Automorphisms of the Cuntz algebras

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arXiv:0805.4654, to appear in TAMS (CS)
arXiv:0805.4655, JMP 2009 (CS)
arXiv:0808.2843, PEMS 2010 (CKS)
arXiv:0910.1304, JFA 2110 (CRS)
arXiv:1003.1815, JAMS 2010
arXiv:1006.4791, to appear in Crelle (CHS)
arXiv:1102.4875, to appear in Banach Center Publ. (CHS)
arXiv:1101.4210 (CHS)
... and work in progress

Outline:

- Overview of the Cuntz algebras
- Weyl groups
- Permutative automorphisms and trees
- Reduced Weyl groups and shift automorphisms
- Exotic endomorphisms
- Outlook: full Weyl group, the above for graph algebras

Cuntz algebras: I met

- Doplicher-Roberts duality (superselection structure of QFT)
- Jones index and sector theory
- Localized automorphisms, permutations and trees (today)
- Representations (ITP-type)
- NCG: $D = \log \Delta$

Isometries S play an important role in functional analysis and operator theory/algebras

Coburn: $C^*(S)$, S proper isometry

Cuntz algebras:

$\mathcal{O}_n = C^*(S_1, \ldots, S_n)$, $2 \leq n < \infty$

$$S_i^* S_i = 1$$
$$\sum_{i=1}^{n} S_i S_i^* = 1$$

In particular, $S_i^* S_j = \delta_{ij} I$, for all $1 \leq i, j \leq n$

$H := \operatorname{span}\{S_1, \ldots, S_n\}$, a Hilbert space of dimension n and support 1 in $\mathcal{O}_n$

Notation/terminology:

$W_n^k := \{1, \ldots, n\}^k$ $(k \geq 1)$, set of k-tuples $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \{1, \ldots, n\}$, $1 \leq i \leq k$

$W_n := \bigcup_{k=0}^{\infty} W_n^k$, $W_n^0 = \{0\}$

multi-indices: elements of W_n

If $\alpha \in W_n^k$, then $l(\alpha) := k$, the length of the word α in the alphabet $\{1, \ldots, n\}$

Given $\alpha = (\alpha_1, \ldots, \alpha_k) \in W_n$ let $S_\alpha := S_{\alpha_1} \ldots S_{\alpha_k}$
($S_0 = I$ by convention).

Facts:

every word in $\{S_i, S_i^* \mid i = 1, \ldots, n\}$ can be uniquely expressed as a normal ordered monomial $S_\alpha S_\beta^*$, for $\alpha, \beta \in W_n$

$\mathcal{O}_n$ is the closed linear span of $S_\alpha S_\beta^*$, for $\alpha, \beta \in W_n$

$\mathcal{O}_n$ unital, simple, separable, nuclear and purely infinite with

$$K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1} \ , \quad K_1(\mathcal{O}_n) = 0$$

(in particular the K-theory of $\mathcal{O}_2$ is trivial)

$\mathcal{O}_n$ not type I

$\mathcal{O}_n$ is a Doplicher-Roberts algebra, a Cuntz-Krieger algebra, a graph algebra and a Pimsner algebra

Cuntz algebras in operator algebras: structure and classification of C^*-algebras and group actions, sectors, subfactors/index theory, entropy, dynamical systems, coding, self-similar sets, wavelets (signal processing), quantum field theory, abstract group duality, twisted cyclic cocycles and Fredholm theory unital C^*-subalgebras of Cuntz algebras:

$$\mathcal{O}_n \supset \mathcal{F}_n \supset \mathcal{D}_n$$

$\mathcal{F}_n$ core UHF-subalgebra
$\mathcal{D}_n$ diagonal, a canonical MASA with Cantor spectrum

$\mathcal{F}_n^k := \mathrm{span}\{S_\alpha S_\beta^*, l(\alpha) = l(\beta) = k\}$, $k \geq 1 \Rightarrow \mathcal{F}_n^k \subset \mathcal{F}_n^{k+1}$ for all k
$\mathcal{F}_n^k \simeq M_{n^k} \simeq M_n \otimes \cdots \otimes M_n$ (k factors), compatile with embeddings $x \mapsto x \otimes 1$
$\mathcal{F}_n := \left(\bigcup_k \mathcal{F}_n^k\right)^{=} \simeq \bigotimes_{k=1}^\infty M_n$ UHF-algebra of Glimm type n^∞, τ unique trace
(in particular, $\mathcal{F}_2 \simeq \bigotimes_{i=1}^\infty M_2$ is the CAR algebra.)
$E : \mathcal{O}_n \to \mathcal{F}_n$ a faithful conditional expectation, obtained by averaging over the canonical gauge action of $\mathbb{T}$
$\mathcal{F}_n = \mathcal{O}_n^{\mathbb{T}}$

$\mathcal{D}_n$, the commutative C^*-subalgebra of $\mathcal{O}_n$ generated by projections $P_\alpha := S_\alpha S_\alpha^*$, $\alpha \in W_n$; it is a regular MASA, both in $\mathcal{F}_n$ and $\mathcal{O}_n$
$\mathcal{D}_n \simeq C(X_n)$, $X_n := \prod^{\mathbb{N}}\{1, \ldots, n\}$ with product topology
$P_\alpha \leftrightarrow \chi_{\sigma_\alpha}$ characteristic function of cylindrical set of sequences starting with α
The Gelfand spectrum X_n, equipped with the product topology, is a Cantor set, i.e. a compact, metrizable, totally disconnected space with no isolated points
$\mathcal{D}_n^k := \mathcal{D}_n \cap \mathcal{F}_n^k$, generated by projections P_α with $\alpha \in W_n^k$, isomorphic to the diagonal matrices in M_{n^k}
Then $\mathcal{D}_n := \left(\bigcup_k \mathcal{D}_n^k\right)^{=}$
There exists a faithful conditional expectation from $\mathcal{F}_n$ onto $\mathcal{D}_n$ and whence from $\mathcal{O}_n$ onto $\mathcal{D}_n$ as well.

Endomorphisms of $\mathcal{O}_n$:

$\mathcal{U}(\mathcal{O}_n)$, the unitary group of $\mathcal{O}_n$

End($\mathcal{O}_n$), the semigroup of unital $*$-endomorphisms of $\mathcal{O}_n$ (they are automatically injective)

Fact: there exists a one-to-one correspondence between elements of $\mathcal{U}(\mathcal{O}_n)$ and of End($\mathcal{O}_n$), denoted

$$U \mapsto \lambda_U \ ,$$

where λ_U is determined by

$$\lambda_U(S_i) = US_i, \quad i = 1, \ldots, n \ .$$

$U \mapsto \lambda_U$ is not a semigroup morphism, rather one has the "fusion rules"

$$\lambda_U \lambda_V = \lambda_{\lambda_U(V)U} \ .$$

For all $x \in \mathcal{F}_n^r$ and $m \geq r$,

$$\lambda_U(x) = U_m x U_m^*$$

where

$$U_m := U\varphi(U) \ldots \varphi^{m-1}(U)$$

satisfies the cocycle relation $U_{m+r} = U_m \varphi^m(U_r)$ for all m, r

Example 0.1. *The canonical endomorphism* $\varphi : \mathcal{O}_n \to \mathcal{O}_n$*,*

$$\varphi(a) := \sum_{i=1}^{n} S_i a S_i^*$$

(it restricts to the unilateral shift $x \mapsto 1 \otimes x$ *on the UHF subalgebra* $\mathcal{F}_n$*)*

Then $\varphi = \lambda_\theta$*, where*

$$\theta := \sum_{1 \leq i,j \leq n} S_i S_j S_i^* S_j^*$$

is the unitary flip operator in $\mathcal{O}_n$*, switching the components of* $H^2 \simeq H \otimes H$

$\mathbb{T} \ni z \mapsto \lambda_{z1} =: \alpha_z$ provides the automorphic "gauge" action of $\mathbb{T}$ (rescaled periodic modular automorphisms w.r.t. $\omega = \tau \circ E$). As stated before, $\mathcal{O}_n^\alpha = \mathcal{F}_n$. $\mathcal{O}_n$ is a $\mathbb{Z}$-graded C^*-algebra.

Definition 0.2. *(Longo) Say that* λ_U *is "localized" (or algebraic) if* $U \in \mathcal{F}_n^k$ *for some* k*.*

If $U \in \mathcal{F}_n^k$ then

$$\lambda_U(\mathcal{F}_n^r) \subset \mathcal{F}_n^{r+k-1}, \quad r \in \mathbb{N}$$

Moreover, $\lambda_U \lambda_{z1} = \lambda_{z1} \lambda_U = \lambda_{zU}$ for all $z \in \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

Localized endomorphisms turn out to have finite index.

Ind > 1 (subfactor theory)

Ind $= 1$?
(Notice that in this talk we will be concerned with C^*-automorphisms)

Cuntz showed that the automorphism group of $\mathcal{O}_n$, $\mathrm{Aut}(\mathcal{O}_n)$, has features similar to semisimple Lie groups, and proposed a definition of the Weyl group as the normalizer of an infinite-dimensional maximal torus in this context.

Question: What can be said about the explicit structure of these Weyl groups?

