Characterization of Local Operators in Factorizing Scattering Models
(work in progress with H. Bostelmann)

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Outline

The Algebraic Formulation of Quantum Field Theory

- We consider a relativistic quantum theory on Minkowski space $\mathbb{R}^d$.
- A model is characterized in terms of its net of algebras $\mathcal{A}$ of local observables, which are given by bounded operators on the Hilbert space of the theory $\mathcal{H}$.
- $\mathcal{A}$ contains all local algebras $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$, which are the von Neumann algebras generated by the observables localized in a spacetime region $\mathcal{O} \subset \mathbb{R}^d$.
- The assignment

$$\mathbb{R}^d \supset \mathcal{O} \longmapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$$

  contains all the physical information of the theory.
- For this assignment to model the observables of a relativistic quantum system, the algebras $\mathcal{A}(\mathcal{O})$ must have...

... a number of properties:

- Isotony

$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \quad \text{for} \quad \mathcal{O}_1 \subset \mathcal{O}_2.$$

- Causality

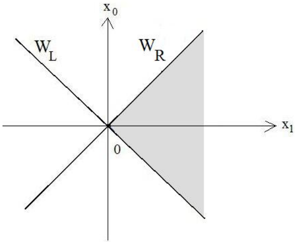
$$\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)' \quad \text{for} \quad \mathcal{O}_1 \subset \mathcal{O}_2'.$$

- There exists a strongly continuous, unitary representation $U : \mathcal{P}_+^\uparrow \to \mathcal{B}(\mathcal{H})$.
- Covariance

$$U(g)\mathcal{A}(\mathcal{O})U(g)^{-1} = \mathcal{A}(g\mathcal{O}), \quad g \in \mathcal{P}_+^\uparrow.$$

- The spectrum of P^μ is in the closed forward light cone.
- unique "Vacuum state" $\Omega \in \mathcal{H}$ invariant under the action of U.
- Ω is cyclic and separating for $\mathcal{A}(\mathcal{O})$.

Wedges



- The right wedge

$$W_R := \{x \in \mathbb{R}^2 : x_1 > |x_0|\}$$

General wedge: Poincaré transform $W = \Lambda W_R + x$.

- "Wedges are big enough to allow for simple observables being localized in them, but also small enough so that two of them can be spacelike separated"

Construction of Local Nets from Wedge Algebras

- Local nets can be constructed from a "wedge algebra" and an action of the Poincaré group.
- Let $\mathcal{N}$ be a $\mathcal{C}^*$-subalgebra of $\mathcal{B}(\mathcal{H})$ such that

$$U(x,\Lambda)\mathcal{N}U(x,\Lambda)^* \subset \mathcal{N} \quad \text{if} \quad \Lambda W_R + x \subset W_R \quad \text{(isotony)}$$
$$U(x,\Lambda)\mathcal{N}U(x,\Lambda)^* \subset \mathcal{N}' \quad \text{if} \quad \Lambda W_R + x \subset W_R' \quad \text{(locality)}$$

  $\mathcal{N}$ is called a wedge algebra.
- Then

$$\mathcal{A} : \Lambda W_R + x \longmapsto U(x,\Lambda)\mathcal{N}U(x,\Lambda)^*$$

  is a well-defined, isotonus, local, covariant net of C^*-algebras.
- Extension to smaller regions:
  if

$$\mathcal{O} = (W_R + x) \cap (W_L + y), \quad y - x \in W_R$$

  then

$$\mathcal{A}(\mathcal{O}) := \mathcal{A}(W_R + x) \cap \mathcal{A}(W_L + y).$$

Outline

Factorizing S-Matrix Model

Example for an explicit contruction of a right wedge algebra: Models with factorizing S-matrix on 2-Minkowski space.

a. Fix the particle spectrum of the theory: one species of particle, massive and scalar.

b. Define the single particle space $\mathcal{H}_1 = L^2(\mathbb{R}^2, d\mu(p))$, together with an irreducible representation U of the Poincaré group on it. (Same as free field theory with the same particle spectrum).

