Matrix Geometries*

Rodrigo Blando
Dublin Institute for Advanced Studies
Dublin Ireland

4th Annual Meeting EU-NCG
Bucharest, Romania

*Joint work with Denjoe O'Connor

April 26, 2011

Table of contents

Motivation

- Matrix models give a non-perturbative definition for string theory,

Motivation

- Matrix models give a non-perturbative definition for string theory,
- IKKT model gives a definition for string theory IIB,

Motivation

- Matrix models give a non-perturbative definition for string theory,
- IKKT model gives a definition for string theory IIB,
- BFSS matrix model definition of M theory,

Motivation

- Matrix models give a non-perturbative definition for string theory,
- IKKT model gives a definition for string theory IIB,
- BFSS matrix model definition of M theory,
- BMN matrix model is a mass deformation of the BFSS model in $d = 10$, admits the fuzzy sphere as a solution of its equations of motion.

Motivation

- Matrix models give a non-perturbative definition for string theory,
- IKKT model gives a definition for string theory IIB,
- BFSS matrix model definition of M theory,
- BMN matrix model is a mass deformation of the BFSS model in $d = 10$, admits the fuzzy sphere as a solution of its equations of motion.
- Mass deformed IKKT Yang-Mills matrix models in various dimensions admit also the fuzzy sphere as a solution.

Motivation

- Matrix models give a non-perturbative definition for string theory,
- IKKT model gives a definition for string theory IIB,
- BFSS matrix model definition of M theory,
- BMN matrix model is a mass deformation of the BFSS model in $d = 10$, admits the fuzzy sphere as a solution of its equations of motion.
- Mass deformed IKKT Yang-Mills matrix models in various dimensions admit also the fuzzy sphere as a solution.
- Matrix models have a very rich phase structure: D-branes, intersecting branes, etc

Motivation

- Matrix models give a non-perturbative definition for string theory,
- IKKT model gives a definition for string theory IIB,
- BFSS matrix model definition of M theory,
- BMN matrix model is a mass deformation of the BFSS model in $d = 10$, admits the fuzzy sphere as a solution of its equations of motion.
- Mass deformed IKKT Yang-Mills matrix models in various dimensions admit also the fuzzy sphere as a solution.
- Matrix models have a very rich phase structure: D-branes, intersecting branes, etc
- Discretisation of supersymmetric models.

Motivation

- Matrix models give a non-perturbative definition for string theory,
- IKKT model gives a definition for string theory IIB,
- BFSS matrix model definition of M theory,
- BMN matrix model is a mass deformation of the BFSS model in $d = 10$, admits the fuzzy sphere as a solution of its equations of motion.
- Mass deformed IKKT Yang-Mills matrix models in various dimensions admit also the fuzzy sphere as a solution.
- Matrix models have a very rich phase structure: D-branes, intersecting branes, etc
- Discretisation of supersymmetric models.
- Understand the origin of geometry as an emergent phenomena

Motivation

- Matrix models give a non-perturbative definition for string theory,
- IKKT model gives a definition for string theory IIB,
- BFSS matrix model definition of M theory,
- BMN matrix model is a mass deformation of the BFSS model in $d = 10$, admits the fuzzy sphere as a solution of its equations of motion.
- Mass deformed IKKT Yang-Mills matrix models in various dimensions admit also the fuzzy sphere as a solution.
- Matrix models have a very rich phase structure: D-branes, intersecting branes, etc
- Discretisation of supersymmetric models.
- Understand the origin of geometry as an emergent phenomena
- Matrix models provide an escenario to understand classical geometry as condensation of random degrees of freedom as the system evolves.

The 2-matrix model: A solvable model (Hoppe '82,Berestein,et.al. '09)

- Consider the following 2-matrix model (X, Y $N \times N$ Hermitian matrices)

The 2-matrix model: A solvable model (Hoppe '82,Berestein,et.al. '09)

- Consider the following 2-matrix model (X, Y $N \times N$ Hermitian matrices)

$$Z[g^2] = \int dX\, dY e^{-Tr[X^2+Y^2-g^2[X,Y]^2]}, \quad dX = \prod dX_{ii} \prod_{i<j} d\mathbb{R}e(X_{ij}) d\mathbb{I}m(X_{ij})$$

The 2-matrix model: A solvable model (Hoppe '82,Berestein,et.al. '09)

- Consider the following 2-matrix model (X, Y $N \times N$ Hermitian matrices)

$$Z[g^2] = \int dX\, dY e^{-Tr\left[X^2+Y^2-g^2[X,Y]^2\right]}, \quad dX = \prod dX_{ii} \prod_{i<j} d\mathbb{R}e(X_{ij}) d\mathbb{I}m(X_{ij})$$

- Exploting the fact that the model is invariant under unitary transformations; the model can be solved diagonalising the matrix X, then $g^2\, Tr[X,Y]^2 = 2g^2\, Tr(XYXY - X^2Y^2) = -g^2 \sum_{ij}(x_i - x_j)^2 \left|Y_{ij}\right|^2$, then integrating over Y

$$Z[g^2] = \int dx_1 \cdots dx_N e^{-\sum_i^N x_i^2 + \frac{1}{2}\sum_{i\neq j}\log(x_i-x_j)^2 - \frac{1}{2}\sum_{i\neq j}\log[1+g^2(x_i-x_j)^2]}$$

The 2-matrix model: A solvable model (Hoppe '82,Berestein,et.al. '09)

- Consider the following 2-matrix model (X, Y $N \times N$ Hermitian matrices)

$$Z[g^2] = \int dX\, dY e^{-Tr[X^2+Y^2-g^2[X,Y]^2]}, \quad dX = \prod dX_{ii} \prod_{i<j} d\mathbb{R}e(X_{ij}) d\mathbb{I}m(X_{ij})$$

- Exploting the fact that the model is invariant under unitary transformations; the model can be solved diagonalising the matrix X, then $g^2 Tr[X, Y]^2 = 2g^2 Tr(XYXY - X^2Y^2) = -g^2 \sum_{ij}(x_i - x_j)^2 \left|Y_{ij}\right|^2$, then integrating over Y

$$Z[g^2] = \int dx_1 \cdots dx_N e^{-\sum_i^N x_i^2 + \frac{1}{2}\sum_{i\neq j}\log(x_i - x_j)^2 - \frac{1}{2}\sum_{i\neq j}\log[1+g^2(x_i-x_j)^2]}$$

- $$S_{eff}(x_i) = \sum_i^N x_i^2 - \frac{1}{2}\sum_{i\neq j}\log(x_i - x_j)^2 + \frac{1}{2}\sum_{i\neq j}\log[1 + g^2(x_i - x_j)^2]$$

The 2-matrix model: A solvable model (Hoppe '82,Berestein,et.al. '09)

