Symmetries of Lévy Processes on compact quantum groups

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Outline

Compact Quantum Groups: definition

Definition (Woronowicz)

A **compact quantum group** $\mathbb{G}$ is a pair (A, Δ), where A is a unital C^*-algebra, $\Delta : \mathsf{A} \longrightarrow \mathsf{A} \otimes \mathsf{A}$ is a unital, $*$-homomorphism which is coassociative, i.e.

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta) \circ \Delta$$

such that the quantum cancellation rules are satisfied

$$\overline{\mathrm{Lin}}((1 \otimes \mathsf{A})\Delta(\mathsf{A})) = \overline{\mathrm{Lin}}((\mathsf{A} \otimes 1)\Delta(\mathsf{A})) = \mathsf{A} \otimes \mathsf{A}.$$

Unitary corepresentations

- n-**dimensional unitary corepresentation** of $\mathbb{G} = (\mathsf{A}, \Delta)$: $U = (u_{jk})_{1 \le j,k \le n} \in M_n(\mathsf{A})$ a unitary such that for all $j, k = 1, \ldots, n$ we have
$$\Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk}.$$
- Let $(U^{(s)})_{s \in \mathcal{I}}$ be a complete family of mutually inequivalent irreducible unitary correpresentations of A
- The algebra of the "polynomial" functions of $\mathsf{A} = \mathrm{Pol}(\mathbb{G})$ is defined as
$$\mathcal{A} = \mathrm{Lin}\{u_{jk}^{(s)}; s \in \mathcal{I}, 1 \le j, k \le n_s\},$$
where n_s is the dimension of $u^{(s)}$.

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where n_s is the dimension of $u^{(s)}$.

$\mathcal{A}$ is a dense $*$-subalgebra of A, which is a Hopf $*$-algebra with
$$\varepsilon(u_{jk}^{(s)}) = \delta_{jk} \quad \text{and} \quad S(u_{jk}^{(s)}) = (u_{kj}^{(s)})^*.$$

Example $SU_q(N)$

The quantum group $SU_q(N)$ is generated by the matrix elements of $U = [u_{ij}]_{i,j=1,\ldots,N}$ satisfying the relations

$$\begin{aligned} u_{ij}u_{kj} &= q u_{kj}u_{ij} \quad \text{for } i<k, & (1)\\ u_{ij}u_{il} &= q u_{il}u_{ij} \quad \text{for } j<l, & (2)\\ u_{ij}u_{kl} &= u_{kl}u_{ij} \quad \text{for } i<k, j>l, & (3)\\ u_{ij}u_{kl} &= u_{kl}u_{ij} + q^{-1}(1-q^2)u_{il}u_{kj} \quad \text{for } i<k, j<l, & (4) \end{aligned}$$

with the additional requirement on the *quantum determinant*

$$D = D(U) := \sum_{\sigma\in S_n} (-q)^{i(\sigma)} u_{1,\sigma(1)}\cdots u_{n,\sigma(n)} = 1.$$

The involution is determined by the relation $UU^* = U^*U = \mathbf{1}$.

Example $SU_q(N)$

We have

$$\mathcal{A} = *\text{-Alg}\{u_{ij}; i,j = 1,\ldots,N\}$$

$$\Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk}, \quad \varepsilon(u_{jk}) = \delta_{jk}, \quad S(u_{jk}) = u_{jk}^*.$$

The matrix U is a corepresenatation and the family of irreducible, inequivalent, unitary corepresetations is indexed by $(\frac{1}{2}\mathbb{N})^{N-1}$.

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E.g., for $SU_q(2)$, $U^{(0)} = (\mathbf{1})$, $U^{(\frac{1}{2})} = U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$, with $\alpha = u_{11}$, $\gamma = u_{21}$,

$$U^{(1)} = \begin{pmatrix} \alpha^2 & -q\sqrt{1+q^2}\gamma^*\alpha & q^2(\gamma^*)^2 \\ \sqrt{1+q^2}\gamma\alpha & 1-(1+q^2)\gamma^*\gamma & -q\sqrt{1+q^2}\alpha^*\gamma^* \\ \gamma^2 & \sqrt{1+q^2}\alpha^*\gamma & (\alpha^*)^2 \end{pmatrix},$$

etc.

