Universal coefficient theorems for C^*-algebras over finite topological spaces

Rasmus Bentmann
(joint work with Manuel Köhler)

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Outline

Classification of simple nuclear purely infinite C^*-algebras

Theorem (Rosenberg-Schochet 1987)

Let A and B be separable C^*-algebras.
If A belongs to the bootstrap class, then there is a short exact sequence of $\mathbb{Z}/2$-graded Abelian groups

$$\mathrm{Ext}^1(\mathrm{K}_{*+1}(A), \mathrm{K}_*(B)) \rightarrowtail \mathrm{KK}_*(A, B) \twoheadrightarrow \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)).$$

Theorem (Kirchberg-Phillips 2000)

A KK-equivalence between two stable, nuclear, separable, purely infinite, simple C^*-algebras lifts to a *-isomorphism.

C^*-algebras over topological spaces

Throughout this talk, X denotes a finite T_0-space.

Definition

A C^*-algebra over X is a pair (A, ψ) consisting of a C^*-algebra A and a continuous map $\psi\colon \operatorname{Prim}(A) \to X$.

- An open subset $U \subseteq X$ gives a distinguished ideal $A(U)$ of A.
- A *-homomorphism $f\colon A \to B$ is X-equivariant if $f(A(U)) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$.
- A subset $Y \subseteq X$ is locally closed if and only if $Y = U \setminus V$ for $V, U \in \mathbb{O}(X)$ with $V \subseteq U$.
  Define distinguished subquotient $A(Y) := A(U)/A(V)$.

Bivariant equivariant K-theory

Kirchberg constructed an X-equivariant version $\mathrm{KK}(X)$ of Kasparov's KK-theory.
The cycles fulfill an appropriate equivariance condition.

Kasparov product

$$\mathrm{KK}_*(X; A, B) \otimes \mathrm{KK}_*(X; B, C) \to \mathrm{KK}_*(X; A, C)$$

Definition

Let $\mathfrak{KK}(X)$ be the category of separable C^*-algebras over X with $\mathrm{KK}_0(X; A, B)$ as morphisms, Kasparov product as composition.

Properties of the category $\mathfrak{KK}(X)$

The category $\mathfrak{KK}(X)$ is

- additive,
- triangulated (by the suspension functor and the class of triangles isomorphic to a mapping cone triangle);

it has

- Bott periodicity,
- six-term exact sequences for semi-split extensions,
- countable coproducts.

Moreover,

- $\mathfrak{KK}(X)$ is functorial in the space variable,
- has exterior products $\otimes\colon \mathfrak{KK}(X) \times \mathfrak{KK}(Y) \to \mathfrak{KK}(X \times Y)$.

Universal property and classification

Higson's Description in the equivariant case (Bonkat)

The canonical functor $\mathfrak{C}^*\mathfrak{sep}(X) \to \mathfrak{KK}(X)$ is the universal split-exact C^*-stable (homotopy invariant) functor.

Theorem (Kirchberg 2000)

A $KK(X)$-equivalence between two stable, nuclear, separable, purely infinite, tight C^*-algebras over X lifts to an X-equivariant *-isomorphism.

Here a C^*-algebra (A, ψ) over X is called tight if $\psi\colon \operatorname{Prim}(A) \to X$ is homeomorphic.

The equivariant bootstrap class

Definition (Meyer-Nest)

The bootstrap class $\mathcal{B}(X) \subseteq \mathfrak{KK}(X)$ is the localising subcategory of $\mathfrak{KK}(X)$ generated by the objects $i_x\mathbb{C}$ for all $x \in X$.
It is the smallest class of objects containing $i_x\mathbb{C}$ closed under

- suspensions,
- KK(X)-equivalence,
- mapping cones,
- countable direct sums.

Proposition (Meyer-Nest)

If $A \in \mathfrak{KK}(X)$ and $A(X)$ is nuclear, then $A \in \mathcal{B}(X)$ if and only if $A(\{x\}) \in \mathcal{B}$ for every $x \in X$.