Few simple and/or known facts about automorphisms of $\mathcal{O}_n$:

- Gauge automorphisms: induced by $U = zI$, $z \in \mathbb{C}$, $|z| = 1$
- Bogolubov automorphisms: if $U \in \mathcal{F}_n^1$, then $\lambda_U \in \mathrm{Aut}(\mathcal{O}_n)$, $(\lambda_U)^{-1} = \lambda_{U^*}$ and, if $U \neq 1$, λ_U is outer. For such unitaries, $\lambda_U\lambda_V = \lambda_{UV}$; this is the quasi-free action of $U(n) \simeq \mathcal{U}(\mathcal{F}_n^1)$ on $\mathcal{O}_n$

  A notable example is Archbold's "flip-flop" automorphism λ_F of $\mathcal{O}_2$, where
  $$F = S_1S_2^* + S_2S_1^* \in \mathcal{F}_2^1$$
- Inner automorphisms: $\lambda_U = \mathrm{Ad}(V) \in \mathrm{Inn}(\mathcal{O}_n)$ if and only if $U = V\varphi(V^*)$, $V \in \mathcal{U}(\mathcal{O}_n)$.
- $\lambda_U \in \mathrm{Aut}(\mathcal{O}_n)$ if and only if $U^* \in \lambda_U(\mathcal{O}_n)$. Therefore, in order to check the surjectivity of λ_U, it is enough to know that a single element, namely U itself, is in the image. (However this statement is somewhat self-referential and thus non terribly useful in practice!)
- It may well be the case that λ_U is an automorphism but λ_{U^*} is not!
- Matsumoto-Tomiyama (outer) automorphism of $\mathcal{O}_4$
- All homeomorphisms of the full n-shift space X_n commuting with the shift transformation extend to automorphisms of $\mathcal{O}_n$.

General problems: find conditions on $U \in \mathcal{U}(\mathcal{O}_n)$, verifiable in practice, ensuring that λ_U is an automorphism. Construct examples.

E.g., this is mandatory in order to get more insight into the structure of $\mathrm{Aut}(\mathcal{O}_n)$, $\mathrm{Out}(\mathcal{O}_n) := \mathrm{Aut}(\mathcal{O}_n)/\mathrm{Inn}(\mathcal{O}_n)$ and some selected subgroups of these groups (see below)

Notation:

$\mathfrak{A}$ unital C^*-algebra
$\mathrm{Aut}(\mathfrak{A})$ group of $*$-automorphisms
$\mathrm{Inn}(\mathfrak{A})$ subgroup of inner automorphisms
$\mathrm{Inn}(\mathfrak{A}) \triangleleft \mathrm{Aut}(\mathfrak{A})$ so can define $\mathrm{Out}(\mathfrak{A}) := \mathrm{Aut}(\mathfrak{A})/\mathrm{Inn}(\mathfrak{A})$,
$\pi : \mathrm{Aut}(\mathfrak{A}) \to \mathrm{Out}(\mathfrak{A})$ canonical projection

$\mathfrak{B} \subset \mathfrak{A}$ unital C^*-subalgebra
$\mathrm{Aut}(\mathfrak{A}, \mathfrak{B}) = \{\alpha \in \mathrm{Aut}(\mathfrak{A}) \mid \alpha(\mathfrak{B}) = \mathfrak{B}\}$
$\mathrm{Aut}_{\mathfrak{B}}(\mathfrak{A}) = \{\alpha \in \mathrm{Aut}(\mathfrak{A}) \mid \alpha(b) = b, b \in \mathfrak{B}\}$
$\mathrm{Aut}_{\mathfrak{B}}(\mathfrak{A}) \triangleleft \mathrm{Aut}(\mathfrak{A}, \mathfrak{B}) \subseteq N_{\mathrm{Aut}(\mathfrak{A})}(\mathrm{Aut}_{\mathfrak{B}}(\mathfrak{A}))$
$N_{\mathfrak{A}}(\mathfrak{B}) = \{u \in \mathcal{U}(\mathfrak{A}) \mid u\mathfrak{B}u^* = \mathfrak{B}\}$ (unitary) normalizer of $\mathfrak{B}$ in $\mathfrak{A}$
$N_{\mathfrak{A}}(\mathfrak{B}) \to \mathrm{Aut}(\mathfrak{B})$
$N_{\mathfrak{A}}(\mathfrak{B}) \to \mathrm{Aut}(\mathfrak{A}, \mathfrak{B}) \cap \mathrm{Inn}(\mathfrak{A})$

The reduced and full Weyl groups:

$N_{\mathcal{F}_n^k}(\mathcal{D}_n^k) \subset N_{\mathcal{F}_n^{k+1}}(\mathcal{D}_n^{k+1}) \subset \ldots \subset N_{\mathcal{F}_n}(\mathcal{D}_n)$ (unitary) normalizers

Theorem 0.3. *(Cuntz, 1980)*

- $\mathcal{U}(\mathcal{D}_n) \simeq \lambda(\mathcal{U}(\mathcal{D}_n)) = \lambda(\mathcal{U}(\mathcal{D}_n))^{-1}$ *"maximal torus" (maximal abelian subgroup, limit of finite-dimensional tori)*
- $\mathcal{O}_n^{\lambda(\mathcal{U}(\mathcal{D}_n))} = \mathcal{D}_n$, $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) = \lambda(\mathcal{U}(\mathcal{D}_n))$ *Galois system*
- $N_{\mathrm{Aut}(\mathcal{O}_n)}(\lambda(\mathcal{U}(\mathcal{D}_n))) = \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) = \lambda(N_{\mathcal{O}_n}(\mathcal{D}_n))^{-1}$ *"Weyl group" (before taking quotient)*
- $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) = \lambda(N_{\mathcal{F}_n}(\mathcal{D}_n))^{-1}$ *"restricted Weyl group"*

For $E \subset \mathcal{U}(\mathcal{O}_n)$,

$$\lambda(E)^{-1} := \{\lambda_U \mid U \in E\} \cap \mathrm{Aut}(\mathcal{O}_n)$$

Problem (Weyl group): which elements of $N_{\mathcal{O}_n}(\mathcal{D}_n)$, resp. of $N_{\mathcal{F}_n}(\mathcal{D}_n)$, induce automorphisms of $\mathcal{O}_n$?

Structure of normalizers :

Theorem 0.4. *(Power, 1998)*

- $N_{\mathcal{O}_n}(\mathcal{D}_n) = \mathcal{S}_n \cdot \mathcal{U}(\mathcal{D}_n)$

  $\mathcal{S}_n := \{u \in \mathcal{U}(\mathcal{O}_n) \mid u = \sum_k^{\text{fin}} S_{\alpha_k} S_{\beta_k}^*\}$ *the subgroup of* $\mathcal{U}(\mathcal{O}_n)$ *of unitaries that can be written as finite sum of words in* S_i *and* S_i^*

- $N_{\mathcal{F}_n}(\mathcal{D}_n) = \left(\bigcup_k N_{\mathcal{F}_n^k}(\mathcal{D}_n^k)\right) \cdot \mathcal{U}(\mathcal{D}_n) = \mathcal{P}_n \cdot \mathcal{U}(\mathcal{D}_n)$

  $\mathcal{P}_n = \{u \in \mathcal{U}(\mathcal{F}_n) \mid u = \sum_k^{\text{fin}} S_{\alpha_k} S_{\beta_k}^*, l(\alpha_k) = l(\beta_k)\ \forall k\}$

Nekrashevych: $\mathcal{S}_n \simeq G_{n,1}$ Higman-Thompson group. In particular, a copy of Thompson group sits naturally in $\mathcal{S}_2 \subset \mathcal{U}(\mathcal{O}_2)$

$\mathcal{P}_n$ direct limit of permutation groups $\mathbb{P}_{n^k}$ w.r.t. strictly diagonal embeddings (see below)

Cuntz problem boils down to recognizing which unitaries in $\mathcal{S}_n$, resp. $\mathcal{P}_n = \mathcal{S}_n \cap \mathcal{F}_n$ induce automorphisms of $\mathcal{O}_n$. We will focus on $\mathcal{P}_n$.

Let P_n^k be the group of permutations of W_n^k, clearly isomorphic to $\mathbb{P}_{n^k}$.

To any $\sigma \in P_n^k$ one associates a unitary in $\mathcal{F}_n^k$ via

$$u_\sigma = \sum_{\alpha \in W_n^k} S_{\sigma(\alpha)} S_\alpha^* \ .$$

Then $\sigma \mapsto u_\sigma$ is an isomorphism of P_n^k with its image $\mathcal{P}_n^k = \mathcal{S}_n \cap \mathcal{F}_n^k$, that can be further identified with the set of permutation matrices in M_{n^k}, and $\mathcal{P}_n = \bigcup_k \mathcal{P}_n^k$.

Proposition 0.5. *Let w be a unitary in $\mathcal{O}_n$.*

(a) If $w \in \mathcal{U}(\mathcal{O}_n)$ then $\lambda_w(\mathcal{F}_n) = \mathcal{F}_n$ if and only if $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ and $w \in \mathcal{U}(\mathcal{F}_n)$;

(b) If $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$ then $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$ if and only if $w \in N_{\mathcal{O}_n}(\mathcal{D}_n)$.

(c) If $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$ then λ_w is an irreducible endomorphism of $\mathcal{O}_n$, i.e. $\lambda_w(\mathcal{O}_n)' \cap \mathcal{O}_n = \mathbb{C}$.

As the endomorphisms of $\mathcal{O}_n$ (with $n \leq 4$) considered below are all induced by unitaries w in $\cup_k \mathcal{P}_n^k \subset N_{\mathcal{F}_n}(\mathcal{D}_n) = N_{\mathcal{O}_n}(\mathcal{D}_n) \cap \mathcal{F}_n$, when they are automorphisms they also provide, by restriction, automorphisms of $\mathcal{D}_n$ and $\mathcal{F}_n$; when they only satisfy the weaker condition $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$ they still act irreducibly on $\mathcal{O}_n$.

Theorem 0.6. *One has*

- $N_{\mathcal{O}_n}(\mathcal{D}_n) \simeq \mathcal{U}(\mathcal{D}_n) \rtimes \mathcal{S}_n$ *(action by conjugation)*
- $N_{\mathcal{F}_n}(\mathcal{D}_n) \simeq \mathcal{U}(\mathcal{D}_n) \rtimes \mathcal{P}_n$
- $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \simeq \lambda(\mathcal{U}(\mathcal{D}_n)) \rtimes \lambda(\mathcal{S}_n)^{-1}$
  In particular, $\lambda(\mathcal{S}_n)^{-1}$ *is a subgroup of* $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$.
- $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) \simeq \lambda(\mathcal{U}(\mathcal{D}_n)) \rtimes \lambda(\mathcal{P}_n)^{-1}$
  In particular, $\lambda(\mathcal{P}_n)^{-1}$ *is a subgroup of* $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$.