The Haar state

Notation: for $a \in \mathsf{A}$ and $\xi, \xi' \in \mathsf{A}'$

$$\begin{aligned} \xi \star \xi'(a) &= (\xi \otimes \xi')\Delta(a) \\ \xi \star a &= (\mathrm{id} \otimes \xi)\Delta(a) \\ a \star \xi &= (\xi \otimes \mathrm{id})\Delta(a) \end{aligned}$$

Theorem (Woronowicz)

Let (A, Δ) be a compact quantum group. There exists unique state (called the **Haar measure**) h on A such that

$$a \star h = h \star a = h(a)I, \quad a \in \mathsf{A}.$$

In general h is not a trace!

Woronowicz characters

Theorem (Woronowicz)

Then there exists a unique family $(f_z)_{z\in\mathbb{C}}$ of linear multiplicative functionals on $\mathcal{A}$, called the **Woronowicz characters**, such that:

1. $f_z(\mathbf{1}) = 1$ for $z \in \mathbb{C}$ and $f_0 = \varepsilon$
2. $\mathbb{C} \ni z \mapsto f_z(a) \in \mathbb{C}$ is an entire holomorphic function.
3. $f_{z_1} \star f_{z_2} = f_{z_1+z_2}$ for any $z_1, z_2 \in \mathbb{C}$.
4. $f_z(S(a)) = f_{-z}(a)$ and $f_{\bar{z}}(a^*) = \overline{f_{-z}(a)}$, for any $z \in \mathbb{C}$, $a \in \mathcal{A}$.
5. $S^2(a) = f_{-1} \star a \star f_1$ for $a \in \mathcal{A}$.
6. $h(ab) = h\left(b(f_1 \star a \star f_1)\right)$ for $a, b \in \mathcal{A}$.

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Example for $SU_q(2)$: $f_z(u^{(s)}_{jk}) = q^{z(j+k)}\delta_{jk}$

Noncommutative Lévy Processes

Let $\mathcal{A}$ be a $*$-bialgebra and let $(\mathcal{P}, \Phi)$ be a noncommutative probability space.

- a **random variable** on $\mathcal{A}$ over $(\mathcal{P}, \Phi)$ is a $*$-algebra homomorphism from $\mathcal{A}$ into the space $(\mathcal{P}, \Phi)$
- the **distribution** of the random variable $j : \mathcal{A} \to \mathcal{P}$ is the state $\varphi_j = \Phi \circ j$
- a **quantum stochastic process** on $\mathcal{A}$ is a family of random variables $(j_t)_{t \in J}$ on $\mathcal{A}$ indexed by a set J
- the **convolution product** of $j_1, j_2 : \mathcal{A} \to \mathcal{P}$ is the random variable $j_1 \star j_2 = m_{\mathcal{P}} \circ (j_1 \otimes j_2) \circ \Delta$, where $m_{\mathcal{P}}$ denotes the product in $\mathcal{P}$.

Noncommutative Lévy Processes

A quantum stochastic process $(j_{st})_{0\leq s\leq t\leq T}$ $(T\in\mathbb{R}_+\cup\{\infty\})$ on a $*$-bialgebra $\mathcal{A}$ over $(\mathcal{P},\Phi)$ is called **Lévy process** if it satisfies:

- (**increment property**)

$$j_{rs}\star j_{st}=j_{rt}\quad\text{for all } 0\leq r\leq s\leq t\leq T$$

  and $j_{tt}=\varepsilon\mathbf{1}_{\mathcal{P}}$ for all $0\leq t\leq T$,
- the increments (j_{st}) are (tensor) **independent**, i.e. for disjoint intervals $(t_i,s_i]$

$$\Phi\big(j_{s_1t_1}(a_1)\ldots j_{s_nt_n}(a_n)\big)=\Phi\big(j_{s_1t_1}(a_1)\big)\ldots\Phi\big(j_{s_nt_n}(a_n)\big)$$

  and $[j_{s_i,t_i}(a_1),j_{s_j,t_j}(a_2)]=0$ for $i\neq j$,
- the increments (j_{st}) are **stationary**, i.e. $\varphi_{st}=\Phi\circ j_{st}$ depends only on $t-s$,
- (**weak continuity**) j_{st} converges to j_{ss} in distribution for $t\searrow s$.

