)) = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad s(\gamma(x_n)) = i \begin{pmatrix} 0 & \pi_{1/2}(x_n) \\ -\pi_{1/2}(x_n) & 0 \end{pmatrix}, \qquad 1 \leq n \leq 3.$$

- The Clifford algebra $\mathrm{cl}(\mathfrak{u}_2)$ has 4 dimensional irreducible representation $\hat{\Sigma}$ given by

$$s(\gamma(x_0)) = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad s(\gamma(x_n)) = i \begin{pmatrix} 0 & \pi_{1/2}(x_n) \\ -\pi_{1/2}(x_n) & 0 \end{pmatrix}, \qquad 1 \leq n \leq 3.$$

- The corresponding representation $\widetilde{\mathrm{ad}}$ of $\mathfrak{u}_2$ has two irreducible components $\hat{\Sigma} = V^{+}_{(1/2,0)} \oplus V^{-}_{(1/2,0)}$

$$\widetilde{\mathrm{ad}}(x_0) = 0, \qquad \widetilde{\mathrm{ad}}(x_i) = \begin{pmatrix} \pi_{1/2}(x_i) & 0 \\ 0 & \pi_{1/2}(x_i) \end{pmatrix}, \qquad x \in \{1,2,3\}, (1)$$

where we have fixed the action of the center to be zero.

- The Clifford algebra $\mathrm{cl}(\mathfrak{u}_2)$ has 4 dimensional irreducible representation $\hat{\Sigma}$ given by

$$s(\gamma(x_0)) = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad s(\gamma(x_n)) = i \begin{pmatrix} 0 & \pi_{1/2}(x_n) \\ -\pi_{1/2}(x_n) & 0 \end{pmatrix}, \quad 1 \leq n \leq 3.$$

- The corresponding representation $\widetilde{\mathrm{ad}}$ of $\mathfrak{u}_2$ has two irreducible components $\hat{\Sigma} = V^+_{(1/2,0)} \oplus V^-_{(1/2,0)}$

$$\widetilde{\mathrm{ad}}(x_0) = 0, \quad \widetilde{\mathrm{ad}}(x_i) = \begin{pmatrix} \pi_{1/2}(x_i) & 0 \\ 0 & \pi_{1/2}(x_i) \end{pmatrix}, \quad x \in \{1, 2, 3\}, (1)$$

where we have fixed the action of the center to be zero.

- Denote by D and D_q the Dirac operators on $SU(2)$ and $SU_q(2)$, Define

$$\hat{D} = \begin{pmatrix} 0 & iD + \partial(x_0) \\ -iD + \partial(x_0) & 0 \end{pmatrix}, \quad \hat{D}_q = \begin{pmatrix} 0 & iD_q + \partial(c) \\ -iD_q + \partial(c) & 0 \end{pmatrix}.$$

where we have applied the twist $F \oplus F$ with (1) and the fact that x_0 is central.

- As a vector space $\mathbb{C}[U_q(2)]$ is spanned by

$$t^{l,k}_{m,n} = |l, k, n\rangle \otimes \langle l, k, m| \in V_{(l,k),q} \otimes V^*_{(l,k),q} \simeq V_{(l,k),q} \otimes V_{(l,k),q}$$

where (l, k) parametrize the representatitions.

- As a vector space $\mathbb{C}[U_q(2)]$ is spanned by

$$t^{l,k}_{m,n} = |l,k,n\rangle \otimes \langle l,k,m| \in V_{(l,k),q} \otimes V^*_{(l,k),q} \simeq V_{(l,k),q} \otimes V_{(l,k),q}$$

where (l,k) parametrize the representatitions.

- The Hilbert space decomposes into irreducible compenents

$$\mathbf{H} = L^2(U_q(2)) \otimes \hat{\Sigma} =$$

$$W^{\uparrow}_{0,+} \oplus W^{\uparrow}_{0,-} \oplus \bigoplus_{2j=1}^{\infty} \bigoplus_{k} W^{\uparrow}_{j,k,+} \oplus W^{\downarrow}_{j,k,+} \oplus W^{\uparrow}_{j,k,-} \oplus W^{\downarrow}_{j,k,-}.$$

For fixed k and $\pm$, the decomposition is given exactly as in the $SU_q(2)$ case. Thus, we can fix a basis

$$\{|j\mu n \uparrow k\pm\rangle, |j'\mu' n \downarrow k\pm\rangle : j, j', \mu, \mu', n; k \in \mathbb{Z} + j\}$$

so that j, j', μ, μ' and n are restricted as earlier.

- As a vector space $\mathbb{C}[U_q(2)]$ is spanned by

$$t^{l,k}_{m,n} = |l,k,n\rangle \otimes \langle l,k,m| \in V_{(l,k),q} \otimes V^*_{(l,k),q} \simeq V_{(l,k),q} \otimes V_{(l,k),q}$$

where (l,k) parametrize the representatitions.

- The Hilbert space decomposes into irreducible compenents

$$\mathbf{H} = L^2(U_q(2)) \otimes \hat{\Sigma} =$$

$$W^{\uparrow}_{0,+} \oplus W^{\uparrow}_{0,-} \oplus \bigoplus_{2j=1}^{\infty} \bigoplus_{k} W^{\uparrow}_{j,k,+} \oplus W^{\downarrow}_{j,k,+} \oplus W^{\uparrow}_{j,k,-} \oplus W^{\downarrow}_{j,k,-}.$$

For fixed k and $\pm$, the decomposition is given exactly as in the $SU_q(2)$ case. Thus, we can fix a basis

$$\{|j\mu n \uparrow k\pm\rangle, |j'\mu' n \downarrow k\pm\rangle : j, j', \mu, \mu', n; k \in \mathbb{Z} + j\}$$

so that j, j', μ, μ' and n are restricted as earlier.

- Action of $\hat{D}_q$:

$$\hat{D}_q|j\mu n \uparrow k\pm\rangle = \mp i(2j + 3N + k)|j\mu n \uparrow k\mp\rangle$$
$$\hat{D}_q|j\mu n \downarrow k\pm\rangle = \mp i(-(2j+2) + 3N + k)|j\mu n \downarrow k\mp\rangle$$

- Theorem: The triple $(\mathbb{C}[U_q(2)], \hat{D}_q, \mathbf{H})$ is a regular 4^+-summable and regular spectral triple.