c. Construct $\mathcal{H}$ as the S-symmetrized Fock space over $\mathcal{H}_1$. $\mathcal{H}$ then automatically has a representation of the Poincaré group. In addition, define a representation of space-time reflection $U(j)$ on $\mathcal{H}$.

d. Introduce the Zamolodchikov algebra (z and z^+ as improper operators on $\mathcal{H}$).

$$z(\vartheta_1)z(\vartheta_2) = S(\vartheta_1 - \vartheta_2)z(\vartheta_2)z(\vartheta_1)$$
$$z^+(\vartheta_1)z^+(\vartheta_2) = S(\vartheta_1 - \vartheta_2)z^+(\vartheta_2)z^+(\vartheta_1)$$
$$z(\vartheta_1)z^+(\vartheta_2) = S(\vartheta_2 - \vartheta_1)z^+(\vartheta_2)z(\vartheta_1) + \delta(\vartheta_1 - \vartheta_2) \cdot 1_{\mathcal{H}}.$$

e. Introduce a quantum field ϕ ("Wightman framework") on 2-Minkowski. Let $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\phi(f) := \int dx f(x) \int d\vartheta \Big(z^+(\vartheta) e^{ip(\vartheta)\cdot x} + z(\vartheta) e^{-ip(\vartheta)\cdot x} \Big)$$

$\phi(f)$ is an "operator valued distribution" over 2-Minkowski, defined on $\mathcal{D}$(=subspace of $\mathcal{H}$ of finite particle number states) and leaves this space invariant. For $\Psi \in \mathcal{D}$,

$$||\phi(f)\Psi|| \leq (||f^+|| + ||f^-||) \cdot ||(N+1)^{1/2}\Psi||$$

...with the following properties:

- if $\Psi \in \mathcal{D}$, then $\phi(f)^*\Psi = \phi(\overline{f})\Psi$.
- if $\Psi \in \mathcal{D}$, then $\phi((\Box + m^2)f)\Psi = 0$.
- U representation of $\mathcal{P}_+^\uparrow$, $U(x,\lambda)\phi(f)U(x,\lambda)^{-1} = \phi(f_{(x,\lambda)})$.
- Ω is cyclic for the polynomial algebra generated by ϕ.
- ϕ is local if and only if $S = 1$.
- We consider $U(j) := J$, $(J\Psi)_n(\vartheta_1,..,\vartheta_n) := \overline{\Psi_n(\vartheta_n,..,\vartheta_1)}$, and we define:

$$\phi'(f) := J\phi(f^j)J.$$

in terms of $z(\psi)' := Jz(\psi)J, \quad z^+(\psi)' := Jz^+(\psi)J$.

Proposition

Wedge locality: for $f \in \mathcal{S}(W_R)$, $g \in \mathcal{S}(W_L)$, there holds

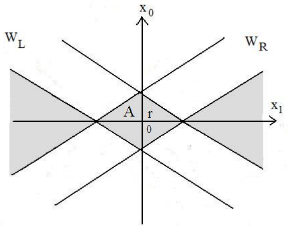
$$[\phi'(f), \phi(g)]\Psi = 0, \qquad \Psi \in \mathcal{D}.$$

f. We construct nets $W \longmapsto \mathcal{A}(W)$ of wedge algebras by

$$\mathcal{A}(W_R) := \left\{ e^{i\phi'(f)} : f \in \mathcal{S}_{\mathbb{R}}(W_R) \right\}'' ,$$
$$\mathcal{A}(W_L) := \left\{ e^{i\phi(f)} : f \in \mathcal{S}_{\mathbb{R}}(W_L) \right\}'' .$$

It is a well-defined wedge-algebra.

g. Define the double cone algebras by intersection of wedge algebras.



h. Apply Haag-Ruelle scattering theory. Result: The S-matrix is factorizing, and the two-particle S-matrix is the function we started with.

$$S_{n,n}(\boldsymbol{\theta};\boldsymbol{\theta}') = S^0_{n,n}(\boldsymbol{\theta};\boldsymbol{\theta}') \prod_{1\le l<k\le n} S(|\theta_k - \theta_l|)$$

The Araki expansion

- Task: Characterization of local operators in the double cone algebras.
- Existence and uniqueness of the Araki expansion for arbitrary bounded (or in fact more general) operator.