Proposition 0.7. *Let* $w \in \mathcal{P}_n^k$ *and suppose that* $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$*, then the inverse* λ_w^{-1} *is also localized. More precisely,* λ_w^{-1} *is induced by a unitary in* $\mathcal{P}_n^h$*, with* $h \leq n^{2(k-1)}$.

Proposition 0.8. *Let* $w \in \mathcal{P}_n$*. If* $\lambda_w \in \mathrm{Inn}(\mathcal{O}_n)$ *then there exists* $z \in \mathcal{P}_n$ *such that* $w = z\varphi(z^*)$*. Moreover, for* $k \geq 2$*, if* $w \in \mathcal{P}_n^k$ *then* $z \in \mathcal{P}_n^{k-1}$.

There is an isomorphism $\mathcal{P}_n \to \lambda(\mathcal{P}_n)^{-1} \cap \mathrm{Inn}(\mathcal{O}_n)$, via $u \mapsto \mathrm{Ad}(u)$.
Thus there is a short exact sequence

$$1 \to \mathcal{P}_n \stackrel{\mathrm{Ad}}{\to} \lambda(\mathcal{P}_n)^{-1} \to \pi(\lambda(\mathcal{P}_n)^{-1}) \to 1$$

$G_n := \lambda(\mathcal{P}_n)^{-1}$

$\pi(G_n)$, $n \geq 3$ non-amenable; actually it contains a copy of $\mathbb{Z}_2 * \mathbb{Z}_3 (\simeq \mathrm{SL}_2(\mathbb{Z}))$

$\pi(G_2)$ non-amenable; (the class of) Archbold's flip-flop is the simplest nontrivial element, the next one being induced by a permutative unitary in $\mathcal{P}_2^4$

$\pi(G_n)$ is residually finite, for every n

(see the next slide)

$$
\begin{array}{ccccccccc}
 & & & & \mathrm{Aut}(X_n) & & & & \\
 & & & & \cap & & & & \\
\frac{\mathrm{Aut}(\mathcal{O}_n,\mathcal{D}_n)\cap\mathrm{Aut}(\mathcal{O}_n,\mathcal{F}_n)}{\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)} & \simeq & G_n & \overset{\simeq}{\to} & \mathfrak{G}_n & \to & \mathfrak{G}_n/I\mathfrak{G}_n \simeq \pi(G_n) & \hookrightarrow & \mathrm{Aut}(\Sigma_n)/\langle\sigma\rangle \\
 & & \cap & & \cap & & \cap & & \\
 & & \mathrm{Aut}(\mathcal{O}_n) & & \mathrm{Aut}(\mathcal{D}_n) & & \mathrm{Out}(\mathcal{O}_n) & &
\end{array}
$$

$G_n := \lambda(\mathcal{P}_n)^{-1}$ the restricted Weyl group of $\mathcal{O}_n$

Theorem 0.9. *G_n is isomorphic (via restriction) with the group of homeomorphisms of the full n-shift space X_n which eventually commute, along with their inverses, with the shift transformation*

Theorem 0.10. *For prime n, the restricted outer Weyl group $\pi(G_n)$ is isomorhic with the automorphism group of the full two-sided n-shift Σ_n divided by its center (generated by the shift)*

Remarks on the automorphisms of $\mathcal{D}_n$ obtained from $\mathcal{O}_n$:

there are proper endomorphisms of $\mathcal{O}_n$ that restrict to automorphisms of $\mathcal{D}_n$

however, any unital endomorphism of $\mathcal{O}_n$ which fixes the diagonal $\mathcal{D}_n$ point-wise is automatically surjective, i.e. it is an element of $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$

the restriction map $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \to \mathrm{Aut}(\mathcal{D}_n)$ is not surjective (and its image is not normal, as $\mathrm{Aut}(\mathcal{D}_n) \simeq \mathrm{Homeo}(\mathcal{C}_n)$ is a simple group)

there are product-type automorphisms of $\mathcal{D}_n$ that do not extend to (possibly proper) endomorphisms of $\mathcal{O}_n$; in case of $\mathcal{D}_2$, consider e.g. $\bigotimes_{i=1}^{\infty} \mathrm{Ad}(u_i)$, where

$$u_i = \begin{cases} 1 & i \quad \text{even} \\ \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) & i \quad \text{odd} \end{cases}$$

and we have realized $\mathcal{D}_2$ as an infinite tensor product over $\mathbb{N}$ of diagonal matrices of size 2.

In particular, it becomes important to characterize those automorphisms of $\mathcal{D}_n$ that can be obtained by restricting automorphisms (or even endomorphisms) of $\mathcal{O}_n$. As recalled above, the automorphisms of $\mathcal{D}_n$ obtained by restriction of elements in $\lambda(\mathcal{P}_n)^{-1}$ have been intrinsically characterized in terms of their action on the Gelfand spectrum.

Explicit computations:

$\#\{\sigma \in P_n^k \; |\lambda_\sigma \in \mathrm{Aut}(\mathcal{O}_n)\} \equiv \#\lambda(\mathcal{P}_n^k)^{-1}$

$\#\{\sigma \in P_n^k \mid \lambda_\sigma|_{\mathcal{D}_n} \in \mathrm{Aut}(\mathcal{D}_n)\}$ (divisible by $n!^{n^{k-1}}$)

completely determined for $n + k \leq 6$

Localized automorphisms:

Let $u \in \mathcal{U}(\mathcal{F}_n^k)$. Define

$$B_u = \{u, \varphi(u), \ldots, \varphi^{k-2}(u)\}' \cap \mathcal{F}_n^{k-1} \tag{1}$$

if $k \geq 2$ and $B_u = \mathbb{C}1$ otherwise.

Then $B_u(= B_{u^*})$ unital $*$-subalgebra of $\mathcal{F}_n^{k-1}$ and if $b \in B_u$ then $\lambda_u(b) = b$.

For $i, j \in \{1, \ldots, n\}$ define maps $a_{ij}^u : \mathcal{F}_n^{k-1} \to \mathcal{F}_n^{k-1}$ by

$$a_{ij}^u(x) = S_i^* u^* x u S_j, \quad x \in \mathcal{F}_n^{k-1}. \tag{2}$$

Denote $V_u = \mathcal{F}_n^{k-1}/B_u$.

Since $a_{ij}^u(B_u) \subseteq B_u$, there are induced maps $\tilde{a}_{ij}^u : V_u \to V_u$.

$$A_u := \quad \text{the subring of } \mathcal{L}(V_u) \text{ generated by } \{\tilde{a}_{ij}^u \mid i, j = 1, \ldots, n\}$$

Given $u \in \mathcal{F}_n^k$, let $K := \lambda_u(H) = uH$ and define inductively

$$\Xi_0 = \mathcal{F}_n^{k-1}, \quad \Xi_r = K^* \Xi_{r-1} K\ ,\ r \geq 1\ , \tag{3}$$

that is $\Xi_r = (K^*)^r \mathcal{F}_n^{k-1} K^r$.

Then $(\Xi_r)_r$ is nonincreasing sequence that stabilizes at the first value p for which $\Xi_p = \Xi_{p+1}$. Let

$$\Xi_u := \bigcap_{r \in \mathbb{N}} \Xi_r = \Xi_p\ ,$$

a self-adjoint subspace of $\mathcal{F}_n^{k-1}$

Theorem 0.11. *Let u be a unitary in $\mathcal{F}_n^k$ for some $k \geq 1$. Then the following conditions are equivalent:*

(1) λ_u is invertible with localized inverse;

(2) the sequence of unitaries

$$\Big(\mathrm{Ad}\big(\varphi^m(u)\varphi^{m-1}(u)\dots\varphi(u)u\big)(u^*)\Big)_{m\geq 1}$$

eventually stabilizes;

(3) the ring A_u is nilpotent;

(4) $\Xi_u \subseteq B_u$;

(5) $\Xi_u = \mathbb{C}1$.

Moreover, if the above equivalent conditions hold, then λ_u^{-1} is induced by a unitary $v \in \mathcal{F}_n^h$ with

$$h \leq n^{2(k-1)}$$

Problem: is this exponential bound optimal?

Problem: if λ_U is a localized automorphism, is λ_U^{-1} still localized ?
(true for permutation automorphisms)

Localized automorphisms of the diagonal

$u \in \mathcal{F}_n^k \cap \mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$

Then both $\mathcal{D}_n^{k-1}$ and $B_u \cap \mathcal{D}_n^{k-1}$ are invariant subspaces for all the operators a_{ij}^u associated with u. Denote the restriction of a_{ij}^u to $\mathcal{D}_n^{k-1}$ by b_{ij}^u.

Each b_{ij}^u induces a linear transformation $\tilde{b}_{ij}^u : V_u^D \to V_u^D$, where $V_u^D = \mathcal{D}_n^{k-1}/B_u \cap \mathcal{D}_n^{k-1}$.

Denote by A_u^D the subring of $\mathcal{L}(V_u^D)$ generated by $\{\tilde{b}_{ij}^u \mid i, j = 1, \ldots, n\}$.

Also, define a subspace of $\mathcal{D}_n^{k-1}$ by

$$\Xi_u^D := \bigcap_r (K^*)^r \mathcal{D}_n^{k-1} K^r$$

Clearly $\Xi_u^D \subset \Xi_u$

Theorem 0.12. *Let u be a unitary in $\mathcal{F}_n^k \cap \mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$. If the ring A_u^D is nilpotent then λ_u restricts to an automorphism of $\mathcal{D}_n$. More precisely, TFAE:*

(1) $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$;

(2) λ_u *restricts to an automorphism of the algebraic part* $\cup_s \mathcal{D}_n^s$ *of* $\mathcal{D}_n$;

(3) the ring A_u^D *is nilpotent;*

(4) $\Xi_u^D \subseteq B_w \cap \mathcal{D}$;

(5) $\Xi_u^D = \mathbb{C}1$.

