Fourier multipliers acting on noncommutative L_p-spaces

Tom Cooney

Universidad Complutense de Madrid

EU - NCG 4, Bucharest, April 30, 2011

Kac algebras

$\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \kappa, \psi)$ is a *Kac algebra* if

1. $L_\infty(\mathbb{G})$ is a von Neumann algebra;

Kac algebras

$\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \kappa, \psi)$ is a *Kac algebra* if

1. $L_\infty(\mathbb{G})$ is a von Neumann algebra;

   locally compact group G with left Haar measure μ
   Commutative case: $L_\infty(G)$
   Cocommutative case: $L(G)$, group von Neumann algebra

Kac algebras

$\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \kappa, \psi)$ is a *Kac algebra* if

1. $L_\infty(\mathbb{G})$ is a von Neumann algebra;
2. Γ is an injective normal unital *-homomorphism from $L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G}) \bar{\otimes} L_\infty(\mathbb{G})$ such that

$$(\iota \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \iota) \circ \Gamma.$$

Kac algebras

$\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \kappa, \psi)$ is a *Kac algebra* if

1. $L_\infty(\mathbb{G})$ is a von Neumann algebra;
2. Γ is an injective normal unital *-homomorphism from $L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G})\bar{\otimes}L_\infty(\mathbb{G})$ such that

$$(\iota \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \iota) \circ \Gamma.$$

Commutative case:

$$\Gamma(f)(s,t) = f(st), \qquad f((st)u) = f(s(tu)),$$

for $f \in L_\infty(G)$, and $s, t, u \in G$.

Cocommutative case:

$$\Gamma(\lambda_s) = \lambda_s \otimes \lambda_s$$

3. κ is an involutive anti-automorphism of $L_\infty(\mathbb{G})$ satisfying

$$(\kappa \otimes \kappa) \circ \Gamma = \zeta \circ \Gamma \circ \kappa,$$

where $\zeta(a \otimes b) = b \otimes a$ for all $a, b \in L_\infty(\mathbb{G})$.

Commutative case:

$$\kappa(f)(s) = f(s^{-1}), \qquad f((st)^{-1}) = f(t^{-1}s^{-1}),$$

for $f \in L_\infty(G)$, and $s, t \in G$.

Cocommutative case:

$$\kappa(\lambda_s) = \lambda_{s^{-1}}, \qquad s \in G.$$

3. κ is an involutive anti-automorphism of $L_\infty(\mathbb{G})$ satisfying

$$(\kappa \otimes \kappa) \circ \Gamma = \zeta \circ \Gamma \circ \kappa,$$

where $\zeta(a \otimes b) = b \otimes a$ for all $a, b \in L_\infty(\mathbb{G})$.

Commutative case:

$$\kappa(f)(s) = f(s^{-1}), \qquad f((st)^{-1}) = f(t^{-1}s^{-1}),$$

for $f \in L_\infty(G)$, and $s, t \in G$.
Cocommutative case:

$$\kappa(\lambda_s) = \lambda_{s^{-1}}, \qquad s \in G.$$

5. ψ is a nsf weight on $L_\infty(\mathbb{G})$ satisfying

$$(\psi \otimes \iota)\Gamma(x) = \psi(x)1, \quad \text{for all } x \in L_\infty(\mathbb{G})^+.$$

Commutative case: right Haar measure ds:

$$\int f(st)\, ds = \int f(s)\, ds,$$

for $f \in L_\infty(G)$, and $s, t \in G$.
Cocommutative case:
Plancherel weight for G.
If G is discrete, this is the canonical trace.

$$\psi(\lambda(f)^*\lambda(f)) = \|f\|_2^2, \qquad f \in L_1(G) \cap L_2(G)$$

5. ψ is a nsf weight on $L_\infty(\mathbb{G})$ satisfying

$$(\psi \otimes \iota)\Gamma(x) = \psi(x)1, \quad \text{for all } x \in L_\infty(\mathbb{G})^+.$$

Commutative case: right Haar measure ds:

$$\int f(st)\, ds = \int f(s)\, ds,$$

for $f \in L_\infty(G)$, and $s, t \in G$.
Cocommutative case:
Plancherel weight for G.
If G is discrete, this is the canonical trace.

$$\psi(\lambda(f)^*\lambda(f)) = \|f\|_2^2\,, \qquad f \in L_1(G) \cap L_2(G)$$

5. ψ is a nsf weight on $L_\infty(\mathbb{G})$ satisfying

$$(\psi \otimes \iota)\Gamma(x) = \psi(x)1, \quad \text{for all } x \in L_\infty(\mathbb{G})^+.$$