The convolution semigroup and the generator of a NC Lévy process

The marginal distributions $\varphi_{s-t} := \varphi_{st} = \Phi \circ j_{st}$ of a Lévy process $(j_{st})_{0\leq s\leq t}$ form a convolution semigroup of states, i.e.

- $\varphi_0 = \varepsilon$, $\varphi_s \star \varphi_t = \varphi_{s+t}$, $\lim_{t\to 0} \varphi_t(b) = \varepsilon(b)$ for all $b \in \mathcal{A}$,
- $\varphi_t(\mathbf{1}) = 1$, $\varphi_t(b^*b) \geq 0$ for all $b \in \mathcal{A}$ and $t \geq 0$.

There exists a unique functional $L : \mathcal{A} \to \mathbb{C}$, called the **generating functional**, such that

$$\varphi_t = \exp_\star tL \quad \text{and} \quad L = \frac{d}{dt}\varphi_t\Big|_{t=0}.$$

Lévy Processes and Markov semigroup

Given a convolution semigroup of states $(\varphi_t)_{t\geq 0}$, we can define a semigroup of operators

$$T_t = (\mathrm{id} \otimes \varphi_t) \circ \Delta, \quad t \geq 0.$$

Its **infinitesimal generator** $G : \mathcal{A} \to \mathcal{A}$ is the convolution operator associated to the generating functional L, i.e.

$$G(a) = (\mathrm{id} \otimes L) \circ \Delta(a) = L \star a.$$

Notation: $G = T_L$.

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Remark

$G : \mathcal{A} \to \mathcal{A}$ is a convolution operator if and only if $\Delta \circ G = (\mathrm{id} \otimes G) \circ \Delta$. In this case $L(a) = \varepsilon \circ G(a)$.

Characterisation of Generators of Convolution Semigroups

Theorem (Schoenberg correspondence):

The functional $L : \mathcal{A} \to \mathbb{C}$ is a generating functional of a convolution semigroup of states if and only if L is

- **hermitian**, i.e. $L(b^*) = \overline{L(b)}$,
- **conditionally positive**, i.e. $L(b^* b) \geq 0$ provided $\varepsilon(b) = 0$,
- and $L(\mathbf{1}) = 0$.

Lévy Processes and the Generators

noncommutative Lévy process
$(j_{st})_{0\leq s\leq t}$
$\updownarrow$
convolution semigroup of states $(\varphi_t)_{t\geq 0}$ $\leftrightarrow$ semigroup of Markov operators $(T_t)_{t\geq 0}$
$\updownarrow$ $\updownarrow$
generating functional $L:\mathcal{A}\to\mathbb{C}$ $\leftrightarrow$ infinitesimal generator $T_L:\mathcal{A}\to\mathcal{A}$
$\updownarrow$
hermitian, cond. positive $L:\mathcal{A}\to\mathbb{C}$, s.t. $L(\mathbf{1})=0$

Lévy Processes and the Generators

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Aim of the project: study the noncommutative **geometry** of a quantum group via its Lévy processes

Ideas / Problems / Questions :

- Which processes (and their generators) give interesting information about the nc geometry?
- Are nc Brownian motions (i.e. Gaussian generators) useful for that?
- What other conditions (symmetries) on the generators would be appropriate?

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- What other conditions (symmetries) on the generators would be appropriate?