The Search of automorphisms

- Hierarchy of matrix equations (general)
- Labeled rooted trees (only for permutations), more precisely n-tuples of such trees, satisfying conditions (b) and (d)

Application to permutation automorphisms

For $\mathcal{O}_n$ at level k one has to consider $n^k!$ permutations. A case-by-case brute force computation is unfeasible as it involves some manipulation of large matrices (with size that could grow up to the order of $n^{n^{2k}}$ or so). Moreover, there are simply too many cases to consider as the following figures illustrate:

$\mathcal{O}_2$:
$2! = 2$
$2^2! = 24$
$2^3! = 40320$
$2^4! = 20,922,789,888,000,$ ...

$\mathcal{O}_3$:
$3! = 6$
$3^2! = 362880$
$3^3! = 10,888,869,450,418,352,160,768,000,000,$ ...

Some simplifications are possible, exploiting the action of inner and Bogolubov automorphisms, but they do not affect significantly the scale of the problem.

Surprisingly enough, (n-tuples of) labeled rooted trees T come to the rescue, where

$$\#V(T) = \#E(T) = n^{k-1}$$

Why trees ?

Given $\sigma \in P_n^k$ define functions

$$f_i^\sigma : W_n^{k-1} \to W_n^{k-1}$$

for $i = 1, \dots n$ by

$$f_i^\sigma(\alpha) = \beta :\Leftrightarrow \text{there exists } m \in \{1, \dots, n\} \text{ such that } (i, \alpha) = \sigma(\beta, m) .$$

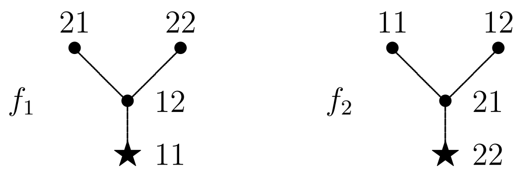
Then a necessary condition for σ to give rise to an automorphism of $\mathcal{O}_n$ (actually of $\mathcal{D}_n$) is that the "diagrams" of all the f_i^σ's are rooted trees, where the root is the unique fixpoint.

In particular, the vertices of these trees are labeled by W_n^{k-1}.

Moreover, in order to get automorphisms, this labeling must also induce a certain partial order relation on $W_n^{k-1} \times W_n^{k-1}$!

Examples:

Example 0.13. The pair of labeled trees corresponding to $\sigma = \mathrm{id}$ in P_2^3. All the edges are downward oriented.



Example 0.14. *Let $u \in \mathcal{P}_n^1$, so that λ_u is a Bogolubov automorphism of $\mathcal{O}_n$. If we view u as an element of $\mathcal{P}_n^k$ then all n unlabeled trees corresponding to u are identical; the root receives $n-1$ edges from other vertices, each other vertex receives either none or n edges, and the height of the tree (the length of the longest path ending at the root) is minimal and equal to $k-1$. In particular, all such unitaries have the corresponding n-tuples of unlabeled trees identical with those of the identity.*

Applications of labeled trees to automorphisms of $\mathcal{O}_2$:

If $w \in \mathcal{P}_2^k$ then the labeled trees associated with f_1^w and f_2^w have the following properties:

- α receives two edges in f_i^w if and only if α receives no edges in f_{3-i}^w;
- α receives one edge in f_i^w if and only if α receives one edge in f_{3-i}^w.

It follows that the numbers of leaves (0-receivers) on both trees are identical and coincide with the number of 2-receivers (including the root) on these trees. In such a case we say these two (unlabeled) trees are matched.

Given $w \in \mathcal{P}_2^k$ with corresponding functions f_1^w, f_2^w and fixed $i \in \{1, 2\}$, we define

$$G(f_i^w) := \{\sigma \in P_2^{k-1} \mid \sigma f_i^w \sigma^{-1} = f_i^w\}, \tag{4}$$

and call it the stabilizing group of f_i^w. Let T be the unlabeled rooted tree corresponding to f_i^w. If $\phi \in P_2^{k-1}$ then we have $G(f_i^w) \cong G(\phi f_i^w \phi^{-1})$, through the map $\sigma \mapsto \phi\sigma\phi^{-1}$. Thus the groups $G(f_i^w)$ do not depend on the choice of labels and we have

$$G(f_i^w) \cong \mathrm{Aut}(T), \tag{5}$$

where $\mathrm{Aut}(T)$ is the automorphism group of the unlabeled rooted tree T. Of course, a similar construction can be carried over for any n.

Case of $\mathcal{P}_2^2$

- $\#\{\sigma \in P_2^2 \mid \lambda_{u_\sigma} \in \mathrm{Aut}(\mathcal{O}_2)\} = 2 \cdot 2 = 4$
- $\#\{\sigma \in P_2^2 \mid \lambda_{u_\sigma}|_{\mathcal{D}_2} \in \mathrm{Aut}(\mathcal{D}_2)\} = 2 \cdot 2^2 = 8 = 4 + 4$

The only pair of labeled trees satisfying our conditions is

Each is realized by 4 permutations and there are 2 such labelings. Thus there are $2! \cdot 2^2 = 2 \cdot 4 = 8$ permutations in P_2^2 yielding elements of $\mathrm{Aut}(\mathcal{D}_2)$. Among these 8 only 4 give automorphisms of $\mathcal{O}_2$. If $F := S_1 S_2^* + S_2 S_1^* \in \mathcal{F}_2^1$ denotes the flip-flop self-adjoint unitary, the four automorphisms are

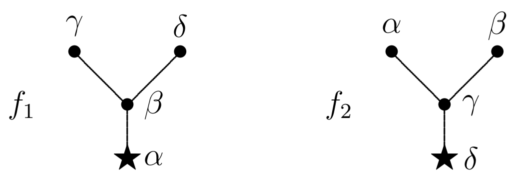
$$\begin{aligned}
&\mathrm{id} \\
&\lambda_F \\
&\mathrm{Ad}(F) = \lambda_{\varphi(F)F} = \lambda_{F\varphi(F)} \\
&\mathrm{Ad}(F)\lambda_F = \lambda_{\varphi(F)}
\end{aligned}$$

They form in $\mathrm{Aut}(\mathcal{O}_2)$ a copy of Klein's four-group. In $\mathrm{Out}(\mathcal{O}_2)$, they give $\mathbb{Z}_2$ with nontrivial generator the class of Archbold's flip-flop (Bogolubov) automorphism λ_F.

Case of $\mathcal{P}_2^3$

Only two graphs are possible (each self-dual), namely

However, there is no labeling of the first graph which yields correct partial order $\leq$ on pairs. So only the second graph remains. The only possible labeling satisfying our conditions is



Given a pair of labeled trees as above, there are 2^4 permutations $\sigma \in P_2^3$ yielding that pair. There are 4! possible choices of labels. Hence

$$\#\{\sigma \in P_2^3 \mid \lambda_{u_\sigma}|_{\mathcal{D}_2} \in \mathrm{Aut}(\mathcal{D}_2)\} = 4! \cdot 2^4 = 24 \cdot 16 = 324 \tag{6}$$

Then considering 16 permutations giving rise to a fixed labeling, as above, one finds that only two of them satisfy the right conditions. Thus, taking into account the action of inner automorphisms corresponding to permutations in P_2^2,

$$\#\{\sigma \in P_2^3 \mid \lambda_{u_\sigma} \in \mathrm{Aut}(\mathcal{O}_2)\} = 24 \cdot 2 = 48 \tag{7}$$

These are precisely the automorphisms inner equivalent to the identity or the flip-flop. Thus, very surprisingly, among $8! = 40,320$ endomorphisms of $\mathcal{O}_2$ from $\lambda(\mathcal{P}_2^3)$ the only outer automorphism is the familiar flip-flop. This is in stark contrast with the case of Cuntz algebras $\mathcal{O}_n$ with $n \geq 3$, where numerous new outer automorphisms appear already in $\lambda(\mathcal{P}_n^2)$.

Case of $\mathcal{P}_2^4$

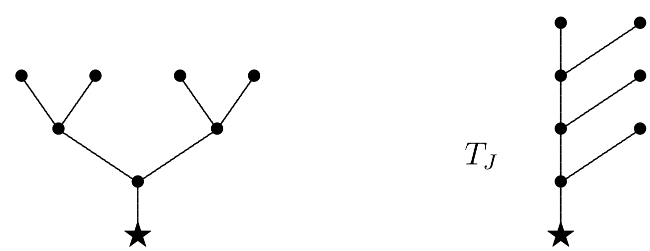
Theorem 0.15. *We have*

$$\#\{\lambda_w \mid w \in \mathcal{P}_2^4 \text{ and } \lambda_w|_{\mathcal{D}_2} \in \mathrm{Aut}(\mathcal{D}_2)\} = 8! \cdot 2^8 \cdot 17 = 175,472,640\ ,$$
$$\#\{\lambda_w \mid w \in \mathcal{P}_2^4 \text{ and } \lambda_w \in \mathrm{Aut}(\mathcal{O}_2)\} = 8! \cdot 14 = 564,480\ .$$

Thus in $\lambda(\mathcal{P}_2^4)^{-1}$ there are exactly 14 representatives of distinct inner equivalence classes.