Filtrated K-theory

- For each locally closed subset $Y \in \mathbb{LC}(X)$, define a functor

$$\mathrm{FK}_Y \colon \mathfrak{KK}(X) \to \mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}, \quad \mathrm{FK}_Y(A) := \mathrm{K}_*(A(Y)).$$

- For $Y, Z \in \mathbb{LC}(X)$, let $\mathcal{NT}(Y, Z)$ be the Abelian group of natural transformations $\mathrm{FK}_Y \Rightarrow \mathrm{FK}_Z$.
- The category $\mathcal{NT}$ with object set $\mathbb{LC}(X)$ and morphisms $\mathcal{NT}(Y, Z)$ is $\mathbb{Z}/2$-graded and pre-additive.
- Filtrated K-theory is the functor

$$\mathrm{FK} = \left(\mathrm{FK}_Y\right)_{Y \in \mathbb{LC}(X)} \colon \mathfrak{KK}(X) \to \mathfrak{Mod}(\mathcal{NT})_{\mathrm{c}},$$
$$A \mapsto \Big(\mathrm{K}_*(A(Y))\Big)_{Y \in \mathbb{LC}(X)}.$$

Canonical generators and relations

Canonical generators

For $Y \in \mathbb{LC}(X)$, $U \in \mathbb{O}(Y)$, $C = Y \setminus U$ there is an extension

$$A(U) \rightarrowtail A(Y) \twoheadrightarrow A(C)$$

and hence a natural sequence

$$\mathrm{FK}_U(A) \xrightarrow{i} \mathrm{FK}_Y(A) \xrightarrow{r} \mathrm{FK}_C(A) \xrightarrow{\delta} \mathrm{FK}_U(A).$$

Canonical relations

- the compositions $r_Y^C \circ i_U^Y$, $\delta_C^U \circ r_Y^C$, $i_U^Y \circ \delta_C^U$ vanish
- various relations following from the naturality of the six-term sequence with respect to morphisms of extensions

Representability and universality

Representability theorem (Meyer-Nest)

For every $Y \in \mathbb{LC}(X)$ there is a separable C^*-algebra $\mathcal{R}_Y$ over X and a natural isomorphism $\mathrm{KK}_*(X; \mathcal{R}_Y, \text{␣}) \cong \mathrm{FK}_Y$.

We have $\mathrm{FK} \cong \mathrm{KK}_*(X; \mathcal{R}, \text{␣})$ for $\mathcal{R} = \bigoplus_{Y \in \mathbb{LC}(X)} \mathcal{R}_Y$ and $\mathcal{NT} \cong \mathrm{KK}_*(X; \mathcal{R}, \mathcal{R})$.

Universality of filtrated K-theory (Meyer-Nest)

Filtrated K-theory $\mathrm{FK}\colon \mathfrak{KK}(X) \to \mathfrak{Mod}(\mathcal{NT})_{\mathrm{c}}$ is universal among the stable homological functors F with $F(f) = 0$ for all $f \in \mathrm{KK}(X; A, B)$ with $\mathrm{FK}(f) = 0$.

UCT with homological condition

Theorem (Meyer-Nest)

Let $A, B \in \mathfrak{KK}(X)$. Suppose that $\mathrm{FK}(A) \in \mathfrak{Mod}(\mathcal{NT})_c$ has a projective resolution of length 1 and that $A \in \mathcal{B}(X)$. Then there are natural short exact sequences

$$\mathrm{Ext}^1_{\mathcal{NT}}(\mathrm{FK}(A)[j+1], \mathrm{FK}(B)) \rightarrowtail \mathrm{KK}_j(X; A, B) \twoheadrightarrow \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A)[j], \mathrm{FK}(B))$$

for $j \in \mathbb{Z}/2$.