6. $(\psi \otimes \iota)((y^* \otimes 1)\Gamma(x)) = \kappa((\psi \otimes \iota)(\Gamma(y^*)(x \otimes 1))$ for all $x, y \in \mathfrak{N}_\psi$;
7. $\kappa\sigma_t^\psi = \sigma_{-t}^\psi\kappa$ for all $t \in \mathbb{R}$;

5. ψ is a nsf weight on $L_\infty(\mathbb{G})$ satisfying

$$(\psi \otimes \iota)\Gamma(x) = \psi(x)1, \quad \text{for all } x \in L_\infty(\mathbb{G})^+.$$

6. $(\psi \otimes \iota)((y^* \otimes 1)\Gamma(x)) = \kappa((\psi \otimes \iota)(\Gamma(y^*)(x \otimes 1))$ for all $x, y \in \mathfrak{N}_\psi$;
7. $\kappa\sigma_t^\psi = \sigma_{-t}^\psi\kappa$ for all $t \in \mathbb{R}$;

which implies

$$\Gamma(\sigma_t^\psi(x)) = (\iota \otimes \sigma_t^\psi)\Gamma(x) = (\sigma_t^\psi \otimes \iota)\Gamma(x)$$

Kac algebra $L_\infty(\mathbb{G})$ acting standardly on $\mathcal{H}_\psi$.
$L_\infty(\mathbb{G})_* = L_1(\mathbb{G})$.

Right fundamental unitary operator V on $\mathcal{H}_\psi \otimes \mathcal{H}_\psi$:

$$V(\Lambda_\psi(x) \otimes \Lambda_\psi(y)) = (\Lambda_\psi \otimes \Lambda_\psi)(\Gamma(x)(1 \otimes y)),$$

for all $x, y \in \mathfrak{N}_\psi$. This operator V satisfies the pentagonal relation

$$V_{12} V_{13} V_{23} = V_{12} V_{23},$$

Comultiplication Γ on $L_\infty(\mathbb{G})$ is given by

$$\Gamma(x) = V(x \otimes 1)V^*.$$

Commutative case:

$$(Vf)(s,t) = f(st,t),$$

for $f \in L_2(G \times G)$, $s, t \in G$.
Cocommutative case:

$$V(\delta_s \otimes \delta_t) = \delta_s \otimes \delta_{st}.$$

Right fundamental unitary operator V on $\mathcal{H}_\psi \otimes \mathcal{H}_\psi$:

$$V(\Lambda_\psi(x) \otimes \Lambda_\psi(y)) = (\Lambda_\psi \otimes \Lambda_\psi)(\Gamma(x)(1 \otimes y)),$$

for all $x, y \in \mathfrak{N}_\psi$. This operator V satisfies the pentagonal relation

$$V_{12} V_{13} V_{23} = V_{12} V_{23},$$

Comultiplication Γ on $L_\infty(\mathbb{G})$ is given by

$$\Gamma(x) = V(x \otimes 1)V^*.$$

Commutative case:

$$(Vf)(s, t) = f(st, t),$$

for $f \in L_2(G \times G)$, $s, t \in G$.
Cocommutative case:

$$V(\delta_s \otimes \delta_t) = \delta_s \otimes \delta_{st}.$$

Representing $L_1(G)$ on $L_p(G)$

Let G be a locally compact group with right Haar measure ds. Then $L_1(G)$ acts contractively by right convolution on $L_p(G)$:

$$\Theta_p^r(f) : L_p(G) \to L_p(G)$$
$$\left(\Theta_p^r(f)(g)\right)(t) = \int_G g(ts)f(s)\, ds$$
$$\left\|\Theta_p^r(f)\right\| \leq \|f\|_1$$

for $f \in L_1(G)$, and $g \in L_p(G)$.

Dual version: Fourier algebra acting on $L_p(L(G))$

(Daws: locally compact group case)
Let G be a discrete group.

$$L(G) = \{\lambda_g : g \in G\}'' \subseteq B(\ell_2(G))$$

Fourier algebra $A(G) = L(G)_*$:

$$A(G) = \{\omega_{\xi,\eta} : \xi, \eta \in \ell_2(G)\}$$
$$\|a\|_{A(G)} = \inf\{\|\xi\|_2 \|\eta\|_2 : a = \omega_{\xi,\eta}\}.$$

with $*$-algebra structure inherited from the following inclusion into $C_0(G)$:

$$\omega_{\xi,\eta}(s) = (\lambda_s \xi \mid \eta) = \int \xi(s^{-1}t)\eta(t)\,dt$$

The noncommutative L_p-space $L_p(L(G))$

Canonical trace on $L(G)$:

$$\tau(\lambda_g) = (\lambda_g \delta_e \mid \delta_e).$$

Tracial noncommutative L_p-spaces:

$$L_p(L(G)) = \overline{L(G)}^{\|\cdot\|_p},$$

where $\|x\|_p = \tau(|x|^p)^{1/p}$.