Proof. There are exactly 23 directed rooted trees (unlabeled) with 8 vertices satisfying our conditions (i.e. each vertex other than the root emits one edge and receives maximum 2 edges, the root is a minimal element and receives one edge from a different vertex). There are only 3 matched pairs of such trees admitting labelings satisfying condition (b): $T_A - T_A$, $T_A - T_J$ and $T_J - T_A$, where T_A and T_J are (downward oriented):

T_A T_J

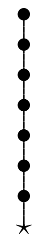


We fix arbitrarily labels on one of the trees in each pair, taking it to be T_J in the second and third case. Then calculation shows the following numbers of labelings of the other tree which satisfy so-called condition (b): 40 for the pair $T_A - T_A$ and 12 for each of the other two pairs. The groups of automorphisms of the rooted trees T_A and T_J have 8 and 2 elements, respectively. Thus, taking into account that each pair of labeled trees under consideration is realized by 2^8 distinct permutations, and factoring in the action of 8! inner automorphisms (which permute the labels simultaneously on both trees), we obtain the following number of distinct permutations in P_2^4 giving rise to automorphisms of the diagonal:

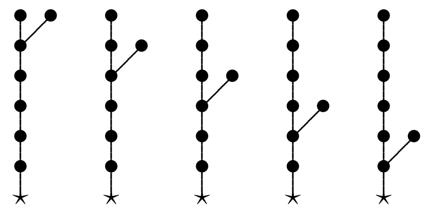
$$2^8 \cdot \frac{8!}{|\mathrm{Aut}(T_A)|} \cdot 40 + 2 \cdot 2^8 \cdot \frac{8!}{|\mathrm{Aut}(T_J)|} \cdot 12 = 2^8 \cdot 8! \cdot 17 = 175,472,640.$$

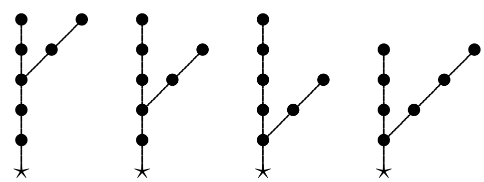
Among these permutations there are only $8! \cdot 14 = 564,480$ yielding automorphisms of $\mathcal{O}_2$. Dividing out 8! inner automorphisms from level 3, we finally get 14 inner equivalence classes of automorphisms in $\lambda(\mathcal{P}_2^4)^{-1}$. □

$x = 1$ [1]

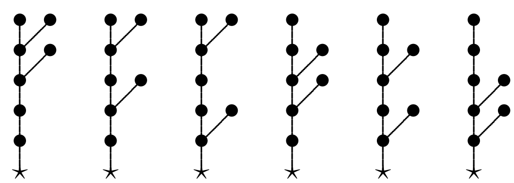


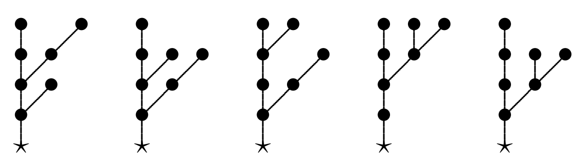
$x = 2$ [9]



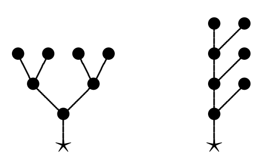


$x = 3$ [11]





$x = 4$ [2]



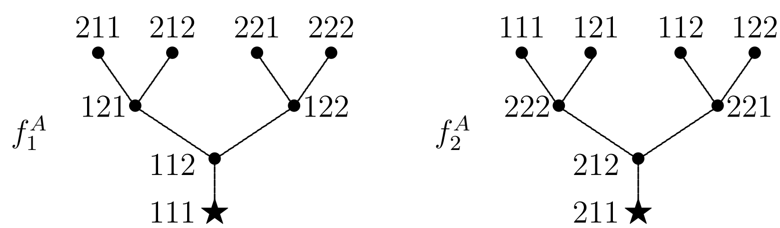
Explicit description of representatives of inner equivalence classes from $\lambda(\mathcal{P}_2^4)^{-1}$ and structure of some infinite subgroups of $\mathrm{Out}(\mathcal{O}_2)$ generated by them.

Consider the permutations A and B of the set W_2^4 given by

$A(1211) = 1211$	$A(1212) = 1212$	$A(1222) = 1222$	$A(1221) = 1221$
$A(1121) = 1121$	$A(1122) = 1122$	$A(1111) = 1112$	$A(1112) = 1111$
$A(2222) = 2111$	$A(2221) = 2121$	$A(2211) = 2112$	$A(2212) = 2122$
$A(2122) = 2222$	$A(2121) = 2221$	$A(2112) = 2212$	$A(2111) = 2211$

$B(1211) = 1211$	$B(1212) = 1212$	$B(1222) = 1222$	$B(1221) = 1221$
$B(1121) = 1121$	$B(1122) = 1122$	$B(1111) = 1112$	$B(1112) = 1111$
$B(2122) = 2111$	$B(2121) = 2112$	$B(2211) = 2121$	$B(2212) = 2122$
$B(2222) = 2212$	$B(2221) = 2221$	$B(2112) = 2222$	$B(2111) = 2211$

Note that the first two rows of these two permutations are identical. That is, $A(1{*}{*}{*}) = B(1{*}{*}{*})$. And of the first eight arguments, six are fixed points. The labeled trees corresponding to A are:



For notational convenience, we equip W_2^k with the reversed lexicographic order and enumerate its elements as $\{1, 2, \ldots, 2^k\}$ accordingly. Then, the permutations A and B above correspond to $A = (1, 9)(2, 4, 10, 12, 14, 16)(6, 8)$ and $B = (1, 9)(2, 4, 6, 10, 16, 12, 14)$. With a slight abuse of notation we also denote simply by A and B the associated unitaries and by λ_A and λ_B the corresponding endomorphisms of $\mathcal{O}_2$.

One can verify that λ_A and λ_B are automorphisms of $\mathcal{O}_2$. One checks by computer calculation that the inverses of the automorphisms λ_A and λ_B are induced by unitaries in $\mathcal{P}_2^7$.

Proposition 0.16. *In* $\mathrm{Out}(\mathcal{O}_2)$*, one has*

$$\lambda_F \lambda_A \lambda_F = \lambda_A^{-1} = \lambda_B \ .$$

Proof. One has $\mathrm{Ad}(z)\lambda_A\lambda_B = \mathrm{id}$, where $z \in \mathcal{P}_2^6$ is given by

$$\begin{aligned} z \sim &(2,4,8)(3,7,15)(5,13,29)(9,25)(10,12) \\ &(18,20,24)(19,23)(26,28)(34,36,40) \\ &(35,39,47)(37,45)(42,44)(50,52,56)(51,55)(58,60). \end{aligned}$$

Also, one has $\mathrm{Ad}(y)\lambda_F\lambda_A = \lambda_B\lambda_F$, where $y \sim (1,3,5,7)(2,4,8) \in P_2^3$. $\square$

Proposition 0.17. λ_A *has infinite order in* $\mathrm{Out}(\mathcal{O}_2)$*.*

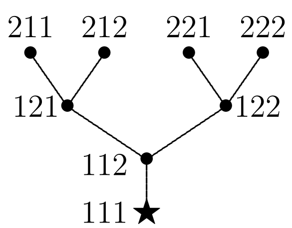
Corollary 0.18. *The subgroup of* $\mathrm{Out}(\mathcal{O}_2)$ *generated by* λ_A *and* λ_F *is isomorphic to the infinite dihedral group* $\mathbb{Z} \rtimes \mathbb{Z}_2$*.*

Let J be a transposition in P_2^4 which exchanges 2112 with 2212 (and fixes all other elements of W_2^4):

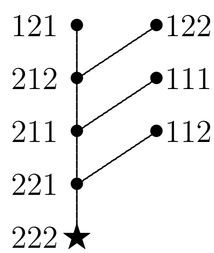
$$J(2112) = 2212 \quad \text{and} \quad J(2212) = 2112.$$

The labeled trees corresponding to J are:

f_1^J



f_2^J



With a slight abuse of notation, we denote by J the associated unitary and by λ_J the corresponding endomorphism of $\mathcal{O}_2$. One checks that

$$\lambda_J^2 = \mathrm{id}. \tag{8}$$

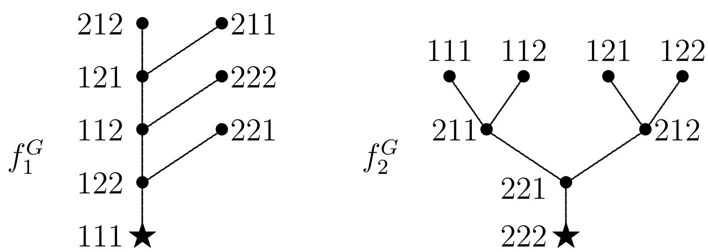
The two trees corresponding to the identity in P_2^4 are both of type T_A. Likewise, both trees corresponding to the flip-flop λ_F are also of type T_A. Since f_2^J is of type $T_J \neq T_A$, it follows that λ_J is an outer automorphism of $\mathcal{O}_2$ not inner equivalent to the flip-flop. (Incidentally, outerness of λ_J can also be deduced from the fact that $\lambda_J(S_1) = S_1$).

Proposition 0.19. *Automorphisms λ_F and λ_J generate a subgroup of* $\mathrm{Out}(\mathcal{O}_2)$ *isomorphic to the free product* $\mathbb{Z}_2 * \mathbb{Z}_2$.

Let G be a 3-cycle in W_2^4 such that

$$G(1112) = 1122, \quad G(1122) = 1222, \quad \text{and} \quad G(1222) = 1112.$$

That is, in the shorthand notation, $G = (9, 13, 15)$. The trees corresponding to G are:



One checks that

$$\lambda_G^6 = \mathrm{id} \tag{9}$$

but none of λ_G, λ_G^2, λ_G^3 is inner. Also note that $\lambda_G(S_2) = S_2$.

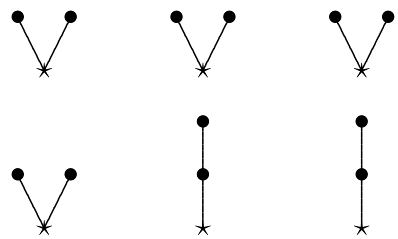
Theorem 0.20. *The following automorphisms give a complete list of representatives of distinct classes in* $\mathrm{Out}(\mathcal{O}_2)$ *appearing in* $\lambda(\mathcal{P}_2^4)^{-1}$*:*

$$\begin{aligned}
&\{\mathrm{id},\ \lambda_F\}, \\
&\{\lambda_A,\ \lambda_A\lambda_F,\ \lambda_F\lambda_A,\ \lambda_F\lambda_A\lambda_F\}, \\
&\{\lambda_J,\ \lambda_J\lambda_F,\ \lambda_F\lambda_J,\ \lambda_F\lambda_J\lambda_F\}, \\
&\{\lambda_G,\ \lambda_G\lambda_F,\ \lambda_F\lambda_G,\ \lambda_F\lambda_G\lambda_F\}.
\end{aligned}$$

Problem: determine $\langle \lambda_F, \lambda_A, \lambda_J, \lambda_G \rangle$ in $\mathrm{Out}(\mathcal{O}_2)$. Is it amenable ?