$$A = \sum_{m,n}^{\infty} \int d^m\theta d^n\eta f^{[A]}_{m,n}(\theta,\eta) z^{+m}(\theta) z^n(\eta)$$

- The coefficients functions:

$$f^{[A]}_{mn}(\theta,\eta) := \frac{1}{m!n!} \sum_{C\in\mathcal{C}_{m+n,m}} (-1)^{|C|}\delta_C S^{(m)}_C \times$$
$$\times < z^{+(n-|C|)}(\hat{\eta})\Omega, JA^*Jz^{+(m-|C|)}(\hat{\theta})\Omega >$$

Lechner 2006

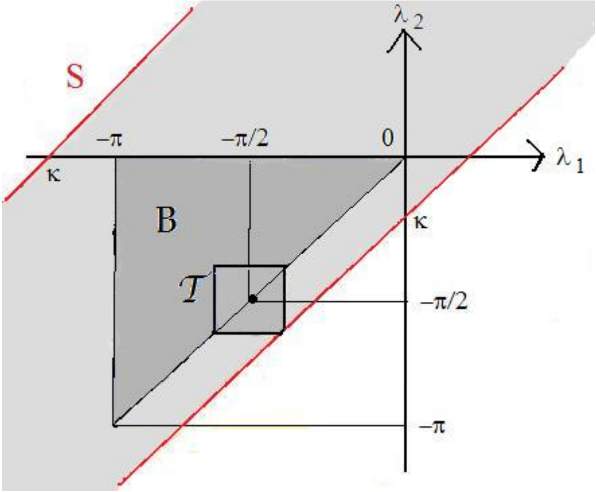
- Define

$$< A >_{n,k}^{con} := \sum_{C \in \mathcal{C}_{n,k}} (-1)^{|C|} \delta_C S_C^{(k)} < \boldsymbol{l}_C | A | \boldsymbol{r}_C >_{n,k}$$

Proposition

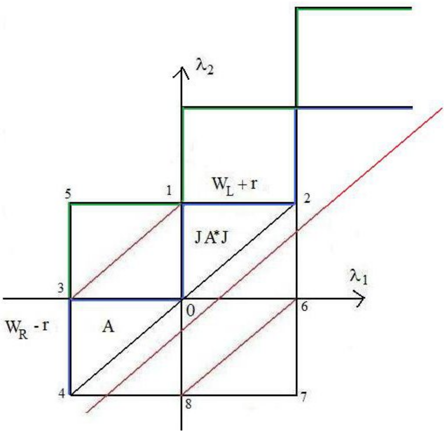
In a model with regular scattering function S, let $A \in \mathcal{A}(W_R)$.
a)$< A >_{n,0}^{con}$ is analytic in the tube $\mathbb{R}^n - i\mathcal{B}_n$.
b) Let $0 < \kappa < \kappa(S)$. There holds the bound,

$$| < A >_{n,0}^{con} (\zeta)| \leq \left(\frac{8}{\pi} \frac{||S||}{\sqrt{\kappa(S) - \kappa}}\right) ||A||, \quad \zeta \in \mathcal{T}.$$


S
λ_2
$-\pi$
$-\pi/2$
0
λ_1
κ
B
κ
T
$-\pi/2$
$-\pi$

Extension

Now, let A be an operator localized in a double cone.



- Based on the interplay between A and JA^*J in the functions $f_{mn}^{[A]}(\theta)$, the Malgrange-Zerner Theorem and the Epstein's Edge of Wedge Theorem.