Intermezzo: finite T_0-spaces as directed graphs

Let X be a finite set. There are the following bijections:

$$\{T_0\text{-topologies on } X\}$$

Alexandrov topology ↑ ↓ specialisation preorder

$$\{\text{partial orders on } X\}$$

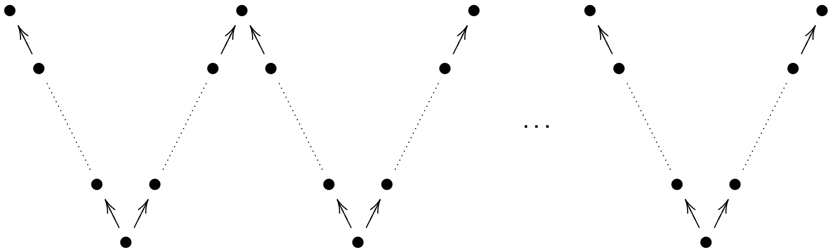
reachability relation ↑ ↓ Hasse diagram

$$\{\text{transitively reduced directed acyclic graphs with vertex set } X\}$$

Examples

1.
   - $X = \bullet$
   - $\mathfrak{C}^*\mathfrak{alg}(X)$ = plain C^*-algebras
   - Universal Coefficient Theorem: Rosenberg-Schochet (1987)
2.
   - $X = \bullet \longrightarrow \bullet$
   - $\mathfrak{C}^*\mathfrak{alg}(X)$ = Extensions of C^*-algebras
   - FK = Six-term exact sequence
   - Rørdam (1997): Classification of extensions of purely infinite simple stable separable C^*-algebras in the bootstrap class
   - UCT: Bonkat (2002)
3.
   - $X = \bullet \longrightarrow \bullet \longrightarrow \bullet$
   - UCT: Restorff (2008)
4.
   - $X = \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet$
   - UCT: Meyer-Nest (2008)

Type (A) spaces



Main result

Theorem (B-Köhler)

Let X be finite T_0-space. The following statements are equivalent:

1. *X is a disjoint union of spaces of type* (A).
2. *Let A and B be separable C^*-algebras over X.*
   Suppose $A \in \mathcal{B}(X)$.
   Then there is a natural short exact UCT sequence

$$\mathrm{Ext}^1_{\mathcal{NT}}(\mathrm{FK}(A)[1], \mathrm{FK}(B)) \rightarrowtail \mathrm{KK}_*(X; A, B) \twoheadrightarrow \mathrm{Hom}_{\mathcal{NT}}(\mathrm{FK}(A), \mathrm{FK}(B)).$$

3. *Let $A, B \in \mathcal{B}(X)$. Then $\mathrm{FK}(A) \cong \mathrm{FK}(B)$ implies $A \simeq_{\mathrm{KK}(X)} B$.*

Classification of certain non-simple C^*-algebras

Corollary

Let X be a disjoint union of spaces of type (A).
Filtrated K-theory induces a bijection between the sets of isomorphism classes of

- *tight, stable, purely infinite, separable, nuclear C^*-algebras over X with simple subquotients in the bootstrap class*

and of

- *countable, exact $\mathcal{NT}$-modules.*

Here M is called exact if the sequence
$M(U) \xrightarrow{i} M(Y) \xrightarrow{r} M(C) \xrightarrow{\delta} M(U)$ is exact
for every $Y \in \mathbb{LC}(X)$, $U \in \mathbb{O}(Y)$, $C = Y \setminus U$.

Proof of UCT for type (A) spaces

Reduce to the totally ordered case proven by Meyer-Nest using the following observation:
Let X be of type (A). Let O be the totally ordered space with the same number of points.

Theorem (B-Köhler)

There is an ungraded isomorphism $\Phi\colon \mathcal{NT}^*(X) \to \mathcal{NT}^*(O)$, *and*

$$\Phi^*\colon \mathfrak{Mod}^{\mathrm{ungr}}\big(\mathcal{NT}^*(O)\big)_{\mathsf{c}} \to \mathfrak{Mod}^{\mathrm{ungr}}\big(\mathcal{NT}^*(X)\big)_{\mathsf{c}}$$

restricts to a bijective correspondence between exact ungraded $\mathcal{NT}^*(O)$*-modules and exact ungraded* $\mathcal{NT}^*(X)$*-modules.*