- Extend the theory of Dirichlet forms associated to Markov semigroups and the construction of their derivations to the non-tracial case (cf. Cipriani & Sauvageot)

Translation invariance

Definition

We call a linear cb map $T : \mathrm{A} \to \mathrm{A}$ **translation invariant** if

$$\Delta \circ T = (\mathrm{id} \otimes T) \circ \Delta.$$

Lemma

If a linear cb map $T : \mathrm{A} \to \mathrm{A}$ is translation invariant, then for all $s \in \mathcal{I}$,

$$T(V_s) \subseteq V_s$$

where $V_s = \mathrm{Lin}\{u_{jk}^{(s)}; 1 \leq j, k \leq n_s\}$, and therefore

$$T(\mathcal{A}) \subseteq \mathcal{A}.$$

Translation invariance

Proposition

A semigroup $(T_t)_{t\geq 0}$ of CP unital maps is the Markov semigroup of a (unique) Lévy process if and only if all T_t are translation invariant.

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See also
M.J. Lindsay and A. Skalski, Convolution semigroups of states, arXiv:0905.1296v2, 2009.

$SU_q(n)$

Let

$$
\begin{aligned}
K_1 &= \ker\varepsilon, \\
K_2 &= \mathrm{Lin}\,\{a_1a_2 : a_1, a_2 \in \ker\varepsilon\}, \\
K_n &= \mathrm{Lin}\,\{a_1a_2\dots a_n : a_1, a_2, \dots, a_n \in \ker\varepsilon\}, \\
K_\infty &= \bigcap_{n\geq 1} K_n.
\end{aligned}
$$

'Commutative' part of $SU_q(n)$

Description of K_∞ for $SU_q(n)$

- $u_{ij}, u_{ij}^* \in K_\infty$ for $i \neq j$
- $u_{ii}u_{jj} = u_{jj}u_{ii}$, $u_{ii}u_{jj}^* = u_{jj}^*u_{ii}$ (modulo K_∞, for $i \neq j$)
- $u_{jj}u_{jj}^* = u_{jj}^*u_{jj} = 1$ (modulo K_∞)
- $u_{11} \ldots u_{nn} = 1$ (modulo K_∞)

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- $u_{jj}u_{jj}^* = u_{jj}^*u_{jj} = 1$ (modulo K_∞)
- $u_{11} \ldots u_{nn} = 1$ (modulo K_∞)

Proposition

The ideal K_∞ is also a coideal in $\mathcal{A}$, $\mathcal{A}/K_\infty$ is a $*$-bialgebra and

$$\mathcal{A}/K_\infty \cong \mathbb{C}(\mathbb{T}^{n-1}).$$

All processes for which $L|_{K_\infty} = 0$ are isomorphic to processes on the $(n-1)$-dimensional torus.

'Commutative' part of $SU_q(n)$

Definition

A generator L is called a **Gaussian** generator if $L|_{K_3} = 0$.

Observation

The gaussian processes on $SU_q(n)$ encode the structure of $(n-1)$-dimensional torus, i.e. they give no information on the **noncommutative** geometry of $SU_q(n)$.

For $SU_q(2)$ this was shown by
M. Schürmann and M. Skeide'1998.

Symmetric generators

We shall consider the inner product induced by the Haar state h

$$\langle a, b\rangle := h(a^* b).$$

Recall: each generator L of a Lévy process induces the operator

$$T_L(a) = L \star a = (\mathrm{id} \otimes L) \circ \Delta(a), \quad a \in \mathcal{A}.$$

Proposition

Each operator $T_L : \mathcal{A} \to \mathcal{A}$ admits unique adjoint, i.e. there exists a unique linear map $T_L^\star : \mathcal{A} \to \mathcal{A}$ such that

$$h\big(a^* T_L(b)\big) = h\big(T_L^\star(a)^* b\big)$$

for all $a, b \in \mathcal{A}$.

Symmetric generators on quantum groups

We say that a generating functional L is symmetric if the operator T_L is self-adjoint, i.e. if

$$h\big(a^* T_L(b)\big) = h\big(T_L(a)^* b\big), \quad a, b \in \mathcal{A}.$$

($\longrightarrow$ **GNS**-symmetry).
One shows

$$T_L^\star = T_{L^\# \circ S}, \quad \text{where } L^\#(a) = \overline{L(a^*)},$$

(if L is hermitian, then $L^\# = L$).