- Consider the following 2-matrix model (X, Y $N \times N$ Hermitian matrices)

$$Z[g^2] = \int dX\, dY e^{-Tr[X^2+Y^2-g^2[X,Y]^2]}, \quad dX = \prod dX_{ii} \prod_{i<j} d\mathbb{R}e(X_{ij}) d\mathbb{I}m(X_{ij})$$

- Exploting the fact that the model is invariant under unitary transformations; the model can be solved diagonalising the matrix X, then $g^2 Tr[X,Y]^2 = 2g^2 Tr(XYXY - X^2Y^2) = -g^2 \sum_{ij}(x_i - x_j)^2 \left|Y_{ij}\right|^2$, then integrating over Y

$$Z[g^2] = \int dx_1 \cdots dx_N e^{-\sum_i^N x_i^2 + \frac{1}{2}\sum_{i\neq j}\log(x_i-x_j)^2 - \frac{1}{2}\sum_{i\neq j}\log[1+g^2(x_i-x_j)^2]}$$

- $$S_{eff}(x_i) = \sum_i^N x_i^2 - \frac{1}{2}\sum_{i\neq j}\log(x_i - x_j)^2 + \frac{1}{2}\sum_{i\neq j}\log[1+g^2(x_i-x_j)^2]$$

- The saddle point equation

$$\frac{dS_{eff}}{dx_k} = 0 \quad \longrightarrow \quad x_i = \sum_{i\neq j}\frac{1}{(x_i-x_j)[1+g^2(x_i-x_j)^2]}$$

The 2-matrix model

- Taking the continuum limit $\sum \to \int \rho(x)dx$ introducing $\rho(x)$ as the eigenvalue density with the normalisation condition $\int \rho(x)dx = N$ the saddle point equation in the large N limit becomes

$$x = \fint \frac{\rho(y)dy}{(x-y)[1+g^2(x-y)^2]}.$$

The 2-matrix model

- Taking the continuum limit $\sum \to \int \rho(x)dx$ introducing $\rho(x)$ as the eigenvalue density with the normalisation condition $\int \rho(x)dx = N$ the saddle point equation in the large N limit becomes

$$x = \int \frac{\rho(y)dy}{(x-y)[1+g^2(x-y)^2]}.$$

- The equation can be solved for $g^2N >> 1$, the solution for $\rho(x)$ is given by

$$\rho(x) = \frac{3N}{r^3}(r^2 - x^2).$$

The 2-matrix model

- Taking the continuum limit $\sum \to \int \rho(x)dx$ introducing $\rho(x)$ as the eigenvalue density with the normalisation condition $\int \rho(x)dx = N$ the saddle point equation in the large N limit becomes

$$x = \fint \frac{\rho(y)dy}{(x-y)[1+g^2(x-y)^2]}.$$

- The equation can be solved for $g^2 N >> 1$, the solution for $\rho(x)$ is given by

$$\rho(x) = \frac{3N}{r^3}(r^2 - x^2).$$

- Model is effectively described by $2N$ dof; the off-diagonal terms become heavy.

The 2-matrix model

- Taking the continuum limit $\sum \rightarrow \int \rho(x)dx$ introducing $\rho(x)$ as the eigenvalue density with the normalisation condition $\int \rho(x)dx = N$ the saddle point equation in the large N limit becomes

$$x = \fint \frac{\rho(y)dy}{(x-y)[1+g^2(x-y)^2]}.$$

- The equation can be solved for $g^2N >> 1$, the solution for $\rho(x)$ is given by

$$\rho(x) = \frac{3N}{r^3}(r^2 - x^2).$$

- Model is effectively described by $2N$ dof; the off-diagonal terms become heavy.
- Adding more matrices is possible, however the model is much more difficult to solve for the fully interacting case even in the $g^2N >> 1$ limit.

The 2-matrix model

- Taking the continuum limit $\sum \to \int \rho(x)dx$ introducing $\rho(x)$ as the eigenvalue density with the normalisation condition $\int \rho(x)dx = N$ the saddle point equation in the large N limit becomes

$$x = \fint \frac{\rho(y)dy}{(x-y)[1+g^2(x-y)^2]}.$$

- The equation can be solved for $g^2N >> 1$, the solution for $\rho(x)$ is given by

$$\rho(x) = \frac{3N}{r^3}(r^2 - x^2).$$

- Model is effectively described by $2N$ dof; the off-diagonal terms become heavy.
- Adding more matrices is possible, however the model is much more difficult to solve for the fully interacting case even in the $g^2N >> 1$ limit.
- Strategy: Use Monte Carlo simulations

The action

$$S[D] = \frac{\tilde{\alpha}^4}{N} Tr \left[-\frac{1}{4}[D_a, D_b]^2 + \frac{i}{3}\epsilon_{abc}[D_a D_b]D_c - \mu D_a^2 \right],$$

The parameters of the model are $\tilde{\alpha}$ and μ. ($\tilde{\alpha}^4 = \frac{1}{g^2} = \beta = \frac{1}{T}$).

The action

$$S[D] = \frac{\tilde{\alpha}^4}{N} Tr \left[-\frac{1}{4}[D_a, D_b]^2 + \frac{i}{3}\epsilon_{abc}[D_a D_b]D_c - \mu D_a^2 \right],$$

The parameters of the model are $\tilde{\alpha}$ and μ. ($\tilde{\alpha}^4 = \frac{1}{g^2} = \beta = \frac{1}{T}$).

Symmetry

The model is invariant under $U(N)$ unitary transformations $D_a \to U D_a U^\dagger$ and global $SO(3)$ rotations. For $\mu = 0$ it is also translational invariant $D_a \to D_a + c_a \mathbb{1}_N$, that can be fixed by imposing traceless condition on D_a.

The action

$$S[D] = \frac{\tilde{\alpha}^4}{N} Tr\left[-\frac{1}{4}[D_a, D_b]^2 + \frac{i}{3}\epsilon_{abc}[D_a D_b]D_c - \mu D_a^2 \right],$$

The parameters of the model are $\tilde{\alpha}$ and μ. ($\tilde{\alpha}^4 = \frac{1}{g^2} = \beta = \frac{1}{T}$).

Symmetry

The model is invariant under $U(N)$ unitary transformations $D_a \to U D_a U^\dagger$ and global $SO(3)$ rotations. For $\mu = 0$ it is also translational invariant $D_a \to D_a + c_a \mathbb{1}_N$, that can be fixed by imposing traceless condition on D_a.

Note that there is no reference whatsoever to a background geometry. The model describes a bunch of matrices that interact in a particular way.