Outline

Characterization Theorem for Local Operators

- If A is a bounded operator localized in a double cone, then the coefficients functions are boundary values of meromorphic functions F_k on $\mathcal{C}^k$, which fulfil bounds

$$|F_k(\zeta)| \leq \left(\frac{C||S||^k}{\sqrt{\rho}}\right) e^{\mu r \cosh(\vartheta_s)|\sin\lambda_s|} \prod_{j\neq s} e^{\mu r \cosh(|\vartheta_j|+\rho)} \times$$

$$\times \prod_{\substack{p,m=1\\ p\neq m}}^{k} |\zeta_p - \zeta_m - i\pi|^{-1} \prod_{n=1}^{k} |M(\zeta_n + i\pi)|^{-1}$$

with $\boldsymbol{\zeta} = \boldsymbol{\theta} + i(0,..,0,\lambda_s,\pi,..,\pi)$, $\lambda_s \in (0,\pi)$.
fulfil the recursion relations

$$F_{m+n}(\boldsymbol{\theta} - i\mathbf{0}, \boldsymbol{\eta} + i\boldsymbol{\pi} + i\mathbf{0}) = \sum_{C\in\mathcal{C}_{m+n,m}} (-1)^{|C|}\delta_C \times$$

$$\times S_C \prod_{j=1}^{|C|} \left(1 - \prod_{p_j=1}^{m+n} S^{(m)}_{r_j,p_j}\right) F_{m+n-2|C|}(\hat{\boldsymbol{\theta}} + i\mathbf{0}, \hat{\boldsymbol{\eta}} + i\boldsymbol{\pi} - i\mathbf{0})$$

Characterization Theorem for Local Operators

...and the properties of S-periodicity and S-symmetry

$$F_k(\theta_1, .., \theta_j + 2i\pi, .., \theta_k) = \Big(\prod_{\substack{i=1 \\ i \neq j}}^{k} S(\theta_i - \theta_j) \Big) F_k(\theta_1, .., \theta_j, .., \theta_k)$$

$$F_k(\vartheta_1, .., \vartheta_j, \vartheta_{j+1}, .., \vartheta_k) = S(\vartheta_{j+1} - \vartheta_j) F_k(\vartheta_1, .., \vartheta_{j+1}, \vartheta_j, .., \vartheta_k)$$

Conversely, given a family of functions F_k with those properties, then the Araki expansion defines an operator (an unbounded quadratic form) which is localized in a double cone.

$$A = \sum_{m,n}^{\infty} \int \frac{d^m\boldsymbol{\theta} d^n\boldsymbol{\eta}}{m!n!} F_{m+n}(\boldsymbol{\theta} + i\mathbf{0}, \boldsymbol{\eta} + i\boldsymbol{\pi} - i\mathbf{0}) z^{+m}(\boldsymbol{\theta}) z^n(\boldsymbol{\eta})$$

Question: encode the norm bounds of A in terms of F_k.

The second part of the theorem

Locality

- Show $[A, \phi'(x)] = 0$ if $x = Re_1, R > r$
- Show $[A, \phi(x)] = 0$ if $R < -r$.

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$$JA^*J = \sum_{m,n}^{\infty} \frac{d^m\boldsymbol{\theta} d^n\boldsymbol{\eta}}{m!n!} F^{\pi}_{m+n}(\boldsymbol{\theta}+i\mathbf{0}, \boldsymbol{\eta}+i\boldsymbol{\pi}-i\mathbf{0}) z^{+m}(\boldsymbol{\theta}) z^n(\boldsymbol{\eta})$$

where $F^{\pi}_k = F_k(\cdot + i\pi)$.
The recursion relations play a fundamental role.

Outline

Conclusion and Outlooks

- We have a general theorem which characterizes explicitly the local operators in Factorizing Scattering Models.
- Open task: to encode $||A||_X$ in terms of F_k. How can this be formulated?
- We provided an example in the case $S = -1$ (Buchholz/Summers 2007).

$$\begin{aligned} A &= \frac{1}{2}\int d\theta d\eta \sinh(\frac{\theta-\eta}{2})\hat{g}(\theta)\hat{g}(\eta)z^{+}(\eta)z^{+}(\theta) \\ &+\frac{1}{2}\int d\eta d\theta \sinh(\frac{\theta-\eta}{2})\hat{g}(\theta)\hat{g}(\eta)z(\eta)z(\theta) \\ &+i\int d\eta d\theta \cosh(\frac{\theta-\eta}{2})\hat{g}(\theta)\hat{g}(\eta)z^{+}(\eta)z(\theta) \end{aligned}$$

Can we find an example for $S = -1$ with non-trivial recursion relations? (Schroer/Truong 1978)
Can we find an example for general S?