ergodic properties of bogoliubov automorphisms in free probability

Francesco Fidaleo
Università di Roma Tor Vergata

April 27, 2011

0–**abstract**

We define mixing–like properties of Unique Weak Mixing (UWM) and Weak Mixing (UM) automorphisms (or merely Markov operators) of C^*–algebras. They are together with unique ergodicity, the topological analogue of ergodicity, weak mixing, mixing respectively in measure theoretical setting (i.e. for W^*–dynamical systems). We show that the last one (UM) cannot have a classical analogue unless the classical dynamical systems is conjugate to the trivial one consisting by a singleton. Conversely, the unique mixing has several nontrivial examples in quantum setting.

We show that the shifts on the reduced C^*–algebras of RD–groups, including the free group on infinitely many generators, and amalgamated free product C^*–algebras are all uniquely mixing. Such a result is extended to the case

of q–relations. We then can exhibit UM mixing C^*–dynamical systems which generate type I_∞ and II_1 von Neumann factors.

By applying the Shlyakhtenko–Hiai construction, we can provide examples in which type III_λ, $\lambda \in (0, 1]$ factors also appear. Namely we are able to exhibit quantum systems based on Bogoliubov automorphisms, which are UE (w.r.t the fixed point algebra) but not UWM, and UWM which are not UM. This is done by applying a standard functorial construction to measure–preserving transformations on probability spaces. Collecting altogether, we provide UE but not UWM, and UWM but not UM examples, generating all the type of von Neumann factors but I_{fin}, II_∞ and III_0. Such results are generalized to the so–called q–relations, provided $|q| < \sqrt{2} - 1$

The present talk is based on the following papers:

–Dykema K., Fidaleo F. *Unique mixing of the shift on the C^*–algebras generated by the q–canonical commutation relations*, Houston J. Math. **36** (2010), 275–281.

–Fidaleo F. *On strong ergodic properties of quantum dynamical systems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **12** (2009), 551–564.

–Fidaleo F., Mukhamedov F. *Strict weak mixing of some C^*–dynamical systems based on free shifts*, J. Math. Anal. Appl. **336** (2007), 180–187.

–Fidaleo F., Mukhamedov F. *Ergodic properties of Bogoliubov automorphisms in free probability*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **13** (2010), 393–411.

1–**strong ergodic properties of dynamical systems**

Let $(\mathfrak{A},\alpha)$ be a C^*–dynamical system made of a unital C^*–algebra $\mathfrak{A}$, and an automorphism α of $\mathfrak{A}$. Consider the following properties:

(i) E–ergodicity:

$$\lim_n \frac{1}{n}\sum_{k=0}^{n-1}\varphi(\alpha^k(x)) = \varphi(E(x))\,,$$

(ii) E–weak mixing:

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \left|\varphi(\alpha^k(x)) - \varphi(E(x))\right| = 0\,,$$

(iii) E–mixing:

$$\lim_n \varphi(\alpha^k(x)) = \varphi(E(x))\,.$$

Here, $a \in \mathfrak{A}$, $\varphi \in \mathfrak{A}^*$, and finally E is a linear map which is seen to coincide with the conditional expectation onto the fixed point subalgebra.

(i) was introduced in B. Abadie and K. Dykema (*Unique ergodicity of free shifts and some other automorphisms of C^*–algebras*, J. Operator Theory, **61** (2009), 279–294), as a natural generalization of unique ergodicity. (ii) was introduced in F. Fidaleo and F. Mukhamedov (Strict weak mixing of some C^*–dynamical systems based on free shifts, J. Math. Anal.

Appl. 336 (2007), 180–187). (iii) was introduced in F. Fidaleo (On strongly ergodic properties of quantum dynamical systems, Infin. Dim. Anal. Quantum Probab. Relat. Top. **12** (2009), 551–564). When the fixed point algebra is $\mathbb{C}\mathbf{I}$, they are called Unique Ergodicity (UE for short), Unique Weak Mixing (UWM), and Unique Mixing (UM). In this case, they are the topological analogues of ergodicity, weak mixing and mixing of the measure theoretical setting, respectively. In the classical situation (i.e. when $\mathfrak{A}$ is made of all the continuous functions on a compact space, and α is generated by a homeomorphism), it is possible to exhibit starting from an ergodic measure preserving tranformation, a vaste class of UE dynamical systems.* It is

*The Jewett–Krieger theorem (cf. R. I. Jewett: *The pervalence of uniquely ergodic systems*, J. Math. Mec. **19** (1970), 717–729, W. Krieger: *On unique ergodicity*, Proceedings on the 6th Berkeley symposium on mathematical statistics and probability, 327–346) asserts that each ergodic classical system (X, T, μ) is equivalent to a uniquely ergodic one. Nothing is known on the corresponding result for weakly mixing classical systems.

known that the irrational rotations on the unit circle are UE but not UWM. It was proven in the last paper that if a classical system is UM, then it is conjugate to a one–point trivial system. On the other hand, there are nontrivial quantum systems enjoying UM (see below). Thus, we can exhibit UM dynamical systems which generates I_∞ and II_1 von Neumann factors. By applying the Shlyakhtenko–Hiai construction, we can provide examples in which type III_λ, $\lambda \in (0,1]$ factors also appear. It is then natural to construct C^*–dynamical systems enjoying one of the mentioned ergodic properties but not the successive (stronger) one, which generate any kind of von Neumann factors.

