Cellularity and the Jones basic construction

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Introduction

The goal of this talk is to describe an application of the Jones basic construction to finite dimensional algebra. The objects of study are certain finite dimensional algebras that arise in invariant theory, knot theory, subfactors, QFT, and statistical mechanics.

The algebras in question have parameters; for generic values of the parameters, they are semisimple, but it is also interesting to study non–semisimple specializations.

It turns out that the Jones basic construction is still useful in the non–semisimple world (although it is not *a priori* evident what it should actually mean in the non–semisimple setting.)

Our objects of study

The algebras in question come in **pairs of towers**

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \quad \text{and} \quad Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \dots,$$

where Q_n is always a quotient of A_n, such as:

- $A_n =$ Brauer algebra, $Q_n =$ symmetric group algebra.
- $A_n =$ BMW algebra, $Q_n =$ Hecke algebra.
- $A_n =$ Jones Temperley Lieb algebra, $Q_n =$ algebra of scalars.
- $A_n =$ cyclotomic BMW algebra, $Q_n =$ cyclotomic Hecke algebra.
- $A_n =$ partition algebra, $Q_n =$ “stuttering" sequence of symmetric group algebras.
- etc.

Role of the generic ground ring

It is necessary to consider these algebras over integral domains, and not just over fields. In fact, in the examples, there is a “generic integral ground ring" R for the algebras, such that:

1. Any “instance" of the algebras is a specialization of that over R. There exist non–semisimple specializations.
2. On the other hand, with F the field of fractions of R, the algebras over F are semisimple.

Somehow, considering the “integral form" of the algebras, over R, one is delicately balanced between the semisimple world and the nonsemisimple world.

I also should note that if we consider algebras over a ring R, it is not automatic that they will be free as R–modules; in fact, existence of R–bases is one of the goals.

Wenzl's semisimplicity method

In the 80's, Wenzl gave a method, applicable to these examples, of proving generic semisimplicity of the algebras A_n from the known generic semisimplicity of the algebras Q_n.

This involved recognizing Jones basic constructions inside the tower $(A_n)_{n \geq 0}$.

Wenzl's method, cont.

The basic idea is that A_n contains an essential idempotent e_{n-1} such that, with $J_n = A_n e_{n-1} A_n$, we have $Q_n \cong A_n/J_n$. Moreover:

1. If A_k is already known to be s.s. for $k \leq n$ and if a certain canonical trace on A_k is also non-degenerate for $k \leq n$, then J_{n+1} is isomorphic to the Jones basic construction for $A_{n-1} \subset A_n$, and is therefore s.s.
2. On the other hand, $Q_{n+1} = A_{n+1}/J_{n+1}$ is already known to be s.s. and thus A_{n+1} is s.s.

To continue the induction, one needs the non-degeneracy of the trace on A_{n+1} and this requires particular methods adapted to each example.

Wenzl's method, cont.

Remarks:

1. Wenzl (1988) used this method to solve a longstanding problem about semisimplicity of the Brauer algebras.
2. The Bratteli diagram for the A_n's is obtained from that of the Q_n's by repeated reflections. Think:

infinite principal graph	$\rightsquigarrow$	Bratteli diagram of tower of relative commutants

   in subfactor theory.

What is cellularity?

The goal of this work is a "cellular version" of Wenzl's method. So what is cellularity?

Cellularity is an organizing framework (due to Graham and Lehrer, 1996) for studying the possibly non-semisimple representation theory of certain algebras, like Hecke algebras, BMW algebras, etc.

The detailed definition is a bit opaque at first sight. I will give a different formulation than the original, and leave out some details. (The description will still be opaque.)

What is cellularity, cont.