Terp's Interpolation Method

Inclusion j_1 of $L(G)$ into $A(G)$:

$$\langle j_1(\lambda_s), \lambda_t \rangle = \tau(\lambda_s \lambda_t) = \begin{cases} 1 & t = s^{-1} \\ 0 & t \neq s^{-1} \end{cases},$$

for $s, t \in G$.
This allows us to interpolate between $L(G)$ and $A(G)$.

Operator space structure of $L_p(L(G))$

Operator space structure: $L_1(L(G)) = A(G)^{op} = L(G)_*^{op}$.
The map

$$\kappa : \lambda_g \mapsto \lambda_{g^{-1}}$$

is a $*$-isomorphism of $L(G)$ onto $L(G)^{op}$ and thus we can completely isometrically identify $L_1(L(G))$ with $\kappa_*(A(G))$:

$$[a_{ij}]_{M_n(L_1(L(G)))} = [\kappa_*(a_{ij})]_{M_n(A(G))}.$$

Operator space structure on $L_p(L(G))$ obtained by interpolation:

$$M_n(L_p(L(G)) = (M_n(L(G)), M_n(L_1(L(G))))_{1/p}.$$

CB-multipliers

A function $\varphi : G \to \mathbb{C}$ is in $M_0A(G)$ if the map

$$m_\varphi : a \mapsto \varphi a$$

maps $A(G)$ into $A(G)$ and m_φ is completely bounded. Equivalently,

$$\lambda_s \mapsto \varphi(s)\lambda_s, \qquad s \in G$$

extends to a cb map $M_\varphi : L(G) \to L(G)$ and then

$$m_\varphi^* = M_\varphi.$$

Given $f \in A(G)$, we have that M_f is a cb multiplier.

CB-multipliers on $L_p(L(G))$

Suppose $m_\varphi \in M_0A(G)$. Let $\check{\varphi}$ denote the function $\check{\varphi}(s) = \varphi(s^{-1})$. It is easily checked that

$$m_{\check{\varphi}} = \kappa_* m_\varphi \kappa_*,$$

and thus $m_{\check{\varphi}}$ is a cb map from $L_1(L(G))$ to $L_1(L(G))$. Then M_φ and $m_{\check{\varphi}}$ are a compatible pair of maps as

$$\langle j_1(M_\varphi \lambda_s), \lambda_t\rangle = \langle j_1(\varphi(s)\lambda_s), \lambda_t\rangle = \varphi(s)\tau(\lambda_s\lambda_t)$$

and

$$\langle m_{\check{\varphi}}(j_1(\lambda_s)), \lambda_t\rangle = \langle j_1(\lambda_s), M_{\check{\varphi}}(\lambda_t)\rangle = \varphi(t^{-1})\tau(\lambda_s\lambda_t) = \varphi(s)\tau(\lambda_s\lambda_t),$$

and thus

$$j_1(M_\varphi \lambda_s) = m_{\check{\varphi}}(j_1(\lambda_t)).$$

We can thus interpolate to get an action of $M_0A(G)$ on $L_p(L(G))$.

Interpolation and the inclusion of $\mathfrak{M}_\psi$ into $L_p(\mathbb{G})$

The intersection of $L_\infty(\mathbb{G})_*$ and $L_\infty(\mathbb{G})$ is given by:

$$\begin{aligned} L =&\{x \in L_\infty(\mathbb{G}) \,:\, \exists \psi_x \in L_\infty(\mathbb{G})_* \text{ such that} \\ &\quad \psi_x(y^* z) = (xJ\Lambda(y) \mid J\Lambda(z))\,, \quad \forall y, z \in \mathfrak{N}_\psi\}. \\ \|x\|_L =& \max\{\|x\|\,, \|\psi_x\|_1\} \end{aligned}$$