2–strong ergodic properties of quantum dynamical systems

Let $\mathbb{F}_\infty$ be the free group on infinitely many generators $\{g_i\}_{i\in\mathbb{Z}}$. Consider the reduced C^*–algebra $C^*_r(\mathbb{F}_\infty) \subset \mathcal{B}(\ell^2(\mathbb{F}_\infty))$. It is generated by the unitaries λ_g associated to the left regular reresentation of $\mathbb{F}_\infty$ on $\ell^2(\mathbb{F}_\infty)$. The shift β on the generators of the free group can be extended to the whole $\mathbb{F}_\infty$, and induces on $C^*_r(\mathbb{F}_\infty)$ an automorphism α called the free shift.

theorem The free shift α on $C^*_r(\mathbb{F}_\infty)$ satisfies

$$\lim_n \varphi(\alpha^n(a)) = \varphi(\mathbf{1})\tau(a)\,,$$

where $a \in C^*_r(\mathbb{F}_\infty)$, $\varphi \in C^*_r(\mathbb{F}_\infty)^*$, and τ is the canonical trace on $C^*_r(\mathbb{F}_\infty)$.

proof By the Haagerup inequality

$$\Big\|\sum_j \alpha_j\lambda_{g_j}\Big\|_{C^*_r(\mathbb{F}_\infty)} \leq C\Big\|\sum_j \alpha_j\delta_{g_j}\Big\|_{\ell^2(\mathbb{F}_\infty)}\,,$$

we get

$$\lim_n \left\|\frac{1}{n}\sum_{j=1}^n \alpha^{k_j}(\lambda_g)\right\|_{C^*_r(\mathbb{F}_\infty)} = 0$$

for each subsequence $\{k_j\}$ of natural numbers. The proof easily follows by the last estimation.

Such a result can be extended to shifts on

(a) RD–groups, which are by definition, discrete groups satisfying the Haagerup inequality (cf. P. Jolissaint: *Rapidly decreasing functions in reduced C^*–algebras of groups*, Trans. Amer. Math. Soc. **317** (1979), 279–293),

(b) reduced amalgamated free product C^*–algebra (cf. F. Fidaleo, F. Mukhamedov) by using *mutatis mutandis* an extension of the Haagerup inequality, proved by B. Abadie and K. Dykema.

(c) the shift α_q on the C^*–algebras generated by the annihilations a_i, $i \in \mathbb{Z}$ fulfilling the q–commutation relations

$$a_i a_j^+ - q a_j^+ a_i = \delta_{ij} 1, \qquad i, j \in \mathbb{Z}$$

where $-1 < q < 1$, as well as their self–adjoint parts generated by $s_i := a_i + a_i^+$. Notice that the the algebra generated by the s_i with $q = 0$ corresponds to the free group C^*–algebra, for which the result directly follows by the Haagerup inequality. For the remaining cases $q \neq 0$, the proof follows by the estimate (cf. K. Dykema and F. Fidaleo):

$$\Big\| \sum_{l=1}^{n} \alpha^{k_l}\Big(a_{\sigma_1}^+ \cdots a_{\sigma_i}^+ a_{\rho_1} \cdots a_{\rho_j}\Big)\Big\| \leq \sqrt{\frac{n}{(1-|q|)^{i+j}}}\,. \tag{1}$$

3–strong ergodic properties of quantum dynamical systems: recent results

Let (X, T, μ) be a classical dynamical system made of a probability space (X, μ), and a measure–preserving transformation $T : X \mapsto X$. Let

$$\mathcal{K} := (L^2_{\mathbb{R}}(X, \mu) \ominus \mathbb{R}1) \bigotimes \Big(\oplus_{\lambda \in G} \mathbb{R}^2 \Big)\,,$$

where G is any (countable) multiplicative subgroup of $\mathbb{R}_+$. Let $uf := f \circ T^{-1}$ and

$$v(t) := \oplus_{\lambda \in G} \begin{pmatrix} \cos(t \ln \lambda) & -\sin(t \ln \lambda) \\ \sin(t \ln \lambda) & \cos(t \ln \lambda) \end{pmatrix} .$$