Case of $\mathcal{P}_3^2$

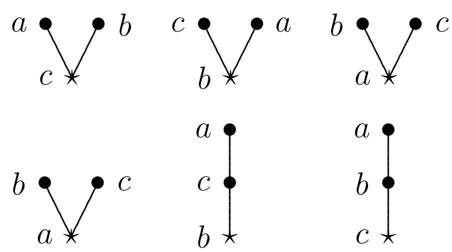
There are only two rooted trees with three vertices. Condition (b) can only be satisfied for the following four 3-tuples of unlabeled trees:



and the other two permutations of the latter.

For each such 3-tuple, there are precisely $3!\,(3!^3) = 6 \cdot 216$ permutations in P_3^2 satisfying condition (b) and among them $3!\,24$ permutations satisfying also condition (d).

The corresponding labeled trees are of the form



etc., where a, b and c are distinct elements in $\{1, 2, 3\}$.

In particular, for each fixed set of labels on a 3-tuple (and there are 3! of them) there are 24 permutations satisfying both the conditions (b) and (d).

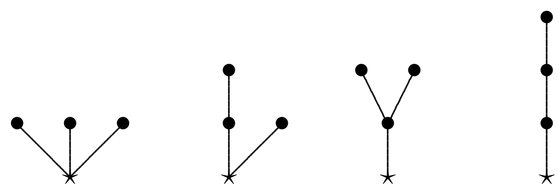
Summarizing:

$$\#\{\sigma \in P_3^2 \ : \ \lambda_{u_\sigma}|_{\mathcal{D}_3} \in \mathrm{Aut}(\mathcal{D}_3)\} = 4 \cdot 3! \cdot 216 = 5184,$$
$$\#\{\sigma \in P_3^2 \ : \ \lambda_{u_\sigma} \in \mathrm{Aut}(\mathcal{O}_3)\} = 4 \cdot 3! \cdot 24 = 576.$$

In particular, there are $4 \cdot 24 = 96$ distinct classes of automorphisms in $\mathrm{Out}(\mathcal{O}_3)$ corresponding to permutations in P_3^2.

Case of $\mathcal{P}_4^2$

There are four (unlabeled) rooted trees with four vertices:



By "type" we mean an unordered set of four trees making up a 4-tuple (two different 4-tuples belong to the same type if one can be obtained from the other by a permutation of the unlabeled trees). One verifies that only eight types of 4-tuples of such trees admit labelings satisfying condition (b); they are listed in the first column of Table 1 below. The second column of this table gives the number of distinct labelings satisfying condition (b) and corresponding to each type. These numbers are factorized as $X \cdot Y \cdot Z$, where X is the number of distinct 4-tuples of unlabeled trees of the given type, $Y = 4!$ is the number of permutations of labels (it corresponds to action of inner automorphisms arising from P_4^1), and Z is the number of orbits under this action. The last column contains the number of all permutations in P_4^2 satisfying both conditions (b) and (d) whose corresponding trees are of the given type.

Table 1:

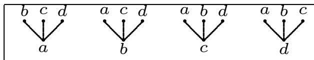
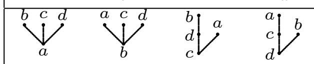
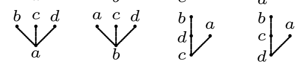
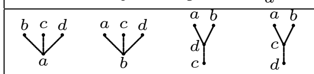
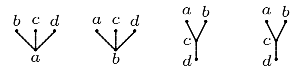
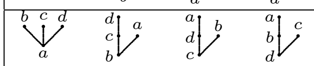
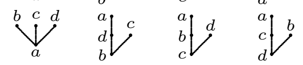
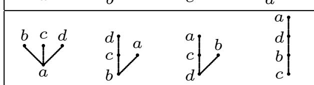
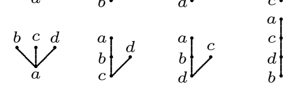
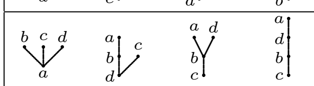
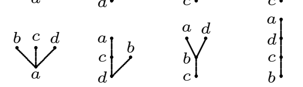
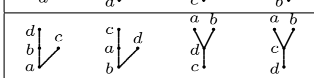
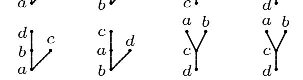
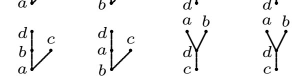
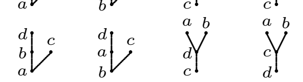
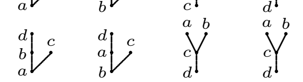
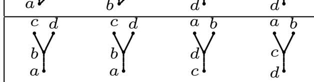
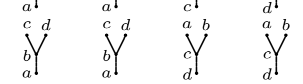
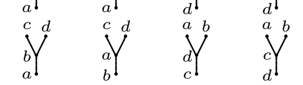
type	# (b)	$\#\sigma$ (d)
$\alpha\alpha\alpha\alpha$	$24 = 1 \cdot 24 \cdot 1$	51,840
$\alpha\alpha\beta\beta$	$576 = 6 \cdot 24 \cdot 4$	787,968
$\alpha\alpha\gamma\gamma$	$288 = 6 \cdot 24 \cdot 2$	311,040
$\alpha\beta\beta\beta$	$768 = 4 \cdot 24 \cdot 8$	746,496
$\alpha\beta\beta\delta$	$1152 = 12 \cdot 24 \cdot 4$	1,575,936
$\alpha\beta\gamma\delta$	$1152 = 24 \cdot 24 \cdot 2$	1,244,160
$\beta\beta\gamma\gamma$	$1152 = 6 \cdot 24 \cdot 8$	787,968
$\gamma\gamma\gamma\gamma$	$288 = 1 \cdot 24 \cdot 12$	266,112
total	5400	5,771,520

The number of permutations satisfying condition (d) depends both on the type of the corresponding 4-tuple of trees and on the specific labeling. However, as it turns out, it does not depend on the permutation of unlabeled trees within the type. Precise information to this effect is provided in Table

2 below. The table has 19 rows, which correspond to all possible distinct labelings (satisfying condition (b)) afforded by each type.

$$\#\{\sigma \in P_4^2 \ : \ \lambda_{u_\sigma}|_{\mathcal{D}_4} \in \mathrm{Aut}(\mathcal{D}_4)\} = 5400 \cdot 4!^4 = 1,791,590,400,$$
$$\#\{\sigma \in P_4^2 \ : \ \lambda_{u_\sigma} \in \mathrm{Aut}(\mathcal{O}_4)\} = 5,771,520.$$

In particular, there are $240,480$ distinct classes of automorphisms in $\mathrm{Out}(\mathcal{O}_4)$ corresponding to permutations in P_4^2.

Labelled Trees	# (b)	# (d)	Example
	$24 = 1 \cdot 24$	2160	Id
	$288 = 12 \cdot 24$	576	(4, 7)
	$288 = 12 \cdot 24$	2160	(7, 8)
	$144 = 6 \cdot 24$	2160	(3, 4)(7, 8)
	$144 = 6 \cdot 24$	0	
	$576 = 24 \cdot 24$	576	(3, 7, 14, 10, 6, 4)(5, 13, 9)(8, 16, 12)(11, 15)
	$192 = 8 \cdot 24$	2160	(2, 3, 4)
	$576 = 24 \cdot 24$	2160	(7, 8)(11, 12)(14, 16, 15)
	$576 = 24 \cdot 24$	576	(3, 4, 7, 10, 8, 11, 6)(5, 9)(14, 16, 15)
	$576 = 24 \cdot 24$	0	
	$576 = 24 \cdot 24$	2160	(2, 3)(6, 8)(10, 12)(14, 15, 16)
	$288 = 12 \cdot 24$	576	(3, 4)(7, 8)(10, 13)
	$288 = 12 \cdot 24$	0	
	$144 = 6 \cdot 24$	0	
	$288 = 12 \cdot 24$	2160	(3, 4)(7, 8)(13, 14)
	$144 = 6 \cdot 24$	0	
	$144 = 6 \cdot 24$	0	
	$72 = 3 \cdot 24$	1536	(2, 9, 5)(4, 11, 7)(6, 10, 13)(8, 12, 15)
	$72 = 3 \cdot 24$	2160	(3, 4)(7, 8)(9, 10)(13, 14)

Case of $\mathcal{P}_3^3$

In this case, there are 286 rooted trees with $n^{k-1} = 9$ vertices, of which 171 satisfy our basic conditions: that each vertex has in-degree at most $n = 3$ (recall that there is a loop at the root, adding 1 to its in-degree). Let us define the *in-degree type* of a rooted tree to be the multiset of the in-degrees of its vertices.

We list the 171 rooted trees in Table 2; they are classified by the eleven in-degree types $\{A \dots K\}$ listed in Table 1.

Table 1: The in-degree types for $\mathcal{P}_3^3$.