Consider an algebra A with involution $*$ over an integral domain S. **The algebra A is cellular if there exist:**

1. A finite partially ordered set Λ.
2. A family $\{\Delta^\lambda : \lambda \in \Lambda\}$ of A–modules, each free of finite rank as an S–module.
3. An increasing map $\Gamma \mapsto A_\Gamma$ from order ideals of Λ to $*$–invariant 2–sided ideals of A, satisfying several properties, the most important of which is:
4. Whenever $\Gamma \subseteq \Gamma'$ are order ideals with $\Gamma' \setminus \Gamma = \{\lambda\}$, there is an isomorphism of A–A bimodules

$$\alpha : A_{\Gamma'}/A_\Gamma \longrightarrow \Delta^\lambda \otimes_S (\Delta^\lambda)^*,$$

with $\alpha \circ * = * \circ \alpha$.

Comments on the definition of cellularity

1. The modules Δ^λ for $\lambda \in \Lambda$ are called the **cell modules**. The map $\Gamma \to A_\Gamma$ is called a Λ**–cell net**.
2. Let $\{c_t^\lambda : t \in \mathscr{T}(\lambda)\}$ be an S–basis of Δ^λ. Then via the maps α, these bases can be lifted to an S–basis of A

$$\{c_{s,t}^\lambda : \lambda \in \Lambda \text{ and } s, t \in \mathscr{T}(\lambda)\}.$$

The usual definition of cellularity is given in terms of properties of this basis, called a **cellular basis.**

What is cellularity good for?

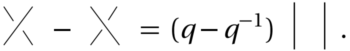
1. When the ground ring is a field, and the algebra A happens to be semisimple, the cell modules Δ^λ are exactly the simple modules.
2. In general, Δ^λ has a canonical bilinear form. With $\mathrm{rad}(\lambda)$ the radical of this form, and with the ground ring a field, $\Delta^\lambda/\mathrm{rad}(\lambda)$ is either zero or simple, and all simples are of this form.
3. Cellularity is preserved under specialization of the ground ring (and specialization takes cell modules to cell modules).

Thus for our examples, it suffices to show cellularity for the algebras over the generic integral ground ring.

Cellularity – Example, the Hecke algebras

Definition 1

Let S be a ring and q an invertible element of S.
The Hecke algebra $H_n^S(q^2)$ over S is the quotient of the braid group algebra over S by the Hecke skein relation:



Properties of the Hecke algebras

- The Hecke algebras $H_n^S(q^2)$ are cellular, with partially ordered set $\Lambda_n =$ the set of Young diagrams with n boxes, ordered by *dominance*.
- The cell modules Δ^λ are known as *Specht modules*. They have bases labelled by the set $\mathscr{T}(\lambda)$ of standard tableaux of shape λ.
- The generic integral ground ring for the Hecke algebras is $R = \mathbb{Z}[\boldsymbol{q}, \boldsymbol{q}^{-1}]$ where $\boldsymbol{q}$ is an indeterminant. The Hecke algebras over the field of fractions $F = \mathbb{Q}(\boldsymbol{q})$ of R are semisimple.

Coherence of cellular structures

It is a general principle that representation theories of the Hecke algebras $H_n(q)$ or of the symmetric group algebras KS_n should be considered all together, that induction/restriction between H_n and H_{n-1} plays a role in building up the representation theory.

Coherence of cellular structures is the cellular version of this principle.

Coherence – definition

Definition 2

A sequence $(A_n)_{n\geq 0}$ *of cellular algebras with is* **coherent** *if restrictions of cell modules from* A_n *to* A_{n-1} *have filtrations by cell modules, and likewise induced modules of cell modules from* A_n *to* A_{n+1} *have filtrations by cell modules.*

Definition 3

Let Λ_n *denote the partially ordered set in the cellular structure of* A_n. **Strong coherence** *adds the requirement that the order in which the cell modules occur in the filtrations is consistent with the order of the partially ordered sets* Λ_n *(and the multiplicities are zero or one.)*

Coherence – example

Example 4

The sequence of Hecke algebras $H_n^S(q^2)$ is a strongly coherent tower of cellular algebras. This follows by results of Dipper, James, Murphy, and Jost.

The point of strong coherence (for a tower of algebras $(A_n)_{n\geq 0}$) is that one gets bases of the cell modules indexed by paths on the Bratteli diagram for the generic semisimple representation theory. Up to a correction, elements of A_k for $k < n$ act on the initial segment of length k of a path.