Then $u \otimes I$ and $I \otimes v(t)$ are orthogonal transformations acting on the real Hilbert space $\mathcal{K}$ satisfying $[u \otimes I, I \otimes v(t)] = 0$. Let $\mathcal{K}_{\mathbb{C}}$ be the complexification of $\mathcal{K}$ together with the positive non singular generator A of the complexification of $I \otimes v(t)$. As $I \otimes v(t) = A^{it}$, let $\mathcal{H}$ be the completion of $\mathcal{K}_{\mathbb{C}}$ with respect the inner product induced by A

$$\langle x, y \rangle := (2A(I + A)^{-1}x, y) .$$

Denote by U and $V(t)$ the (unitary) extension of the corresponding orthogonal operators to the whole $\mathcal{H}$. Let $\mathcal{F}(\mathcal{H})$ be the full Fock space generated by $\mathcal{H}$ together with the Fock vacuum vector Ω, and $\mathfrak{G}$ the C^*–algebra acting on $\mathcal{F}(\mathcal{H})$, generated by $\{s(f) := a(f) + a^+(f) : f \in \mathcal{K}\}$. Notice that Ω is cyclic for $\mathfrak{G}$ and $\mathfrak{G}'$ (i.e. Ω is a standard vector for $\mathfrak{G}''$). We have the following result (F. Fidaleo, F. Mukhamedov):

theorem Let α be the automorphism in $\mathfrak{G}$ induced by $\alpha(s(f)) := s(Uf)$ (i.e. the Bogoliubov automorphism induced by U). Then the following assertions hold true.

(i) If (X, T, μ) is ergodic but not weakly mixing, then $(\mathfrak{G}, \alpha)$ is E–ergodic, E being the conditional expectation onto the fixed point algebra which is (in the ergodic case) always nontrivial.

(ii) If (X, T, μ) is weakly mixing but not mixing, then $(\mathfrak{G}, \alpha)$ is UWM but not UM, with $\omega := \langle \cdot\Omega\,, \Omega\rangle$ the unique invariant state.

(iii) If (X, T, μ) is mixing, then $(\mathfrak{G}, \alpha)$ is UM, with ω the unique invariant state.

(a) $\mathfrak{G}''$ is a von Neumann factor wich is type II_1, III_λ, $\lambda \in (0,1)$ or III_1, whenever G is $\{1\}$, $\{\lambda^n : n = 0,1,2,\dots\}$ or $\mathbb{Q}_+$, respectively.†

The proof relies on the following estimate

$$\begin{aligned}&\left\| \sum_{l=1}^{N} \alpha_U^{k_l}\Big(a^+(f_1)\cdots a^+(f_m)a(g_1)\cdots a(g_n)\Big)\right\| \\ &\leq \left\| \sum_{l=1}^{N} U^{-k_l}f_1 \otimes \cdots \otimes U^{-k_l}f_m \otimes U^{k_l}g_n \otimes \cdots \otimes U^{k_l}g_1 \right\|\end{aligned} \tag{2}$$

which reduces the ergodic behavior of the Bogoliubov automorphism generated by U to the ergodic properties of (tensors of) U itself.

3–the case of q–commutation relations

After defining the q–Fock space $\mathcal{F}_q(\mathcal{H})$, and the annihilators $a_q(f)$ acting on it, $-1 < q < 1$,

†We have by construction, $\mathfrak{G}'' = \pi_\omega(\mathfrak{G})''$, π_ω being the GNS representation relative to ω.

we consider the concrete unital C^*–algebra $\mathfrak{R}_q$ generated by $\{a_q(f) \mid f \in \mathcal{K}_\mathbb{R}\}$, and the unital subalgebra $\mathfrak{G}_q$ generated by the selfadjoint part $s_q(f) := a_q(f)+a_q^+(f)$ of the annihilators. The Bogoliubov automorphism α_q generated by U, is defined in the standard way. All the above results concerning the construction and the properties of the C^*–dynamical systems associated to the Bogoliubov automorphisms extend to the q–commutation relations case, provided $|q| < \sqrt{2}-1$. The proof relies on the following theorem.

theorem There exists an isomorphism $\theta : \mathfrak{R}_q \to \mathfrak{R}_0$ which intertwines any Bogoliubov automorphism: $\theta\circ\alpha_q = \alpha_0\circ\theta$, provided $|q| < \sqrt{2}-1$.

The above results (concerning the q–commutation relations) can be extended to all the q if the Bogoliubov transformation under consideration is the shift. Namely, fix any $q \in$

$(-1,1)$. By using the previuos Shlyakhtenko–Hiai constructions for the shift on the q–commutation relations, the GNS representation w.r.t. the unique invariant state ω_q, the Fock vacuum, of the resulting C^*–dynamical systems $(\mathfrak{G}_q, \alpha_q)$ (which are UM) generate von Neumann factors of all types III_λ, $\lambda \in (0,1]$. The proof relies on the estimate (1). The conjecture is that the previuos results can be extended to Bogoliubov automorphisms for all the q–commutation relations, for all $q \in (-1,1)$. Unfortunately, it is unclear if the involved C^*–algebras are all isomorphic for generic $q \in (-1,1)$, and such isomorphisms intertwine the corresponding Bogoliubov automorphisms. Another way is to extend the estimates (1) or (2) to generic Bogoliubov automorphisms, or to generic $q \in (-1,1)$, respectively. Unfortunately, also such results are not yet available.