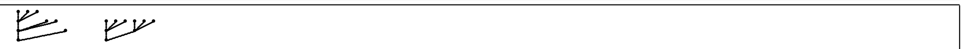
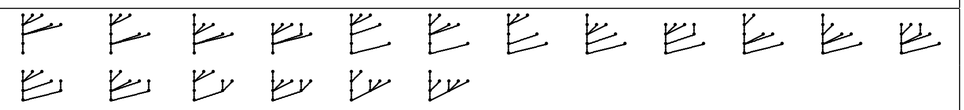
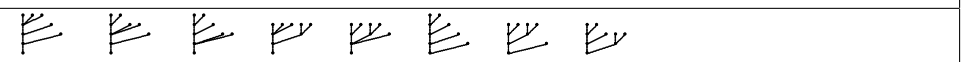
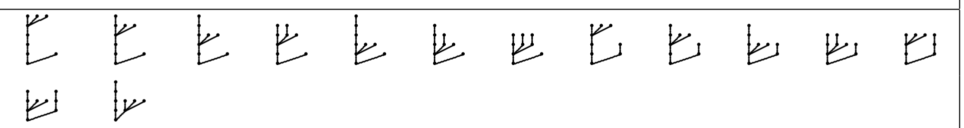
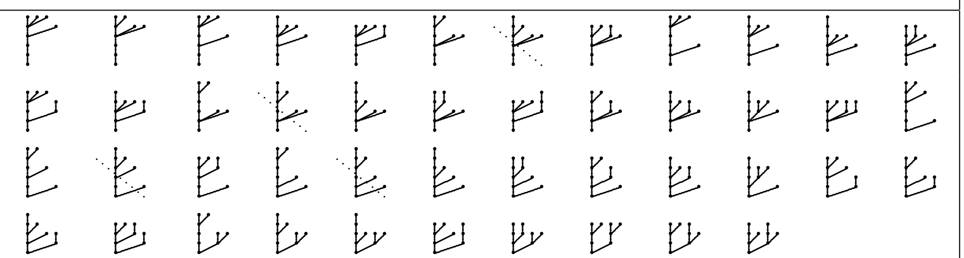
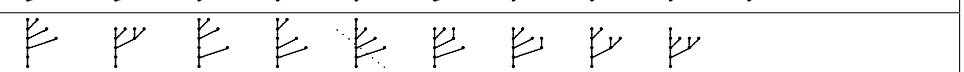
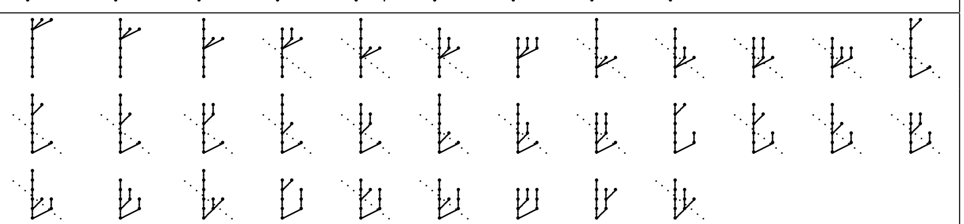
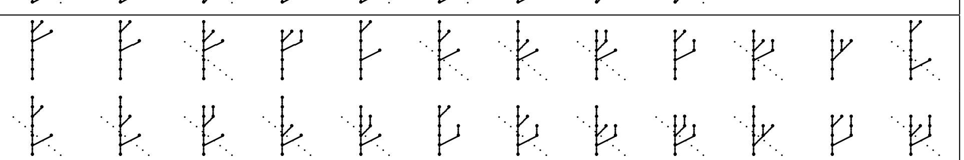
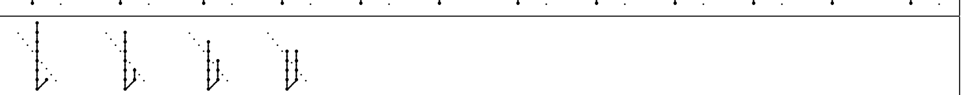
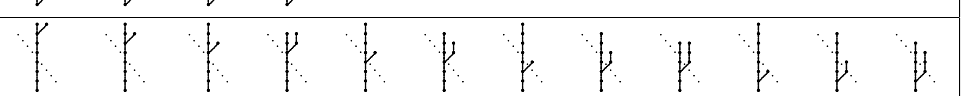
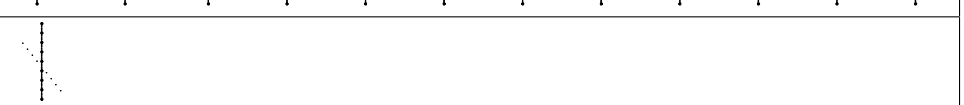
Type	In-Degree 0	1	2	3	# trees
	Multiplicities:				
A	6	0	0	3	2
B	5	1	1	2	18
C	5	0	3	1	8
D	4	3	0	2	14
E	4	2	2	1	46
F	4	1	4	0	9
G	3	4	1	1	33
H	3	3	3	0	24
I	2	6	0	1	4
J	2	5	2	0	12
K	1	7	1	0	1

We wish to find 3-tuples f of labeled trees such that

$$\forall j \in \{1..9\} \quad f_1(j) + f_2(j) + f_3(j) = 3.$$

For such an f we define the in-degree *alignment* matrix M where M_{ij} is the in-degree of the vertex labeled j in the tree f_i. Every in-degree alignment matrix has each row adding to $n^{k-1} = 9$ and each column adding to $n = 3$. In order to find all required f we first determine the possible in-degree alignments of our 11 types.

Table 2: In-degree types and trees.

	Trees of given in-degree type
A	
B	
C	
D	
E	
F	
G	
H	
I	
J	
K	

Now the number of size three multisets with elements chosen from a set of eleven is 286 (the eleventh tetrahedral number)[1]. Of the 286 size three multisets of in-degree types, we compute that 100 have at least one alignment. The number of alignments (up to consistent relabeling) is 133.

After about 200 processor days we report that condition (b) is satisfied for a set $\mathcal{F}$ of 7 390 three-tuples of labeled trees, up to permutation of tree position (action of S_3) and consistent relabeling of all trees (action of S_9). Only 110 of the 171 unlabeled trees appear in $\mathcal{F}$ (those which do not appear in $\mathcal{F}$ are marked in Table 2 with a dotted backslash); they have the first eight, $\{A \dots H\}$, of the eleven in-degree types. In these $\mathcal{F}$ there occur 474 multisets of three unlabeled trees; they have the six distinct three element multisets of in-degree types listed in the first column of Table 3. The second column contains the number of three element multisets of unlabeled trees having the respective types; the third column, $\mathcal{F}_{\text{types}}$, is the partition of $\mathcal{F}$ according to the respective types; the fourth column of each row in Table 3, $\#f$ covered, is the inner product of the last two columns of the corresponding table in the Appendix.

Table 3: Three element multisets of in-degree types.

ID types	# tree triples	# $\mathcal{F}_{\text{types}}$	$\#f$ covered
$A \quad A \quad A$	4	2 168	$12924 \cdot 9!$
$A \quad B \quad B$	176	2 782	$16650 \cdot 9!$
$A \quad C \quad D$	75	950	$5700 \cdot 9!$
$A \quad E \quad E$	180	1 072	$6396 \cdot 9!$
$A \quad F \quad G$	31	392	$2352 \cdot 9!$
$A \quad H \quad H$	8	26	$150 \cdot 9!$
Total	474	7 390	$44\,172 \cdot 9!$

In total, we have

$$44\,172 \times 9! = 16\,029\,135\,360$$

three-tuples of labeled trees satisfying condition (b). Therefore we have

$$44\,172 \times 9! \times n!^{n^{k-1}} = 44\,172 \times 9! \times 6^9 = 161\,536\,753\,300\,930\,560$$

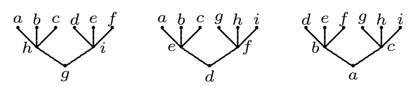
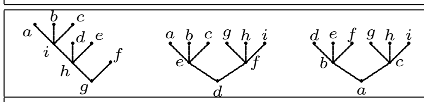
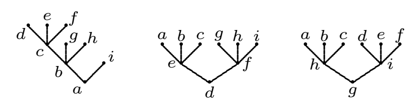
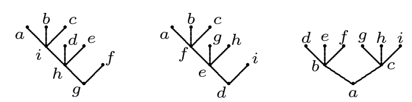
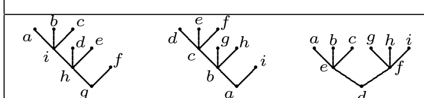
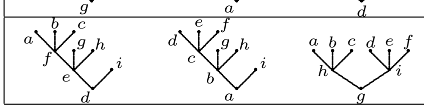
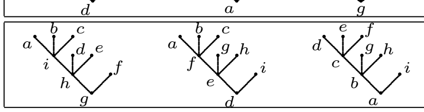
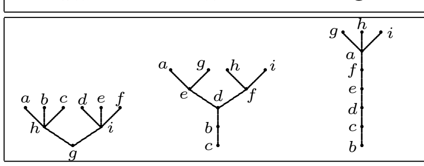
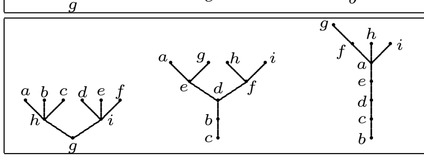
[1]Is it just a combinatorial accident that the eleventh tetrahedral number is the same as the number of all (unordered) rooted trees on nine-vertices.

permutations $\sigma \in P_3^3$ satisfying condition (b).

With additional work, one can even determine the precise number of permutations satisfying condition (d), i.e. providing automorphisms of $\mathcal{O}_3$

Some examples of 3-tuples of labeled trees satisfying condition (b) are listed in the first column of Table 4; the second column contains the size of the orbit of the combined actions of S_3 and S_9 on the first entry of each row; the third column is a count of the number of permutations corresponding to f that satisfy condition (d); the last column contains (when one exists) an example permutation satisfying condition (d) with labels $(a, b, c, \ldots, i)$ chosen to be $(1, 2, 3, \ldots, 9) = (\,(1, 1),\ (2, 1),\ (3, 1),\ \ldots, (3, 3)\,)$.

Table 4: Examples for $\mathcal{P}_3^3$

Labeled Trees	# (b)	# (d)	Example
A A A			
	$1 \cdot 9!$	312	(1, 6, 26, 7, 22, 17) (2, 12, 24, 20, 18, 13, 14) (3, 27, 16, 25, 19, 9, 10) (4, 11, 15, 23, 8)
	$6 \cdot 9!$	0	
	$6 \cdot 9!$	240	(1, 25, 24, 23, 2, 19) (3, 16, 27, 15, 26) (4, 17, 9, 18, 12, 10) (6, 20, 22, 14, 8) (7, 21, 13, 11)
	$3 \cdot 9!$	216	(1, 3, 27, 4, 26, 10, 9) (2, 18, 7, 16, 19, 6) (5, 20, 12, 21, 24) (8, 25, 22, 11, 15) (13, 14, 17)
	$6 \cdot 9!$	0	
	$6 \cdot 9!$	0	
	$6 \cdot 9!$	0	
A F G			
	$6 \cdot 9!$	0	
	$6 \cdot 9!$	0	

More examples:

The following examples, show that G_n is non-amenable for $n > 2$.