Proof of UCT for type (A) spaces

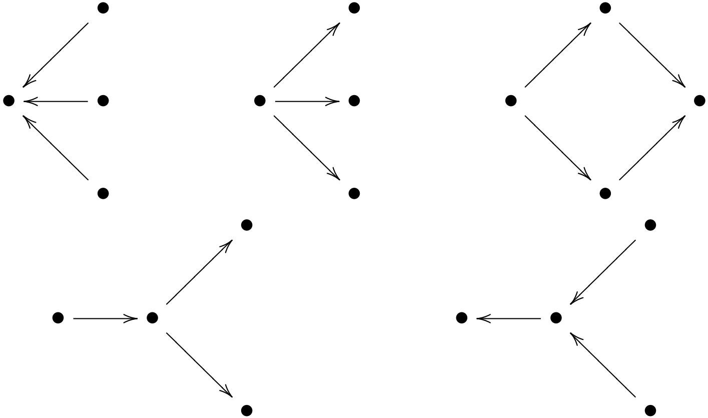
This allows to carry over the arguments by Meyer-Nest.

Lemma

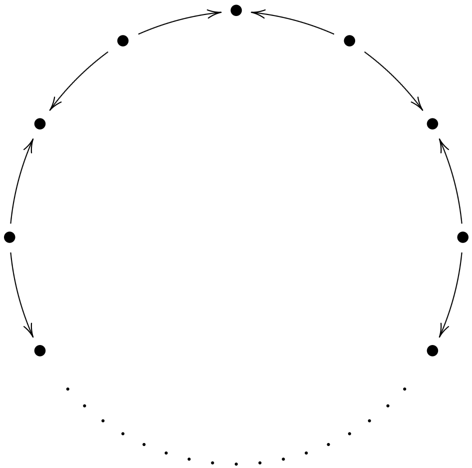
Let $M \in \mathfrak{Mod}(\mathcal{NT}^*(X))_{\mathrm{c}}$.

- M is projective $\iff$ M is exact and has free entries.
- M has a projective resolution of length 1
  $\iff$ M is exact,
  $\iff$ $M = \mathrm{FK}(A)$ for some $A \in \mathfrak{KK}(X)$.

Some non-type (A) spaces

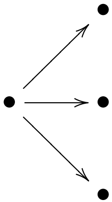


Some non-type (A) spaces



Counterexamples

Meyer-Nest construct to non-equivalent objects with isomorphic filtrated K-theory for the space



Counterexamples

- We copy their method to exhibit counterexamples for all spaces in the previous list.
- We reduce to spaces from this list:

Lemma

1. *Let* $Y \in \mathbb{LC}(X)$. $\neg UCT(Y) \Longrightarrow \neg UCT(X)$.
2. *Let* $f: X \to Y$, $g: Y \to X$ *be continuous with* $f \circ g = \mathrm{id}_Y$. $\neg UCT(Y) \Longrightarrow \neg UCT(X)$.

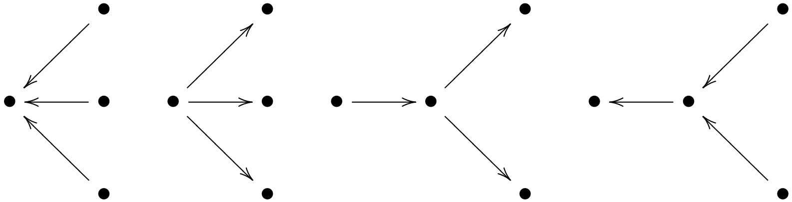
Observation

If X is a connected non-type (A) space, then either

- some vertex of $\Gamma(X)$ has unoriented degree 3 or more;
- every vertex of $\Gamma(X)$ has unoriented degree 2;

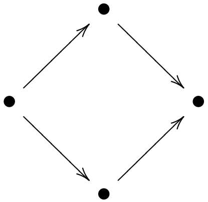
Counterexamples

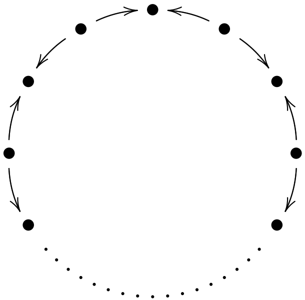
If there is a vertex of $\Gamma(X)$ with unoriented degree 3, "transport" counterexamples from



Counterexamples

If every vertex of $\Gamma(X)$ has unoriented degree 2 "transport" counterexamples from





Thank you for your attention!