Proposition: $T_L = T_L^\star$ iff $L = L \circ S$

The Haar state is KMS

Theorem (Woronowicz):

The formula

$$\sigma_t(a) = f_{it} \star a \star f_{it}$$

defines a one parameter group of modular automorphisms of $\mathcal{A}$ and the Haar measure h is a $(\sigma, -1)$-KMS state, i.e.,

$$h(ab) = h(b(f_1 \star a \star f_1)) = h(b\sigma_i(a)), \quad a, b \in \mathcal{A}.$$

KMS-symmetry

We say that T_L is KMS-symmetric if

$$h\left(a^* T_L(b)\right) = h\Big(\big(\sigma_{-\frac{i}{2}} \circ T_L \circ \sigma_{\frac{i}{2}}\big)(a)^*\, b\Big).$$

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Using $T_L = L \star a$, $T_L^\star = (L^\# \circ S) \star a$ and $\sigma_t(a) = f_{it} \star a \star f_{it}$, we have

$$L \star a = f_{-\frac{1}{2}} \star (L^\# \circ S) \star f_{\frac{1}{2}} \star a.$$

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If L is hermitian, this reduces to

$$L(a) = \varepsilon \circ (L \star a) = (L \circ S)\big(f_{\frac{1}{2}} \star a \star f_{-\frac{1}{2}}\big)$$

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If L is hermitian, this reduces to

$$L(a) = \varepsilon \circ (L \star a) = (L \circ S)(f_{\frac{1}{2}} \star a \star f_{-\frac{1}{2}}) = (L \circ R)(a).$$

Recall: $\overline{S} = R \circ \tau_{\frac{i}{2}}$, $R(a) = S(f_{\frac{1}{2}} \star a \star f_{-\frac{1}{2}})$

KMS-symmetry

Proposition

Let $L \in \mathcal{A}'$ be hermitian. Then

$$T_L \text{ is self-adjoint} \quad \text{iff} \quad L \circ S = L.$$

KMS-symmetry

Proposition

Let $L \in \mathcal{A}'$ be hermitian. Then

$$\begin{aligned} T_L \text{ is self-adjoint} \quad &\text{iff} \quad L \circ S = L. \\ T_L \text{ is KMS-symmetric} \quad &\text{iff} \quad L \circ R = L. \end{aligned}$$

KMS-symmetry

Proposition

Let $L \in \mathcal{A}'$ be hermitian. Then

$$T_L \text{ is self-adjoint} \quad \text{iff} \quad L \circ S = L.$$
$$T_L \text{ is KMS-symmetric} \quad \text{iff} \quad L \circ R = L.$$

Remark

If L is a generating functional, then

- $L + L \circ R$ is a generating functional (it is conditionally positive!)
- $T_{L+L\circ R}$ is KMS-symmetric.

Relations between symmetry and KMS-symmetry

Relations between symmetry and KMS-symmetry

Theorem

For $L \in \mathcal{A}'$ the following are equivalent:

1. T_L commutes with the modular group σ: $T_L \circ \sigma_t = \sigma_t \circ T_L$,
2. L commutes with the Woronowicz characters: $L \star f_z = f_z \star L$ for $z \in \mathbb{C}$,
3. L is invariant under $\tau_{\frac{i}{2}}$: i.e. $L \circ \tau_{\frac{i}{2}} = L$.

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Remark

- If L is symmetric, then L commutes with the Woronowicz characters and is also KMS-symmetric.
- If the algebra is of Kac type ($S^2 = \mathrm{id}$), then $R = S$ and the symmetric and KMS-symmetric generators coincide.