The action

$$S[D] = \frac{\tilde{\alpha}^4}{N} Tr \left[-\frac{1}{4}[D_a, D_b]^2 + \frac{i}{3}\epsilon_{abc}[D_a D_b]D_c - \mu D_a^2 \right],$$

The parameters of the model are $\tilde{\alpha}$ and μ. ($\tilde{\alpha}^4 = \frac{1}{g^2} = \beta = \frac{1}{T}$).

Symmetry

The model is invariant under $U(N)$ unitary transformations $D_a \to U D_a U^\dagger$ and global $SO(3)$ rotations. For $\mu = 0$ it is also translational invariant $D_a \to D_a + c_a \mathbb{1}_N$, that can be fixed by imposing traceless condition on D_a.

Note that there is no reference whatsoever to a background geometry. The model describes a bunch of matrices that interact in a particular way.

Equations of Motion, $\delta S = 0$

$$[D_b, iF_{ab}] - 2\mu D_a = 0, \qquad \text{with} \quad F_{ab} = i[D_a, D_b] + \epsilon_{abc} D_c$$

Equations of Motion, $\delta S = 0$

$$[D_b, iF_{ab}] - 2\mu D_a = 0, \qquad \text{with} \quad F_{ab} = i[D_a, D_b] + \epsilon_{abc} D_c$$

Solutions for matrix configurations

The extrema of the model are clearly given by the trivial solution $D_a = 0$ and representations of $SU(2)$, commuting matrices are also solutions.

Equations of Motion, $\delta S = 0$

$$[D_b, iF_{ab}] - 2\mu D_a = 0, \qquad \text{with} \quad F_{ab} = i[D_a, D_b] + \epsilon_{abc} D_c$$

Solutions for matrix configurations

The extrema of the model are clearly given by the trivial solution $D_a = 0$ and representations of $SU(2)$, commuting matrices are also solutions.

Ground State

The ground state of the model is given by $D_a = L_a$; irreducible representations of $SU(2)$:

$$[L_a, L_b] = \epsilon_{abc} L_c, \quad \text{and} \quad \sum_{i=1}^{3} L_a^2 = \frac{N^2 - 1}{4} \mathbb{1}_N,$$

for which $S(D_a = L_a) = -\tilde{\alpha}^4 \frac{N^2-1}{4} \left(\frac{1}{6} + \mu\right)$.

Equations of Motion, $\delta S = 0$

$$[D_b, iF_{ab}] - 2\mu D_a = 0, \qquad \text{with} \quad F_{ab} = i[D_a, D_b] + \epsilon_{abc} D_c$$

Solutions for matrix configurations

The extrema of the model are clearly given by the trivial solution $D_a = 0$ and representations of $SU(2)$, commuting matrices are also solutions.

Ground State

The ground state of the model is given by $D_a = L_a$; irreducible representations of $SU(2)$:

$$[L_a, L_b] = \epsilon_{abc} L_c, \quad \text{and} \quad \sum_{i=1}^{3} L_a^2 = \frac{N^2 - 1}{4} \mathbb{1}_N,$$

for which $S(D_a = L_a) = -\tilde{\alpha}^4 \frac{N^2-1}{4} \left(\frac{1}{6} + \mu\right)$.

Fuzzy sphere

Fuzzy sphere

- Define a matrix $X_a = \frac{2L_a}{\sqrt{N^2-1}}$, where L_a are irreducible representations of $SU(2)$. Then

$$[X_a, X_b] = \frac{2i\epsilon_{abc}}{\sqrt{N^2-1}} X_c, \quad \text{and} \quad \sum_{i=1}^{3} X_a^2 = \frac{N^2-1}{4} \mathbb{1}_N,$$

relations which define the fuzzy sphere; a finite matrix approximation for the commutative sphere. The commutative limit is when $N \to \infty$

Stability of the fuzzy sphere

We are interested in tha stability of the fuzzy sphere matrix solution. For this reason we consider the classical potential parametrising $D_a = \phi L_a$.

Fuzzy sphere

Fuzzy sphere

- Define a matrix $X_a = \frac{2L_a}{\sqrt{N^2-1}}$, where L_a are irreducible representations of $SU(2)$. Then

$$[X_a, X_b] = \frac{2i\epsilon_{abc}}{\sqrt{N^2-1}} X_c, \quad \text{and} \quad \sum_{i=1}^{3} X_a^2 = \frac{N^2-1}{4} \mathbb{1}_N,$$

relations which define the fuzzy sphere; a finite matrix approximation for the commutative sphere. The commutative limit is when $N \to \infty$

Stability of the fuzzy sphere

We are interested in tha stability of the fuzzy sphere matrix solution. For this reason we consider the classical potential parametrising $D_a = \phi L_a$.

Classical Potential

- The classical potential given in terms of the configuration $D_a = \phi L_a$

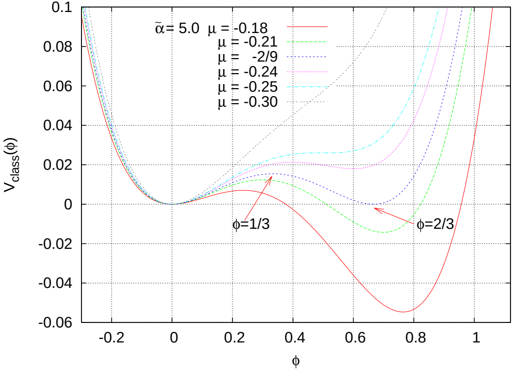
$$\frac{V_{class}}{2c_2} = \tilde{\alpha}\left(\frac{1}{4}\phi^4 - \frac{1}{3}\phi^3 - \frac{\mu}{2}\phi^2\right)$$

with $c_2 \equiv L_a^2 = \mathcal{S}(\mathcal{S}+1) = (N^2-1)/4$ is the quadratic Casimir for the $\mathcal{S}$-irrep of $SU(2)$ ($\mathcal{S} = \frac{N-1}{2}$, the spin)

- The condition $\frac{dV_{class}}{d\phi} = 0$ gives the extrema for the potential
-

$$\phi = \left\{0, \frac{1+\sqrt{1+4\mu}}{2}, \frac{1-\sqrt{1+4\mu}}{2}\right\}$$

- For fix $\tilde{\alpha}$ we can plot the potential for different values of μ. For $\mu = -2/9$ $\phi = (0, 2/3, 1/3)$



Quantum corrections

- Taking into account quantum fluctuations

Quantum corrections

- Taking into account quantum fluctuations
- The one-loop quantum effective potential in the large N limit is given by

$$\frac{V_{\text{eff}}}{2c_2} = \tilde{\alpha}^4 \left[\frac{\phi^4}{4} - \frac{\phi^3}{3} - \mu \frac{\phi^2}{2} \right] + \log \phi^2$$

