Non-formal deformation quantization of the Heisenberg supergroup

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Introduction

- Non-formal deformation quantization of a manifold M. $(C^\infty(M), \star)$: noncommutative algebra.
- For an abelian Lie group G, Rieffel formula $\rightsquigarrow$ deformation of $C^\infty(G)$.
- Universal deformation formula: it deforms also any algebra $\mathbf{A}$ on which G acts (Drinfeld twist).
- QFT point of view: the real scalar ϕ^4 theory on the deformation of $\mathbb{R}^m$ is not renormalizable.

$\Rightarrow$ Rieffel formula is not universal for the ϕ^4 theory.

Goal: To find a universal formula for ϕ^4.
$\rightarrow$ deformation of the Heisenberg supergroup.

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1. Basics of Supergeometry
2. Deformation Quantization
3. Universal Deformation Formula
4. Application to renormalization

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Basic notions

- Essence of the concrete approach of Supergeometry: replace real field $\mathbb{R}$ by a real supercommutative algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \simeq \Lambda V = \mathbb{R} \oplus$ nilpotents $\quad ab = (-1)^{|a||b|} ba$

- Superspace of dim $m|n$ $\mathbb{R}^{m|n} := (\mathcal{A}_0)^m \times (\mathcal{A}_1)^n$

- Smooth map $f : \mathbb{R}^{m|n} \to \mathcal{A}$ if $\exists f_I \in C^\infty(\mathbb{R}^m)$ $(I \subset \{1, .., n\})$ $\forall (x, \xi) \in \mathbb{R}^{m|n}$, $\quad (\xi^I = \Pi_{i \in I} \xi^i)$

$$f(x, \xi) = \sum_I \tilde{f}_I(x) \xi^I$$

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Structure on superfunctions

- Complex-valued smooth functions: $C^\infty(\mathbb{R}^{m|n}) \simeq C^\infty(\mathbb{R}^m) \otimes \bigwedge \mathbb{R}^n$.
- Berezin integration: $\int d\xi\, f(x,\xi) = f_{\{1,..,n\}}(x)$
- Product: $\xi^I \xi^J = \varepsilon(I,J)\xi^{I\cup J}$
- Natural superhermitian scalar product:

$$\langle f,g\rangle = \int dx d\xi\, \overline{f(x,\xi)} g(x,\xi) = \sum_I \varepsilon(I,\complement I) \int dx \overline{f_I(x)} g_{\complement I}(x)$$

- Hodge operation: $*\xi^I = \varepsilon(I,\complement I)\xi^{\complement I}$
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Heisenberg Supergroup

- Even symplectic form on $\mathbb{R}^{m|n}$ (with m even):

$$\omega \equiv \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$$

- Heisenberg supergroup $G = \mathbb{R}^{m|n} \times \mathbb{R}^{1|0}$ with

$$(x_1, \xi_1, a_1)\cdot(x_2, \xi_2, a_2) = (x_1 + x_2, \xi_1 + \xi_2, a_1 + a_2 + \frac{1}{2}\omega((x_1, \xi_1), (x_2, \xi_2)))$$

- Non-abelian, neutral element: 0, $(x + aZ)^{-1} = -x - aZ$.
- Quotient: $\mathbb{R}^{m|n} = G/\mathbb{R}^{1|0}$ abelian supergroup $(x_1, \xi_1)\cdot(x_2, \xi_2) = (x_1 + x_2, \xi_1 + \xi_2)$.

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Quantization map

- Choice of a polarization and Kirillov's orbits method.

$\rightarrow$ (induced) Schrödinger representation of G

- Generalization of Weyl's quantization:

$$\Omega : L^1(\mathbb{R}^{m|n}) \rightarrow \mathcal{L}(L^2(\mathbb{R}^{\frac{m}{2}|n}))$$

- $\mathcal{B}$-space of Schwartz:

$$\mathcal{B}(\mathbb{R}^{m|n}) = \{f \in C^\infty(\mathbb{R}^{m|n}), \forall D^\alpha, \|f\|_\alpha = \sup_{x \in \mathbb{R}^m} \sum_I |D^\alpha f_I(x)| < \infty\}$$

- Extension of the quantization map (oscillating integrals):

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Deformed product

Product given by von Neumann-Rieffel formula.
$\star : \mathcal{B}(\mathbb{R}^{m|n}) \times \mathcal{B}(\mathbb{R}^{m|n}) \to \mathcal{B}(\mathbb{R}^{m|n})$

$$(f \star g)(z) = \kappa \int_{(\mathbb{R}^{m|n})^2} \mathrm{d}z_1 \mathrm{d}z_2 \, f(z_1) g(z_2) e^{\frac{-2i}{\theta}(\omega(z_1,z_2)+\omega(z_2,z)+\omega(z,z_1))}$$

Proposition (Bieliavsky A.G. Tuynman '10)

- $\Omega(f \star g) = \Omega(f)\Omega(g)$
- Ω is injective. $\Rightarrow \star$ is associative.
- $(\mathcal{B}(\mathbb{R}^{m|n}), \star) \simeq (\mathcal{B}(\mathbb{R}^{m}), \star) \otimes Cl(n, \mathbb{C})$.
- Symmetries: $OSP(n, \frac{m}{2}) \ltimes \mathbb{R}^{m|n}$.