Ready for the main theorem

The theorem on the following slide represents a cellular adaptation of Wenzl's method. It gives a way of lifting cellular structures from the quotient algebras Q_n to the algebras A_n. just as Wenzl's method lifted semisimplicity from the Q_n's to the A_n's.

Main theorem

Theorem 5

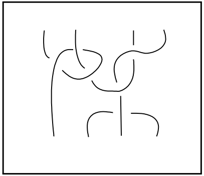
Suppose $(A_n)_{n\geq 0}$ and $(Q_n)_{n\geq 0}$ are two sequence of $*$ - algebras over R. Let F be the field of fractions of R. Assume:

1. $(Q_n)_{n\geq 0}$ *is a (strongly) coherent tower of cellular algebras.*
2. $A_0 = Q_0 \cong R$, $A_1 \cong Q_1$.
3. *For each $n \geq 2$, A_n has an essential idempotent $e_{n-1} = e^*_{n-1}$ and with $J_n = A_n e_{n-1} A_n$, one has $A_n/J_n \cong Q_n$ as $*$–algebras.*
4. $A_n^F = A_n \otimes_R F$ *is split semisimple.*
5. (a) e_{n-1} commutes with A_{n-2}, and $e_{n-1}A_{n-1}e_{n-1} \subseteq A_{n-2}e_{n-1}$, (b) $A_n e_{n-1} = A_{n-1}e_{n-1}$ and $x \mapsto xe_{n-1}$ is injective from A_{n-1} to $A_{n-1}e_{n_1}$, and (c) $e_{n-1} = e_{n-1}e_n e_{n-1}$.

Then $(A_n)_{n\geq 0}$ is a (strongly) coherent tower of cellular algebras.

Example: The BMW algebras

The introduction of the BMW algebra was motivated by the Kauffman link invariant, which is an invariant defined by certain skein relations.
The BMW algebra is an algebra of braid like objects, namely

framed (n, n)*–tangles* :  .

Tangles can be represented by quasi-planar diagrams as shown here. Tangles are multiplied by stacking (like braids).

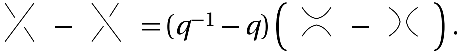
Important note: closed loops, which may be knotted and linked with each other and with the n non–closed strands are allowed. In particular $(0, 0)$–tangles are framed link diagrams.

Definition of BMW algebras

Definition 6

Let S be a commutative unital ring with invertible elements ρ, q and an element δ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$.

The BMW algebra W_n^S is the S–algebra of framed (n, n)–tangles, modulo the Kauffman skein relations:

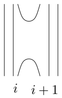
1. *(Crossing relation)*
   

2. *(Untwisting relation)* 

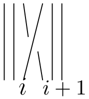
3. *(Free loop relation)* $T \cup \bigcirc = \delta T$, *where* $T \cup \bigcirc$ *is the union of a tangle T and an additional closed loop with zero framing (and with no crossings with the components of T).*

The BMW algebras, properties

- The following tangles generate the BMW algebra

$e_i =$ 

and

$g_i =$ 

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- The element e_i is an essential idempotent with $e_i^2 = \delta e_i$.
- Let $J_n = W_n e_{n-1} W_n$. One has $W_n / J_n \cong H_n(q^2)$, where H_n is the Hecke algebra.
- That $W_0^S \cong S$ is equivalent to the existence of the Kauffman link invariant.

BMW algebras, properties cont.

- When we specialize to $q = 1$ and $\rho = 1$, we get the *Brauer algebras*, introduced by Brauer in connection with invariant theory of orthogonal and symplectic groups.
- W_n^S imbeds in W_{n+1}^S by adding an additional strand.
- The BMW algebras have an S–linear algebra involution, acting by turning tangle diagrams upside down.
- There is a generic ground ring for the BMW algebras, namely

$$R = \mathbb{Z}[\boldsymbol{q}^{\pm 1}, \boldsymbol{\rho}^{\pm 1}, \boldsymbol{\delta}]/J,$$

where J is the ideal generated by

$$\boldsymbol{\rho}^{-1} - \boldsymbol{\rho} - (\boldsymbol{q}^{-1} - \boldsymbol{q}^{-1})(\boldsymbol{\delta} - 1)$$

and where the bold symbols denote indeterminants.