Quantum corrections

- Taking into account quantum fluctuations
- The one-loop quantum effective potential in the large N limit is given by

$$\frac{V_{\text{eff}}}{2c_2} = \tilde{\alpha}^4 \left[\frac{\phi^4}{4} - \frac{\phi^3}{3} - \mu \frac{\phi^2}{2} \right] + \log \phi^2$$

- The configurations are given by $D_a = \phi L_a$ where ϕ is solution of the equation

$$\phi^4 - \phi^3 - \mu\phi^2 + 2\tilde{\alpha}^{-4} = 0$$

Quantum corrections

- Taking into account quantum fluctuations
- The one-loop quantum effective potential in the large N limit is given by

$$\frac{V_{\mathrm{eff}}}{2c_2} = \tilde{\alpha}^4 \left[\frac{\phi^4}{4} - \frac{\phi^3}{3} - \mu \frac{\phi^2}{2} \right] + \log \phi^2$$

- The configurations are given by $D_a = \phi L_a$ where ϕ is solution of the equation

$$\phi^4 - \phi^3 - \mu\phi^2 + 2\tilde{\alpha}^{-4} = 0$$

- The effective potential for μ fixed

Quantum corrections

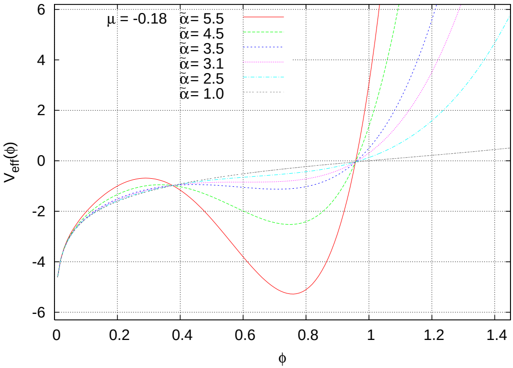
- Taking into account quantum fluctuations
- The one-loop quantum effective potential in the large N limit is given by

$$\frac{V_{\text{eff}}}{2c_2} = \tilde{\alpha}^4 \left[\frac{\phi^4}{4} - \frac{\phi^3}{3} - \mu \frac{\phi^2}{2} \right] + \log \phi^2$$

- The configurations are given by $D_a = \phi L_a$ where ϕ is solution of the equation

$$\phi^4 - \phi^3 - \mu\phi^2 + 2\tilde{\alpha}^{-4} = 0$$

- The effective potential for μ fixed



Quantum corrections

- Taking into account quantum fluctuations
- The one-loop quantum effective potential in the large N limit is given by

$$\frac{V_{\text{eff}}}{2c_2} = \tilde{\alpha}^4 \left[\frac{\phi^4}{4} - \frac{\phi^3}{3} - \mu \frac{\phi^2}{2} \right] + \log \phi^2$$

- The configurations are given by $D_a = \phi L_a$ where ϕ is solution of the equation

$$\phi^4 - \phi^3 - \mu\phi^2 + 2\tilde{\alpha}^{-4} = 0$$

- The effective potential for μ fixed
- The effective potential for $\tilde{\alpha}$ fixed

Quantum corrections

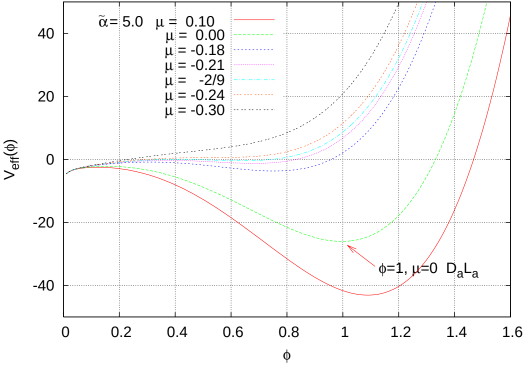
- Taking into account quantum fluctuations
- The one-loop quantum effective potential in the large N limit is given by

$$\frac{V_{\text{eff}}}{2c_2} = \tilde{\alpha}^4 \left[\frac{\phi^4}{4} - \frac{\phi^3}{3} - \mu\frac{\phi^2}{2}\right] + \log\phi^2$$

- The configurations are given by $D_a = \phi L_a$ where ϕ is solution of the equation

$$\phi^4 - \phi^3 - \mu\phi^2 + 2\tilde{\alpha}^{-4} = 0$$

- The effective potential for μ fixed
- The effective potential for $\tilde{\alpha}$ fixed



Quantum corrections

- Taking into account quantum fluctuations
- The one-loop quantum effective potential in the large N limit is given by

$$\frac{V_{\text{eff}}}{2c_2} = \tilde{\alpha}^4 \left[\frac{\phi^4}{4} - \frac{\phi^3}{3} - \mu \frac{\phi^2}{2} \right] + \log \phi^2$$

- The configurations are given by $D_a = \phi L_a$ where ϕ is solution of the equation

$$\phi^4 - \phi^3 - \mu\phi^2 + 2\tilde{\alpha}^{-4} = 0$$

- The effective potential for μ fixed
- The effective potential for $\tilde{\alpha}$ fixed
- A first order phase transition occurs.

Quantum corrections

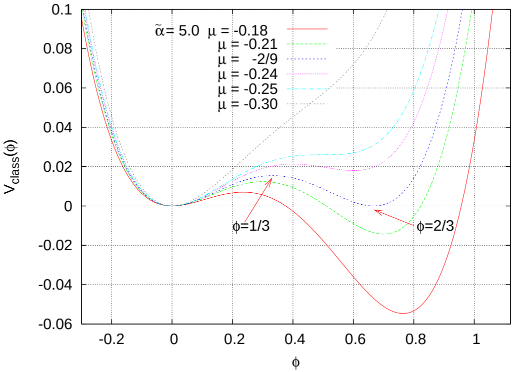
- Taking into account quantum fluctuations
- The one-loop quantum effective potential in the large N limit is given by

$$\frac{V_{\text{eff}}}{2c_2} = \tilde{\alpha}^4 \left[\frac{\phi^4}{4} - \frac{\phi^3}{3} - \mu\frac{\phi^2}{2} \right] + \log \phi^2$$

- The configurations are given by $D_a = \phi L_a$ where ϕ is solution of the equation

$$\phi^4 - \phi^3 - \mu\phi^2 + 2\tilde{\alpha}^{-4} = 0$$

- The effective potential for μ fixed
- The effective potential for $\tilde{\alpha}$ fixed
- A first order phase transition occurs.