$L_\infty(\mathbb{G}) \hookrightarrow L^*$ and $L_\infty(\mathbb{G})_* \hookrightarrow L^*$ given by for $x \in L_\infty(\mathbb{G}), \psi \in L_\infty(\mathbb{G})_*$,

$$\begin{aligned} \langle x, y\rangle_{L^*,L} &= \langle \psi_y, x\rangle, \quad y \in L \\ \langle \psi, y\rangle_{L^*,L} &= \langle \psi, y\rangle, \quad y \in L. \end{aligned}$$

Then

$$L_p(\mathbb{G}) \simeq (L_\infty(\mathbb{G}), L_\infty(\mathbb{G})_*)_{1/p}.$$

The noncommutative L_p-spaces $L_p(\mathbb{G}) = L_p(L_\infty(\mathbb{G}), \psi)$

Notation: $\mathfrak{M}_\psi = \operatorname{span}\{x \in L_\infty(\mathbb{G})_+ : \psi(x) < \infty\}$.

For a certain positive, self-adjoint, invertible operator D, we have that

$$\overline{\{D^{1/2p} x D^{1/2p} : x \in \mathfrak{M}_\psi\}}^{\|\cdot\|_p} = L_p(\mathbb{G})$$

Inclusion of $\mathfrak{M}_\psi \subset L_\infty(\mathbb{G})$ into $L_1(\mathbb{G})$:

$$\langle \mu_1(x), y \rangle = \langle D^{1/2} x D^{1/2}, y \rangle = \psi(\sigma^\psi_{i/2}(x) y),$$

for all $x \in \mathfrak{M}_\psi \cap \mathfrak{N}_\infty$ and $y \in \mathfrak{N}_\psi$.
$\mathfrak{N}_\infty = \{x \in L_\infty(\mathbb{G}), x \text{ analytic and } \sigma^\psi_\alpha(x) \in \mathfrak{N}_\psi, \forall \alpha \in \mathbb{C}\}$

The inclusion of $\mathfrak{M}_\psi$ into $L_p(\mathbb{G})$ is given by

$$\mu_p : \mathfrak{M}_\psi \to L_p(\mathbb{G}), \qquad \mu_p(x) = D^{1/2p} x D^{1/2p},$$

and these inclusions are compatible with Terp's interpolation method.

The map $\Theta^r(f)$

Proposition (Junge, Neufang, Ruan)

Let $\mathbb{G}$ be a locally compact quantum group. Let $f \in L_1(\mathbb{G})$ and define the map $\Theta^r(f)$ by

$$\Theta^r(f)(x) = \langle \iota \otimes f, V(x \otimes 1)V^* \rangle, \qquad x \in B(\mathcal{H}).$$

Then Θ^r is an injective completely contractive homomorphism from $L_1(\mathbb{G})$ into $CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(B(\mathcal{H}))$.
In fact, there exists a completely isometric algebra isomorphism

$$M^r_{cb}(L_1(\mathbb{G})) \simeq CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(B(\mathcal{H})).$$

Let $f \in L_\infty(\mathbb{G})_*$.
We define $\Theta^r(f) : L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G})$ by

$$\Theta^r(f)(x) = (\iota \otimes f)\Gamma(x) = (\iota \otimes f)V(x \otimes 1)V^*.$$

Extending $\Theta^r(f)$ to $L_p(\mathbb{G})$

Let $a \in \mathfrak{M}_\psi$. Then $x = D^{1/2p} a D^{1/2p}$ is an operator on $L_2(\mathbb{R}) \otimes \mathcal{H}$, affiliated to $L_\infty(\mathbb{G}) \rtimes_{\sigma^\psi} \mathbb{R}$.
With some work, it can be shown that

$$(\iota \otimes \iota \otimes \xi^*)(\iota \otimes V)(x \otimes 1)(\iota \otimes V^*)(\iota \otimes \iota \otimes \xi) = D^{1/2p} \Theta^r(f)(a) D^{1/2p}.$$

Thus our extension of $\Theta^r(f)$ to $\Theta^r_p(f) : L_p((G)) \to L_p(\mathbb{G})$ should satisfy

$$\Theta^r_p(f)(D^{1/2p} a D^{1/2p}) = D^{1/2p} \Theta^r(f)(a) D^{1/2p}, \qquad a \in \mathfrak{M}_\psi.$$

We know $\Theta^r(f)$ is bounded as a map from $L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G})$. If we show it is bounded as a map from $L_1(\mathbb{G}) \to L_1(\mathbb{G})$, then we can interpolate to get bounded maps $\Theta^r_p(f) : L_p(\mathbb{G}) \to L_p(\mathbb{G})$.