$W_n^1 = R_1 \cup \ldots \cup R_r$ nontrivial r-partition, $1 < r \leq n$
$(\sigma_i)_{i=1}^r \in P_n^1$ such that $\sigma_i \sigma_j^{-1}(R_m) = R_m$, $\forall i, j, m$
Then define $\psi \in P_n^2$ by

$$\psi(\alpha, \beta) = (\alpha, \sigma_i(\beta)), \quad \alpha \in R_i, \beta \in W_n^1$$

Claims:

- λ_ψ invertible, with inverse induced by an element $\overline{\psi}$ in P_n^3, namely

$$\overline{\psi}(\alpha, \beta, \gamma) = (\alpha, \sigma_i^{-1}(\beta), \sigma_j \sigma_k^{-1}(\gamma)) \ ,$$

  where $\alpha \in R_i, \beta \in R_k, \sigma_i^{-1}(\beta) \in R_j$

- $\lambda_\psi \in \mathrm{Inn}(\mathcal{O}_n)$ if and only if $\psi = id$

Example 0.21. $n = r = 2$, $W_2^1 = \{1\} \cup \{2\}$, $\sigma_1 = \sigma_2 \neq \mathrm{id}$, *then* $\mathrm{Ad}(u_{\sigma_1})\lambda_\psi = \lambda_{u_{\sigma_1}}$ *is Archbold flip-flop automorphism of* $\mathcal{O}_2$

Example 0.22. $n = 4$, $W_4^1 = \{1, 2\} \cup \{3, 4\}$, $\sigma_1 = (23), \sigma_2 = (1243)$, *then* $\mathrm{Ad}(u_{\sigma_1})\lambda_\psi$ *is Matsumoto-Tomiyama outer automorphism of* $\mathcal{O}_4$*, namely:*
$S_1 \to S_1$,
$S_2 \to S_3$,
$S_3 \to S_2S_1S_3^* + S_2S_2S_4^* + S_2S_3S_1^* + S_2S_4S_2^*$,
$S_4 \to S_4S_1S_3^* + S_4S_2S_4^* + S_4S_3S_1^* + S_4S_4S_2^*$

All the associated trees are the same as for Bogolubov automorphisms

A family of matrix equations:

Let $U \in \mathcal{F}_n^k$ and $V \in \mathcal{F}_n^h$ be such that

$$\lambda_U \lambda_V = \lambda_V \lambda_U = \mathrm{id}$$

i.e. $\lambda_U(V) = U^*$, $\lambda_V(U) = V^*$

Therefore

$$U_h V U_h^* = U^*, \quad V_k U V_k^* = V^*$$

and thus

$$V = U_h^* U^* U_h, \quad U = V_k^* V^* V_k$$

Now replace both h and k with $h \vee k$

For $r \in \mathbb{N}$ and $U \in \mathcal{F}_n^r$, get an equation for U alone:

$$(U_r^* U^* U_r)_r^* U_r^* U U_r (U_r^* U^* U_r)_r = U$$

(where $U_r^* = (U_r)^*$)

Example:

$r = 2$:

$$\big(\varphi(U^*)U^*\varphi(U)\varphi(\varphi(U^*)U^*\varphi(U))\big)^*\varphi(\varphi(U^*)U^*\varphi(U)) = U \ ,$$

i.e.

$$U\varphi(U\varphi(U^*)U^*) = \varphi(U\varphi(U^*)U^*)U$$

$r = 3$:

U commutes with

$$\varphi^2\big(U\varphi(U)\varphi^2(U)\big)\varphi\big(\varphi^2(U^*)\varphi(U^*)U\varphi(U)\varphi^2(U)\big)\varphi^2(U^*)\varphi(U^*)$$

It is quite intriguing that in the case of permutation matrices the solutions to the above equations can be described in terms of rooted trees.

We are not aware of any other occurrence of these polynomial matrix equations...

Tabulated results (small values of n and k)

$d_n^k := \#\lambda(\mathcal{P}_n^k)^{-1}$, number of permutative automorphisms of $\mathcal{O}_n$ at level k

c_n^k number of classes modulo inner equivalence; it holds

$$d_n^k = n^{k-1}!\, c_n^k\ .$$

$b_n^k = \#\{u \in \mathcal{P}_n^k : \lambda_u|_{\mathcal{D}_n} \in \mathrm{Aut}(\mathcal{D}_n)\}$

s_n^k number of square-free automorphisms in $\lambda(\mathcal{P}_n^k)^{-1}$

$d_n^k/(b_n^k)$:

$k \setminus n$	2	3	4
1	2 (2)	6 (6)	24 (24)
2	4 (8)	576 (5184)	5,771,520 (1,791,590,400)
3	48 (324)	329,148,126,720 (161,536,753,300,930,560)	
4	564,480 (175,472,640)		

c_n^k:

$k \setminus n$	2	3	4
1	2	6	24
2	2	96	240,480
3	2		
4	14		

s_n^k:

$k \setminus n$	2	3	4
1	2	4	10
2	4	52	2,032
3	20		
4	1,548		

Entropy vs. Index (cf. Kawamura, Skalski-Zacharias)

Table 1. Entropy and index of the 'rank 2' permutation endomorphisms of $\mathcal{O}_2$.

ρ_σ	$\rho_\sigma(s_1)$	$\rho_\sigma(s_2)$	*property*	$\mathrm{ht}(\rho_\sigma)$	$\mathrm{ht}(\rho_\sigma\|_{\mathcal{D}_2})$	$\mathrm{Ind}(\rho_\sigma)$
ρ_{id}	s_1	s_2	inn	0	0	1
ρ_{12}	$s_{12,1}+s_{11,2}$	s_2	irr	$\log 2$	0	2
ρ_{13}	$s_{21,1}+s_{12,2}$	$s_{11,1}+s_{22,2}$	irr	$\log 2$	$\log 2$	2
ρ_{14}	$s_{22,1}+s_{12,2}$	$s_{21,1}+s_{11,2}$	red	$\log 2$	$\log 2$	4
ρ_{23}	$s_{11,1}+s_{21,2}$	$s_{12,1}+s_{22,2}$	red	$\log 2$	$\log 2$	4
ρ_{24}	$s_{11,1}+s_{22,2}$	$s_{21,1}+s_{12,2}$	irr	$\log 2$	$\log 2$	2
ρ_{34}	s_1	$s_{22,1}+s_{21,2}$	irr	$\log 2$	0	2
ρ_{123}	$s_{12,1}+s_{21,2}$	$s_{11,1}+s_{22,2}$	red	$\log 2$	$\log 2$	4
ρ_{132}	$s_{21,1}+s_{11,2}$	$s_{12,1}+s_{22,2}$	red	$\log 2$	$\log 2$	4
ρ_{124}	$s_{12,1}+s_{22,2}$	$s_{21,1}+s_{11,2}$	red	$\log 2$	$\log 2$	4
ρ_{142}	$s_{22,1}+s_{11,2}$	$s_{21,1}+s_{12,2}$	irr	$\log 2$	$\log 2$	4
ρ_{134}	$s_{21,1}+s_{12,2}$	$s_{22,1}+s_{11,2}$	irr	$\log 2$	$\log 2$	4
ρ_{143}	$s_{22,1}+s_{12,2}$	$s_{11,1}+s_{21,2}$	red	$\log 2$	$\log 2$	4
ρ_{234}	$s_{11,1}+s_{21,2}$	$s_{22,1}+s_{12,2}$	red	$\log 2$	$\log 2$	4
ρ_{243}	$s_{11,1}+s_{22,2}$	$s_{12,1}+s_{21,2}$	red	$\log 2$	$\log 2$	4
ρ_{1234}	$s_{12,1}+s_{21,2}$	$s_{22,1}+s_{11,2}$	irr	$\log 2$	$\log 2$	2
ρ_{1243}	$s_{12,1}+s_{22,2}$	$s_{11,1}+s_{21,2}$	red	$\log 2$	$\log 2$	4
ρ_{1324}	s_2	$s_{12,1}+s_{11,2}$	irr	$\log 2$	0	2
ρ_{1342}	$s_{21,1}+s_{11,2}$	$s_{22,1}+s_{12,2}$	red	$\log 2$	$\log 2$	4
ρ_{1423}	$s_{22,1}+s_{21,2}$	s_1	irr	$\log 2$	0	2
ρ_{1432}	$s_{22,1}+s_{11,2}$	$s_{12,1}+s_{21,2}$	irr	$\log 2$	$\log 2$	2
$\rho_{(12)(34)}$	$s_{12,1}+s_{11,2}$	$s_{22,1}+s_{21,2}$	out	0	0	1
$\rho_{(13)(24)}$	s_2	s_1	out	0	0	1
$\rho_{(14)(23)}$	$s_{22,1}+s_{21,2}$	$s_{12,1}+s_{11,2}$	inn	0	0	1

Remark: $ht = 0$ precisely when λ_U (resp. $\lambda_U|_{\mathcal{D}_2}$) is an automorphism (4 + 4 cases)

"Exotic" endomorphisms of $\mathcal{O}_n$:
see arXiv:0910.1304 (JFA 2010)

Full Weyl group $\lambda(\mathcal{S}_n)^{-1}$ and outer full Weyl group

Problem: find necessary and sufficient conditions for $w \in \mathcal{S}_n$ such that

- $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$
- $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$
- characterize intrinsically the group of homeomorphisms of X_n arising in this way

... work in progress ...

The above for graph algebras $C^*(E)$, with E finite (CK-algebras):
see e.g. arXiv:1101.4210 for a first set of results