Phase Diagram

- The conditions $V''_{eff} = 0$

Phase Diagram

- The conditions $V''_{eff} = 0$
- and $V'_{eff} = 0$ give the critical values for ϕ and $\tilde{\alpha}$ given by

Phase Diagram

- The conditions $V''_{eff} = 0$
- and $V'_{eff} = 0$ give the critical values for ϕ and $\tilde{\alpha}$ given by
- $$\phi_* = \frac{3}{8}\left(1 + \sqrt{1 + \frac{32\mu}{9}}\right)$$
  $$\frac{1}{\tilde{\alpha}_*^4} = \frac{\phi_*^2(\phi_* + 2\mu)}{8}$$

Phase Diagram

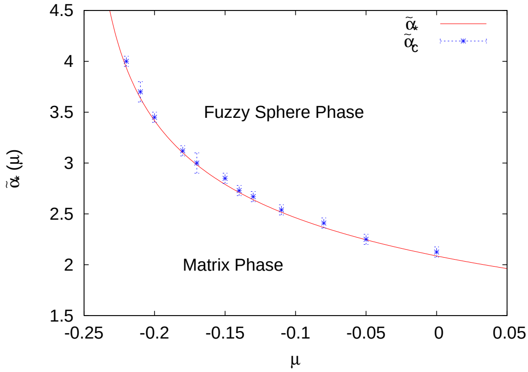
- The conditions $V''_{eff} = 0$
- and $V'_{eff} = 0$ give the critical values for ϕ and $\tilde{\alpha}$ given by
-
$$\phi_* = \frac{3}{8}\left(1 + \sqrt{1 + \frac{32\mu}{9}}\right)$$
$$\frac{1}{\tilde{\alpha}_*^4} = \frac{\phi_*^2(\phi_* + 2\mu)}{8}$$

Phase Diagram

- The conditions $V''_{eff} = 0$
- and $V'_{eff} = 0$ give the critical values for ϕ and $\tilde{\alpha}$ given by
- $$\phi_* = \frac{3}{8}\left(1 + \sqrt{1 + \frac{32\mu}{9}}\right)$$
  $$\frac{1}{\tilde{\alpha}_*^4} = \frac{\phi_*^2(\phi_* + 2\mu)}{8}$$
- For $\mu = -1/4$, $\tilde{\alpha}_*$ is sent to infinity and there is no fuzzy sphere for $\mu < -1/4$.

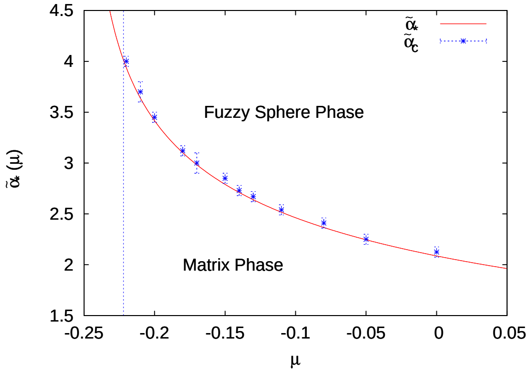
Phase Diagram

- The conditions $V''_{eff} = 0$
- and $V'_{eff} = 0$ give the critical values for ϕ and $\tilde{\alpha}$ given by
- $$\phi_* = \frac{3}{8}\left(1 + \sqrt{1 + \frac{32\mu}{9}}\right)$$
$$\frac{1}{\tilde{\alpha}_*^4} = \frac{\phi_*^2(\phi_* + 2\mu)}{8}$$
- For $\mu = -1/4$, $\tilde{\alpha}_*$ is sent to infinity and there is no fuzzy sphere for $\mu < -1/4$.



Phase Diagram

- The conditions $V''_{eff} = 0$
- and $V'_{eff} = 0$ give the critical values for ϕ and $\tilde{\alpha}$ given by
- $$\phi_* = \frac{3}{8}\left(1 + \sqrt{1 + \frac{32\mu}{9}}\right)$$
  $$\frac{1}{\tilde{\alpha}_*^4} = \frac{\phi_*^2(\phi_* + 2\mu)}{8}$$
- For $\mu = -1/4$, $\tilde{\alpha}_*$ is sent to infinity and there is no fuzzy sphere for $\mu < -1/4$.



Geometrical Observables

Geometrical observables

- The matrices X_a and $i[X_a, X_b]$. X_a are $N \times N$ Hermitian matrices.
- The matrix $C = \sigma_a X_a$, where σ_a are the Pauli matrices. C is a $2N \times 2N$ Hermitian matrix.
- The Dirac operator $\mathcal{D} = \sigma_a [X_a, \cdot]$, acting on a $2N^2$ Hilbert space.
- We will be interested in the eigenvalue distribution for all these matrices. Which is precisely the information which encodes the geometry.

Specific backgrounds

Fuzzy sphere

- The spectrum for those matrices are known for specific backgrounds. From the equation of motion we see that the ground state is given by irrep $SU(2)$, $X_a = \frac{2L_a}{\sqrt{N^2-1}}$. Then

$$[X_a, X_b] = \frac{2i\epsilon_{abc}}{\sqrt{N^2-1}} X_c, \quad \text{and} \quad \sum_{i=1}^{3} X_a^2 = \frac{N^2-1}{4} \mathbb{1}_N,$$

  relations which define the fuzzy sphere; a finite matrix approximation for the commutative sphere. The commutative limit is when $N \to \infty$
- In this case we have $C = \left(\pm\frac{N}{2} - \frac{1}{2}\right) \mathbb{1}_N$
- and $spec\,\{\mathcal{D}\} = \{-N, -(N-1), \ldots, -3, -2, 0, 1, 2, 3, \ldots, (N-1)\}$

Eigenvalue distributions

- Consider the model

$$S[D] = \frac{\tilde{\alpha}^4}{N} Tr \left[-\frac{1}{4}[D_a, D_b]^2 + \frac{i}{3}\epsilon_{abc}[D_a D_b]D_c - \mu D_a^2 \right] ,$$

- The critical line $\tilde{\alpha}(\mu)_*$ defines two phases; the low temperature phase where the underlying geometry is a fuzzy sphere and a the high temperature phase or matrix phase.

From random matrices to geometry

We emphasize that we start from a random matrix configuration and then the system is brought into the equilibrium and is when we measure the distributions.