BMW algebras, properties 3

- The generic ground ring R is an integral domain, with field of fractions $F = \mathbb{Q}(\boldsymbol{q}, \boldsymbol{\rho})$, and

$$\boldsymbol{\delta} = (\boldsymbol{\rho}^{-1} - \boldsymbol{\rho})/(\boldsymbol{q}^{-1} - \boldsymbol{q}) + 1$$

  in F.
- BMW algebras over F are semisimple by Wenzl's theorem.

The BMW algebras, application of our theorem

Now let's see what's involved in applying the theorem to the BMW algebras (with $A_n = W_n^R$, and $Q_n = H_n^R(q^2)$).

- Hypothesis (1) is the (strong) coherence of the sequence of Hecke algebras, which is a significant but known theorem about Hecke algebras.
- Hypothesis (2) is equivalent to the existence of Kauffman's link invariant, another significant but known theorem.
- Hypothesis (4) on the semisimplicity of W_n^F is Wenzl's theorem.
- Everything else is elementary.

All the other examples work pretty much the same way (except hypothesis (2) is trivial in other examples).

The Jones basic construction

Let's try to see where the Jones basic construction is hiding in our cellular context. It is not *a priori* clear what it should mean in our setting.
In more familiar contexts, the basic construction is a machine which, given a pair of algebras $\mathbf{1} \in A \subseteq B$, will produce a third algebra J, with $A \subseteq B \subseteq J$.

If A and B are split semisimple over a field F, then it is clear what J should be, namely $J = \mathrm{End}(B_A)$.

Suppose now that A and B are not only split s.s., but also that we have an F–valued trace on B, which is faithful on B and has faithful restriction to A.

Basic construction, cont.

In the case just described, we have a conditional expectation $\varepsilon : B \to A$ determined by $\mathrm{tr}(ba) = \mathrm{tr}(\varepsilon(b)a)$ for $b \in B$ and $a \in A$. Then we also have

$$\mathrm{End}(B_A) = B\varepsilon B \cong B \otimes_A B,$$

where the latter isomorphism is as B–B bimodules

Now consider, for example, three successive **BMW algebras** $A_{n-1} \subseteq A_n \subseteq A_{n+1}$. We would like to understand the ideal $J_{n+1} = A_{n+1} e_n A_{n+1}$ in A_{n+1}.

Basic construction, slide 3

If we are in Wenzl's setting, all the algebras in sight are s.s. and

$$J_{n+1} \cong \mathrm{End}((A_n)_{A_{n-1}}) \cong A_n \otimes_{A_{n-1}} A_n.$$

That is, J_{n+1} is the basic construction for $A_{n-1} \subseteq A_n$.

Does some part of this persist in the non–s.s. case? If we work over the generic integral ground ring for the BMW algebras, J_{n+1} is no longer even a unital algebra and A_n is not even projective as an A_{n-1} module. Nevertheless, it remains true that

$$J_{n+1} \cong A_n \otimes_{A_{n-1}} A_n.$$

Quick sketch of the inductive proof of cellularity

Assume A_k ($k \le n$) is already known to be a (strongly) coherent tower of cellular algebras. If Λ_{n-1} is the partially ordered set in the cellular structure of A_{n-1}, and $\Gamma \mapsto J(\Gamma)$ is the Λ_{n-1}–cell net, we attempt to show that

$$A_n e_n J(\Gamma) A_n \cong A_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} A_n.$$

and that

$$\Gamma \mapsto A_n e_n J(\Gamma) A_n$$

is a Λ_{n-1}–cell net in J_{n+1}. This establishes that J_{n+1} is a "cellular ideal" in A_{n+1}. It is already known that A_{n+1}/J_{n+1} is cellular, and cellularity is preserved under extensions of this sort, so A_{n+1} is cellular.
To continue the induction, one still has to show that one has (strong) coherence up to level $n+1$.