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Deformed product

Product given by von Neumann-Rieffel formula.
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Universality for Fréchet algebras

- Let $(\mathbf{A}, \|\cdot\|_j)$ be a Fréchet algebra.

$\rightarrow$ Deformed product on $\mathcal{B}(\mathbb{R}^{m|n})$ can be extended to $\mathcal{B}_{\mathbf{A}}(\mathbb{R}^{m|n})$.

- Let $\rho : \mathbb{R}^{m|n} \times (\mathbf{A} \otimes \mathcal{A}) \to (\mathbf{A} \otimes \mathcal{A})$ be an action of $\mathbb{R}^{m|n}$ on $\mathbf{A}$:

$$\rho_{z_1+z_2} = \rho_{z_1}\rho_{z_2}, \quad \rho_0 = \mathrm{id}, \quad \rho_z(ab) = \rho_z(a)\rho_z(b),$$

$+$ some technical axioms on continuity...

- Smooth vectors space $\mathbf{A}^\infty$ is dense in $\mathbf{A}$

$$\mathbf{A}^\infty = \{a \in \mathbf{A},\ z \mapsto \rho_z(a) \in C^\infty_{\mathbf{A}}(\mathbb{R}^{m|n})\}$$

Theorem (Bieliavsky A.G. Tuynman '10)

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Definition (Bieliavsky A.G. Tuynman '10)

A Hilbert superspace of parity n is a $\mathbb{Z}_2$-graded Hilbert space $(\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, (\cdot,\cdot))$ with $(\mathcal{H}_0, \mathcal{H}_1) = 0$, endowed with a unitary operator $J \in \mathcal{L}(\mathcal{H})$ of degree n, satisfying $J^2(x) = (-1)^{(n+1)|x|}x$.

- Superhermitian scalar product: $\langle x, y\rangle := (J(x), y)$
- Example: $\mathcal{H} = L^2(\mathbb{R}^{m|n})$, $J = *$, $(f,g) = \int dz \overline{f(z)}(*g)(z)$.
- Superadjoint: $\forall T \in \mathcal{L}(\mathcal{H})$, $\exists T^\dagger \in \mathcal{L}(\mathcal{H})$, $\forall x, y \in \mathcal{H}$,

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C*-superalgebras

Superinvolution of a $\mathbb{Z}_2$-graded algebra $\mathbf{A}$: map $^\dagger : \mathbf{A} \to \mathbf{A}$ of degree 0 such that $(a^\dagger)^\dagger = a$, and $(a \cdot b)^\dagger = (-1)^{|a||b|} b^\dagger \cdot a^\dagger$

Definition (Bieliavsky A.G. Tuynman '10)

A C*-superalgebra is a superinvolutive $\mathbb{Z}_2$-graded Banach algebra $\mathbf{A}$ which can be represented on a Hilbert superspace $(\mathcal{H}, J)$ by an isometric representation (compatible with the superinvolution) of degree 0:

$$\varrho : \mathbf{A} \to \mathcal{L}(\mathcal{H})$$

- $(\mathbf{A} = L^\infty(\mathbb{R}^{m|n}), \|f\| = \sum_I \|f_I\|_\infty)$: C*-superalgebra, represented by multiplication on $\mathcal{H} = L^2(\mathbb{R}^{m|n})$.

$\to$ Notion of noncommutative superspace.

- Compatible with the universal deformation.
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QFT on the Moyal space

- Renormalizable action-functional on the Moyal space $\mathbb{R}^m_\theta$:

(Grosse Wulkenhaar '04) $\phi : \mathbb{R}^m \to \mathbb{R}$

$$S[\phi] = \int d^m x \big(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{\Omega^2}{2} x^2 \phi^2 + \frac{M^2}{2}\phi^2 + \lambda\, \phi \star \phi \star \phi \star \phi \big)$$

- Renormalizable ϕ^4-action on $\mathbb{R}^{m|1}$, with $\eta = 1 + b\xi$ ($b \in \mathbb{R}$):

$$\mathrm{tr}\Big(\frac{1}{2}|[-\frac{i}{2}\widetilde{x}_\mu \eta, \phi\eta]_\star|^2 + \frac{M^2}{2}(\phi\eta)^{\star 2} + \lambda(\phi\eta)^{\star 4}\Big)$$

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- Change the space, not the ϕ^4-action.

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$$= \int d^m x \Big(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{b^4}{8} x^2 \phi^2 + \frac{M^2}{2}\phi^2 + \lambda(1 + \frac{b^4 \theta}{16}) \phi \star \phi \star \phi \star \phi \Big)$$

- Change the space, not the ϕ^4-action.

$\to$ Universality of the deformation for the ϕ^4-action: $\mathbb{R}^m \to \mathbb{R}^{m|1}_\theta$.

$\to$ potentially renormalizable theories on other NC spaces.

Outlook

Results:

- Deformation of $\mathcal{B}(\mathbb{R}^{m|n})$, $\simeq$ Moyal$\otimes$Clifford
- Notion of C*-superalgebra: noncommutative superspace
- Deformation of C*-superalgebras on which $\mathbb{R}^{m|n}$ is acting
- ϕ^4-action consistency: $\mathbb{R}^4 \to \mathbb{R}^{4|1}_\theta$

Perspectives:

- Example of supertorus, classification of foliations
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