Keep in mind the series of approximations have been made to derive the effective potential and its consecuences; one-loop calculation, N large limit

Eigenvalue distributions: Fuzzy sphere phase

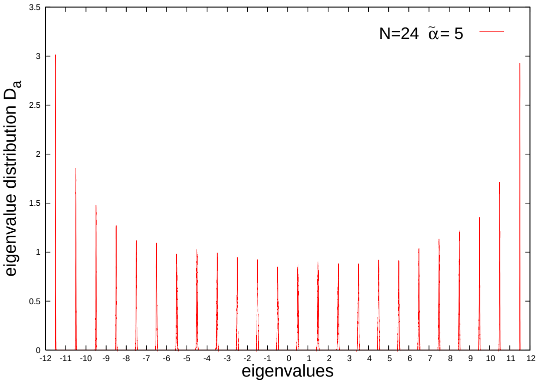
- Eigenvalue distributions in the fuzzy sphere $\tilde{\alpha} > \tilde{\alpha}_*(\mu)$

Eigenvalue distributions: Fuzzy sphere phase

- Eigenvalue distributions in the fuzzy sphere $\tilde{\alpha} > \tilde{\alpha}_*(\mu)$
- D_a $(\sim L_a)$ $(s = \frac{N-1}{2})$

Eigenvalue distributions: Fuzzy sphere phase

- Eigenvalue distributions in the fuzzy sphere $\tilde{\alpha} > \tilde{\alpha}_*(\mu)$
- D_a $(\sim L_a)$ $(s = \frac{N-1}{2})$

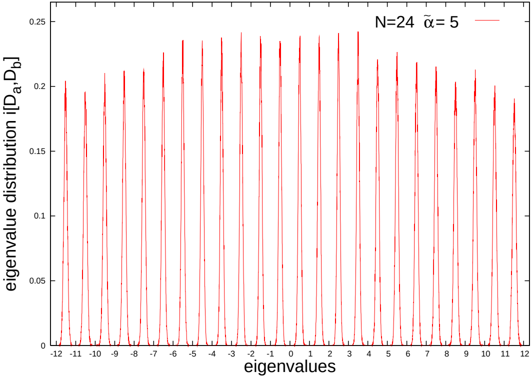


Eigenvalue distributions: Fuzzy sphere phase

- Eigenvalue distributions in the fuzzy sphere $\tilde{\alpha} > \tilde{\alpha}_*(\mu)$
- D_a $(\sim L_a)$ $(s = \frac{N-1}{2})$
- $i[D_a, D_b]$ $(\sim L_a)$

Eigenvalue distributions: Fuzzy sphere phase

- Eigenvalue distributions in the fuzzy sphere $\tilde{\alpha} > \tilde{\alpha}_*(\mu)$
- D_a $(\sim L_a)$ $(s = \frac{N-1}{2})$
- $i[D_a, D_b]$ $(\sim L_a)$

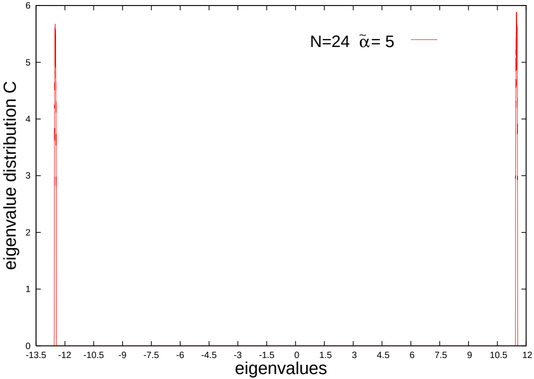


Eigenvalue distributions: Fuzzy sphere phase

- Eigenvalue distributions in the fuzzy sphere $\tilde{\alpha} > \tilde{\alpha}_*(\mu)$
- D_a ($\sim L_a$) ($s = \frac{N-1}{2}$)
- $i[D_a, D_b]$ ($\sim L_a$)
- $C = \sigma_a D_a$ ($\left(\pm\frac{N}{2} - \frac{1}{2}\right) \mathbf{1}_N$)

Eigenvalue distributions: Fuzzy sphere phase

- Eigenvalue distributions in the fuzzy sphere $\tilde{\alpha} > \tilde{\alpha}_*(\mu)$
- D_a $(\sim L_a)$ $(s = \frac{N-1}{2})$
- $i[D_a, D_b]$ $(\sim L_a)$
- $C = \sigma_a D_a$ $\left(\left(\pm\frac{N}{2} - \frac{1}{2}\right) \mathbf{1}_N\right)$

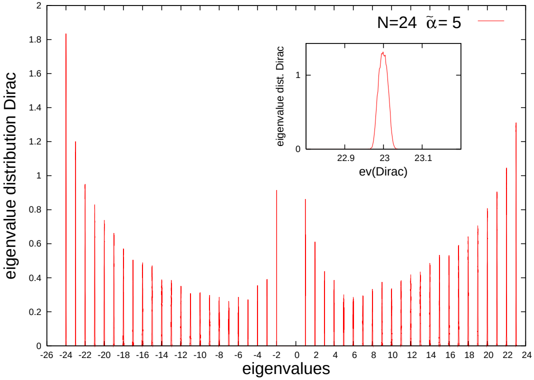


Eigenvalue distributions: Fuzzy sphere phase

- Eigenvalue distributions in the fuzzy sphere $\tilde{\alpha} > \tilde{\alpha}_*(\mu)$
- D_a ($\sim L_a$) ($s = \frac{N-1}{2}$)
- $i[D_a, D_b]$ ($\sim L_a$)
- $C = \sigma_a D_a$ $(\left(\pm\frac{N}{2} - \frac{1}{2}\right) \mathbf{1}_N)$
- $\mathcal{D} = \sigma_a[D_a, \cdot]$ $(-N, -(N-1), \ldots, -3, -2, 0, 1, 2, 3, \ldots, (N-1))$

Eigenvalue distributions: Fuzzy sphere phase

- Eigenvalue distributions in the fuzzy sphere $\tilde{\alpha} > \tilde{\alpha}_*(\mu)$
- D_a $(\sim L_a)$ $(s = \frac{N-1}{2})$
- $i[D_a, D_b]$ $(\sim L_a)$
- $C = \sigma_a D_a$ $\left(\left(\pm\frac{N}{2} - \frac{1}{2}\right) \mathbf{1}_N\right)$
- $\mathcal{D} = \sigma_a [D_a, \cdot]$ $(-N, -(N-1), \ldots, -3, -2, 0, 1, 2, 3, \ldots, (N-1))$



Eigenvalue distributions: Pure Yang-Mills model

- Lets now use the matrix $X_a = \alpha D_a$, with $\alpha = \frac{\tilde{\alpha}}{\sqrt{N}}$.