Another symmetry: ad-Invariance

Definition

The *adjoint action* of a Hopf algebra is defined by $\mathrm{ad} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$,

$$\mathrm{ad}(a) = a_{(1)} S(a_{(3)}) \otimes a_{(2)}, \quad a \in \mathcal{A}.$$

Remarks

- The adjoint action is a left corepresentation, i.e. we have

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{ad}) \circ \mathrm{ad} &= (\Delta \otimes \mathrm{id}) \circ \mathrm{ad}, \\ (\varepsilon \otimes \mathrm{id}) \circ \mathrm{ad} &= \mathrm{id}.\end{aligned}$$

- ad is not an algebra homomorphism.

ad-Invariance

Definition

A linear map $T \in \mathrm{Lin}(\mathcal{A})$ is called ad-*invariant*, if

$$(\mathrm{id} \otimes T) \circ \mathrm{ad} = \mathrm{ad} \circ T.$$

A linear functional $L \in \mathcal{A}'$ is called ad-*invariant*, if

$$(\mathrm{id} \otimes L) \circ \mathrm{ad} = L\mathbf{1}_{\mathcal{A}}.$$

Remarks

- The counit ε and the Haar state h are ad-invariant.
- For $L \in \mathcal{A}'$, T_L is ad-invariant if and only if L is ad-invariant.
- If $L, L' \in \mathcal{A}'$ are ad-invariant then $L \star L'$ is ad-invariant.

ad-Invariance

Denote by $\mathrm{ad}_h \in \mathrm{Lin}(\mathcal{A})$ the linear map given by

$$\mathrm{ad}_h = (h \otimes \mathrm{id}) \circ \mathrm{ad}.$$

Properties

- $L \circ \mathrm{ad}_h$ is ad-invariant for all $L \in \mathcal{A}'$.
- $L \in \mathcal{A}'$ is ad-invariant if and only if $L = L \circ \mathrm{ad}_h$.
- A functional L is ad-invariant iff it is of the form $L(u^{(s)}_{jk}) = c_s \delta_{jk}$ for some $c_s \in \mathbb{C}$.

ad-Invariance

Remarks

- If L is ad-invariant and hermitian, then L is symmetric if and only if $c_s \in \mathbb{R}$ for all $s \in \mathcal{I}$.
- $L \to L \circ \mathrm{ad}_h$ does not preserve the hermitianity, neither positivity!

From Lévy Processes to Dirichlet Forms and beyond...

What next?

Lévy process $\longrightarrow$ Markov semigroup

From Lévy Processes to Dirichlet Forms and beyond...

What next?

$$\begin{array}{ccc} \text{Lévy process} & \longrightarrow & \text{Markov semigroup} \\ & \longrightarrow & \text{Dirichlet form } \mathcal{E} \end{array}$$

From Lévy Processes to Dirichlet Forms and beyond...

What next?

$$\begin{array}{ccc} \text{Lévy process} & \longrightarrow & \text{Markov semigroup} \\ & \longrightarrow & \text{Dirichlet form } \mathcal{E} \\ & \longrightarrow & \text{derivation } \partial \end{array}$$

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What next?

Lévy process	$\longrightarrow$	Markov semigroup
	$\longrightarrow$	Dirichlet form $\mathcal{E}$
	$\longrightarrow$	derivation ∂
	$\longrightarrow$	Dirac operator D

Open problems

- Find **interesting** explicit examples of symmetric or KMS-symmetric generators on $SU_q(n)$.
- Construct the related derivations and Dirac operators (need to extend Cipriani & Sauvageot's construction to the non-tracial case).

References

- F. Cipriani, *Dirichlet Forms on Noncommutative Spaces* in Lecture Notes in Math. 1954, Springer 2008.
- U. Franz, *Levy Processes on Quantum Groups and Dual Groups* in Lecture Notes in Math. 1866, Springer 2006.
- M. Schürmann, *White noise on Bialgebras*, Lecture Notes in Math. 1544, Springer 1993.
- M. Schürmann, M. Skeide, Infinitesimal generators on the quantum group $SU_q(2)$, *IDAQP* 1(4) (1998), 573598.
- S. L. Woronowicz, *Compact Quantum Groups*, Les Houches, Session LXIV, 1995, Quantum Symmetries, Elsevier 1998.