The pre-adjoint of $\Theta^r(f)$ and the boundedness of $\Theta^r_1(f)$

Theorem

For $f \in L_1(\mathbb{G})$, $\Theta^r_1(f)^* = \Theta^r(f \circ \kappa)$. *Thus*

$$\|\Theta^r_1(f)\| = \|\Theta^r(f \circ \kappa)\| \leq \|f \circ \kappa\| = \|f\|.$$

Thus we have bounded maps $\Theta^r_p(f) : L_p(\mathbb{G}) \to L_p(\mathbb{G})$ *satisfying*

$$\Theta^r_p(f)(D^{1/2p} a D^{1/2p}) = D^{1/2p} \Theta^r(f)(a) D^{1/2p}, \qquad a \in \mathfrak{M}_\psi.$$

This yields a contractive representation of $L_\infty(\mathbb{G})_*$ *on* $CB(L_p(\mathbb{G}))$.

- $\langle D^{1/2} \Theta^r(f)(x) D^{1/2}, y \rangle = \psi(\sigma_{i/2}(\Theta^r(f)(x))y)$,
- $(\psi \otimes \iota)((y^* \otimes 1)\Gamma(x)) = \kappa((\psi \otimes \iota)(\Gamma(y^*)(x \otimes 1))$
- Compatible with inclusions of Terp's interpolation method.

Convolution and the inclusion of L into $L_1(\mathbb{G})$

For $f \in L_1(\mathbb{G})$ and $y \in L_\infty(\mathbb{G})$, we have

$$\begin{aligned}
\langle \psi_x * f, y \rangle &= \langle \psi_x, \Theta^r(f)(y) \rangle \\
&= \langle \mu_1(x), \Theta^r(f)(y) \rangle \\
&= \langle \Theta^r(f)_* \mu_1(x), y \rangle \\
&= \langle \mu_1(\Theta^r(f \circ \kappa)(x)), y \rangle \\
&= \langle \psi_{\Theta^r(f \circ \kappa)(x)}, y \rangle
\end{aligned}$$

Application

(Kraus & Ruan: AP for Kac algebras
Junge & Ruan: AP for noncommutative L_p-spaces associated with discrete groups)

An element a in $L_\infty(\mathbb{G})$ is a *left multiplier* if

$$a\hat{\lambda}(\hat{\omega}) \in \hat{\lambda}(A(\mathbb{G})) \text{ for all } \hat{\omega} \in A(\mathbb{G}) = L_\infty(\hat{\mathbb{G}})_*.$$

The set of all left multipliers will be denoted by $M^l(A(\mathbb{G}))$. Given $a \in M^l(A(\mathbb{G})$, we have a bounded map m_a^l on $A(\mathbb{G})$ given by

$$m_a^l(\hat{\omega}) = \hat{\lambda}^{-1}(a\hat{\lambda}(\hat{\omega}))$$

for all $\hat{\omega} \in A(\mathbb{G})$.
The set of completely bounded left multipliers of $A(\mathbb{G})$ will be denoted by $M_0^l(A(\mathbb{G})) \subset CB(A(\mathbb{G}))$.

We then have

$$M_a^l = (m_a^l)^* \in CB(L_\infty(\hat{\mathbb{G}})),$$

and

$$M_a^l\Big|_{C_\lambda^*(\mathbb{G})} \in CB(C_\lambda^*(\mathbb{G}))$$

There is a contractive inclusion of $A(\mathbb{G})$ into $M_0^l(A(\mathbb{G}))$ given by

$$m_{\hat{\omega}}^l(\hat{\omega}') = \hat{\omega} * \hat{\omega}',$$

for $\hat{\omega}, \hat{\omega}' \in A(\mathbb{G})$.
We have that for $\hat{\omega} \in A(\mathbb{G})$

$$M_{\hat{\omega}}^l = (m_{\hat{\omega}}^l)^* = \hat{\Theta}^l(\hat{\omega}), \text{ and}$$
$$m_{\hat{\omega}}^l = \hat{\Theta}^l(\hat{\omega})_*.$$

AP for Kac algebras

$$M_0^l A(\mathbb{G}) = Q^l(\mathbb{G})^*$$

Definition

$\mathbb{G}$ has the *approximation property* if $A(\mathbb{G})$ has a left stable weak* approximate identity (i.e., a net $\{\hat{\omega}_i\}$ in $A(\mathbb{G})$ such that $\hat{\Theta}^l(\hat{\omega}_i) \to id_{L_\infty(\hat{\mathbb{G}})}$ in the stable point weak* topology of $CB(L_\infty(\hat{\mathbb{G}}))$.