Eigenvalue distributions: Pure Yang-Mills model

- Lets now use the matrix $X_a = \alpha D_a$, with $\alpha = \frac{\tilde{\alpha}}{\sqrt{N}}$. The model now looks like

$$S[X] = N \text{ Tr } \left[-\frac{1}{4}[X_a, X_b]^2 + \frac{i}{3}\alpha\epsilon_{abc}[X_a X_b]X_c - \mu\alpha^2 X_a^2 \right]$$

Eigenvalue distributions: Pure Yang-Mills model

- Lets now use the matrix $X_a = \alpha D_a$, with $\alpha = \frac{\tilde{\alpha}}{\sqrt{N}}$. The model now looks like

$$S[X] = N\ \mathrm{Tr}\ \left[-\frac{1}{4}[X_a, X_b]^2 + \frac{i}{3}\alpha\epsilon_{abc}[X_a X_b]X_c - \mu\alpha^2 X_a^2\right]$$

- Thus we can now set $\alpha = 0$, the action is a pure Yang-Mills model

$$S[X] = -\frac{N}{4}\ \mathrm{Tr}\ [X_a, X_b]^2.$$

Eigenvalue distributions: Pure Yang-Mills model

- Lets now use the matrix $X_a = \alpha D_a$, with $\alpha = \frac{\tilde{\alpha}}{\sqrt{N}}$. The model now looks like

$$S[X] = N\ \mathrm{Tr}\ \left[-\frac{1}{4}[X_a, X_b]^2 + \frac{i}{3}\alpha\epsilon_{abc}[X_a X_b]X_c - \mu\alpha^2 X_a^2 \right]$$

- Thus we can now set $\alpha = 0$, the action is a pure Yang-Mills model

$$S[X] = -\frac{N}{4}\ \mathrm{Tr}\ [X_a, X_b]^2.$$

- For larger number of matrices this model is the bosonic part of the IKKT matrix model, which is well defined for $N > \frac{d}{d-1}$ (Hotta '98), where d is the number of matrices.

Eigenvalue distributions: Pure Yang-Mills model

- Lets now use the matrix $X_a = \alpha D_a$, with $\alpha = \frac{\tilde{\alpha}}{\sqrt{N}}$. The model now looks like

$$S[X] = N \, \mathrm{Tr} \left[-\frac{1}{4}[X_a, X_b]^2 + \frac{i}{3}\alpha\epsilon_{abc}[X_a X_b]X_c - \mu\alpha^2 X_a^2 \right]$$

- Thus we can now set $\alpha = 0$, the action is a pure Yang-Mills model

$$S[X] = -\frac{N}{4} \, \mathrm{Tr} \, [X_a, X_b]^2.$$

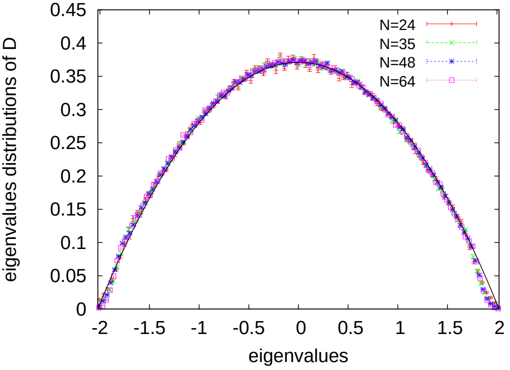
- For larger number of matrices this model is the bosonic part of the IKKT matrix model, which is well defined for $N > \frac{d}{d-1}$ (Hotta '98), where d is the number of matrices.
- It is a Yang-Mills model defined in zero volume.

Eigenvalue distributions: Pure Yang-Mills model

- X_a

Eigenvalue distributions: Pure Yang-Mills model

- X_a



Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

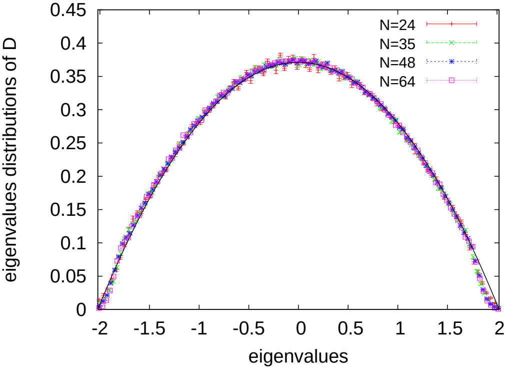
r is found to be $R = 2.0208 \pm 0.015$

Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

r is found to be $R = 2.0208 \pm 0.015$



Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

  r is found to be $R = 2.0208 \pm 0.015$
- The eigenvalues distribute uniformly inside a solid 3-ball with radius r.

Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

  r is found to be $R = 2.0208 \pm 0.015$
- The eigenvalues distribute uniformly inside a solid 3-ball with radius r.
- $i[X_a, X_b]$

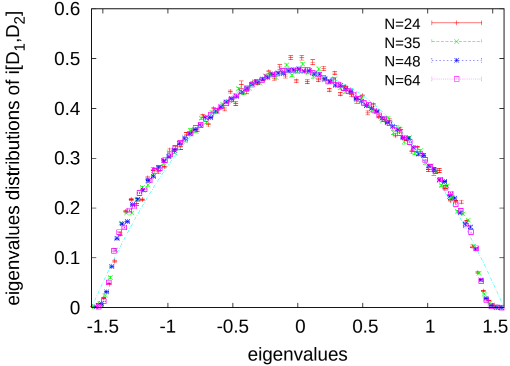
Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

r is found to be $R = 2.0208 \pm 0.015$

- The eigenvalues distribute uniformly inside a solid 3-ball with radius r.
- $i[X_a, X_b]$



Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

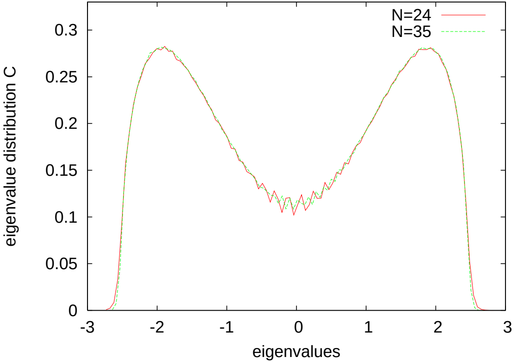
  r is found to be $R = 2.0208 \pm 0.015$
- The eigenvalues distribute uniformly inside a solid 3-ball with radius r.
- $i[X_a, X_b]$
- $C = \sigma_a X_a$

Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

  r is found to be $R = 2.0208 \pm 0.015$
- The eigenvalues distribute uniformly inside a solid 3-ball with radius r.
- $i[X_a, X_b]$
- $C = \sigma_a X_a$



Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

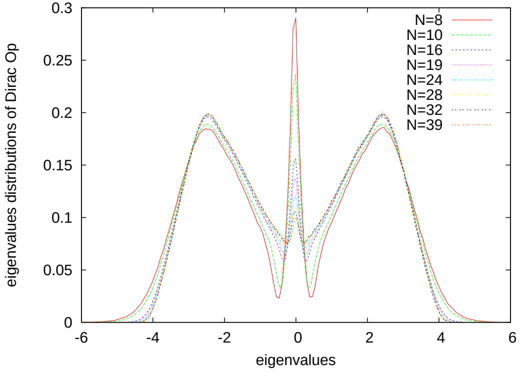
  r is found to be $R = 2.0208 \pm 0.015$
- The eigenvalues distribute uniformly inside a solid 3-ball with radius r.
- $i[X_a, X_b]$
- $C = \sigma_a X_a$
- $\mathcal{D} = \sigma_a [X_a, \cdot]$

Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

  r is found to be $R = 2.0208 \pm 0.015$
- The eigenvalues distribute uniformly inside a solid 3-ball with radius r.
- $i[X_a, X_b]$
- $C = \sigma_a X_a$
- $\mathcal{D} = \sigma_a [X_a, \cdot]$



Eigenvalue distributions: Pure Yang-Mills model

- X_a
- The fit corresponds to the parabola

$$\rho(x) = \frac{3}{4r^3}(r^2 - x^2)$$

  r is found to be $R = 2.0208 \pm 0.015$
- The eigenvalues distribute uniformly inside a solid 3-ball with radius r.
- $i[X_a, X_b]$
- $C = \sigma_a X_a$
- $\mathcal{D} = \sigma_a [X_a, \cdot]$
- We observe continuous and symmetric distributions.

Eigenvalue distributions:Mass deformation

- Consider the model

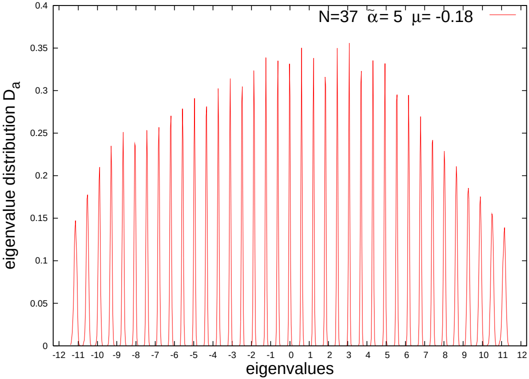
$$S[D] = \frac{\tilde{\alpha}^4}{N} Tr \left[-\frac{1}{4}[D_a, D_b]^2 + \frac{i}{3}\epsilon_{abc}[D_a D_b]D_c - \mu D_a^2 \right],$$

Eigenvalue distributions: Matrix phase

- Lets see the effect of the parameter μ on X_a

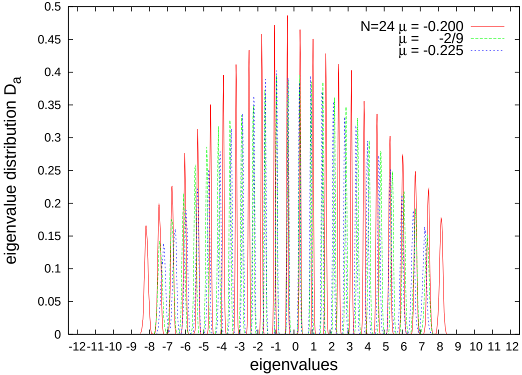
Eigenvalue distributions: Matrix phase

- Lets see the effect of the parameter μ on X_a



Eigenvalue distributions: Matrix phase

- Lets see the effect of the parameter μ on X_a



Eigenvalue distributions: Matrix phase

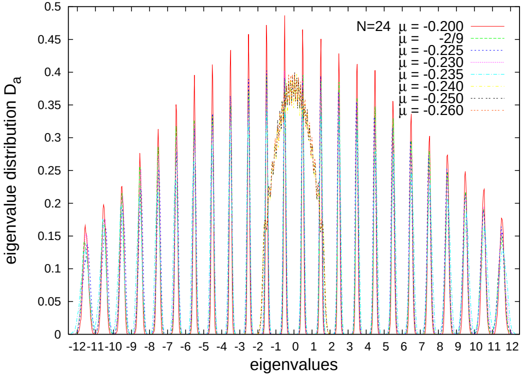
- Lets see the effect of the parameter μ on X_a
- The effect of μ in the fuzzy sphere is to squeeze the configuration

Eigenvalue distributions: Matrix phase

- Lets see the effect of the parameter μ on X_a
- The effect of μ in the fuzzy sphere is to squeeze the configuration
- Eventually the μ_* is reached and the fuzzy sphere dissapears and a commutative sphere with fixed radius pops up!

Eigenvalue distributions: Matrix phase

- Lets see the effect of the parameter μ on X_a
- The effect of μ in the fuzzy sphere is to squeeze the configuration
- Eventually the μ_* is reached and the fuzzy sphere dissapears and a commutative sphere with fixed radius pops up!

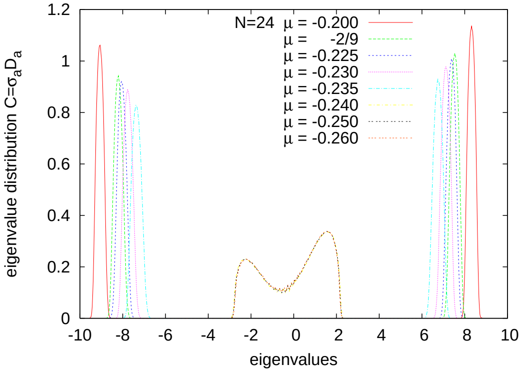


Eigenvalue distributions: Matrix phase

- Lets see the effect of the parameter μ on X_a
- The effect of μ in the fuzzy sphere is to squeeze the configuration
- Eventually the μ_* is reached and the fuzzy sphere dissapears and a commutative sphere with fixed radius pops up!
- $C = \sigma_a X_a$

Eigenvalue distributions: Matrix phase

- Lets see the effect of the parameter μ on X_a
- The effect of μ in the fuzzy sphere is to squeeze the configuration
- Eventually the μ_* is reached and the fuzzy sphere dissapears and a commutative sphere with fixed radius pops up!
- $C = \sigma_a X_a$



Eigenvalue distributions: Matrix phase

- Lets see the effect of the parameter μ on X_a
- The effect of μ in the fuzzy sphere is to squeeze the configuration
- Eventually the μ_* is reached and the fuzzy sphere dissapears and a commutative sphere with fixed radius pops up!
- $C = \sigma_a X_a$
- The transition is that of a non-commutative sphere to a commutative sphere of radius 2.

Conclusions

- We have shown a simple example of how geometry emerges dynamically from matrix models.
- Starting from a matrix model with no background geometry, random degrees of freedom condensate under the effect of quantum fluctuations giving origin to geometry.
- The geometry in matrix models is then understood as an emergent concept.
- In this concrete example, in the matrix phase or high temperature phase the matrices are effectively commutative and dominated by diagonal elements which form a solid ball.
- The low temperature phase has an non-commutative sphere as underlying geometry, the fuzzy sphere.
- The transition observed is that of a non-commutative sphere to a commutative sphere of radius 2.
- It is believed that such transitions belong to a new universality class.
- The excellent agreement of theoretical predictions and simulations suggest the existence of an exact solution for this type of matrix models.