$\mathbb{G}$ has the *weak approximation property* if 1 is in the $\sigma(M_0^l(A(\mathbb{G}), Q^l(\mathbb{G}))$-closure of $A(\mathbb{G})$ in $M_0^l(A(\mathbb{G}))$.

If $\mathbb{G}$ is a discrete Kac algebra, these conditions are equivalent.

$$\begin{aligned} A_c(\mathbb{G}) &= \{\hat{\omega} \in A(\mathbb{G}) \,:\, \Theta^l_\infty(\hat{\omega}) \in F(L_\infty(\hat{\mathbb{G}}))\} \\ &= \{\hat{\omega} \in A(\mathbb{G}) \,:\, \Theta^l_\infty(\hat{\omega}) \in F(C^*_\lambda(\hat{\mathbb{G}}))\}, \end{aligned}$$

Definition

$\mathbb{G}$ is said to be weakly amenable if $A(\mathbb{G})$ has a left approximate identity $\{\hat{\omega}_i\}$ such that

$$\sup \left\| \hat{\lambda}(\hat{\omega}_i) \right\|_{M_0^l(A(\mathbb{G}))} \leq L$$

for some positive number L.

Definition

An operator space V has the *operator space approximation property (OAP)* if there exists a net of finite rank maps $T_\alpha : V \to V$ such that $T_\alpha \to id_V$ in the *stable point-norm topology*; that is, we have $\|(T_\alpha \otimes id_\infty)(x) - x\| \to 0$ for all $x \in V \check{\otimes} K_\infty$.

An operator space V has the *completely bounded approximation property (CBAP)* if there exists a net of finite rank maps $T_\alpha : V \to V$ with $\|T_\alpha\|_{cb} \leq \lambda$ for some positive λ such that $T_\alpha \to id_V$ in the *point-norm topology* on V.

Theorem

Let $\mathbb{G}$ be a discrete Kac algebra with the AP. Then there exists a net $\{\hat{\omega}_\alpha\}$in $A_c(\mathbb{G}) \cap Z(A(\mathbb{G}))$ such that $\hat{\Theta}^l_1(\hat{\omega}_\alpha) \to id_{A(\mathbb{G})}$ in the stable point-norm topology on $A(\mathbb{G})$ and $\hat{\Theta}^l_\infty(\hat{\omega}_\alpha) \to id_{C^*_\lambda(\hat{\mathbb{G}})}$ in the stable point-norm topology on $C^*_\lambda(\hat{\mathbb{G}})$.

(and a similar version for weak-amenability)

The map $\hat{\Theta}^l_1(\hat{\omega} \circ \hat{\kappa}) \in CB(L_1(\hat{\mathbb{G}}))$ as

$$\hat{\Theta}^l_1(\hat{\omega} \circ \hat{\kappa}) = \kappa_* \Theta^r_1(\hat{\omega})\kappa_* = \kappa_* \Theta^l_1(\hat{\omega})\kappa_*.$$

Given $x \in L \subset L_\infty(\hat{\mathbb{G}})$, we have that the corresponding element in $L_\infty(\mathbb{G})_*$ is φ_x and $\hat{\Theta}^l_\infty(\hat{\omega})(x)$ corresponds to $\hat{\varphi}_{\hat{\Theta}^l_\infty(\hat{\omega})(x)}$.
By earlier proposition about the preadjoint of $\Theta^l(f)$, we have

$$\begin{aligned}\hat{\Theta}^l_1(\hat{\omega} \circ \hat{\kappa})(\hat{\varphi}_x) =& (\hat{\omega} \circ \hat{\kappa}) * (\hat{\varphi}_x)\\ =& \hat{\varphi}_{\hat{\Theta}^l_\infty(\hat{\omega})(x)}.\end{aligned}$$

Theorem

Let $1 < p < \infty$*. If* $\mathbb{G}$ *is a discrete Kac algebra with the approximation property, then* $L_p(\hat{\mathbb{G}})$ *has the operator space approximation property.*

Let $\mathbb{G}$ *be a weakly amenable discrete Kac algebra and let* $1 < p < \infty$*. Then* $L_p(\hat{\mathbb{G}})$ *has the completely bounded approximation property.*

Thanks!!

Thanks for your attention!

Fourier multipliers acting on noncommutative L_p-spaces

Tom Cooney

Universidad Complutense de Madrid

EU - NCG 4, Bucharest, April 30, 2011