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An Institution-Independent Proof of the Robinson Consistency Theorem

Abstract. We prove an institutional version of A. Robinson's *Consistency Theorem*. This result is then applied to the institution of many-sorted first-order predicate logic and to two of its variations, infinitary and partial, obtaining very general syntactic criteria sufficient for a signature square in order to satisfy the Robinson consistency and Craig interpolation properties.

Keywords: institution, Robinson consistency, Craig interpolation, elementary diagram, many-sorted first-order logic.

1. Introduction

The many-sorted, rather than unsorted, versions of logical systems (such as equational logic, first-order logic, etc.) are acknowledged as being particularly suitable for applications to computer science, in areas like semantics of programming languages and formal specifications. However, in pure mathematical logic, many-sorted logics tend to be classified as "inessential variations" [33] of their unsorted forms. While this might be true w.r.t. some classical logical aspects such as *compactness*, *completeness*, *Löwenheim properties*, or *axiomatizability*, there is at least one important class of properties that become significantly more intricate when passing from the unsorted to the many-sorted case: those involving the concept of *translation* between languages (signatures), also known as *signature morphism*. Although classical logic, dealing usually just with the very simple case of unsorted language *inclusions*, very rarely cared about these problems, nevertheless any kind of study aiming at providing logical support for diverse areas of theoretical computer science has to consider them, due to the crucial importance of translation between languages in the latter field.

In order to point out the difference between unsorted and many-sorted w.r.t. signature morphisms, we consider two examples in first-order logic. As noticed in [20], the functor *Mod*, taking signatures into their corresponding classes of models and signature morphisms into corresponding "forgetful"

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functors, preserves arbitrary colimits in the many-sorted case, but only some colimits, such as pushouts, in the unsorted case. Another example regards the Craig interpolation property [13], abbreviated CIP, which is classically stated as follows: if $e_1 \vdash e_2$ for two first-order sentences e_1 and e_2, then there exists a sentence e, called the *interpolant* of e_1 and e_2, that uses only logical symbols which appear both in e_1 and e_2 and such that $e_1 \vdash e \vdash e_2$. An equivalent expression of the above property assumes $e_1 \vdash e_2$ in the *union* language $L_1 \cup L_2$ and asks from e to be in the *intersection language* $L_1 \cap L_2$, where L_i is the language of e_i. If, following an approach originating in [49], we naturally generalize the inclusion square

$$\begin{tikzcd} & L_1 \arrow[dr, hook] & \\ L_1 \cap L_2 \arrow[ur, hook] \arrow[dr, hook] & & L_1 \cup L_2 \\ & L_2 \arrow[ur, hook] & \end{tikzcd}$$

to a pushout of language translations (signature morphisms)

$$\begin{tikzcd} & L_1 \arrow[dr, "\varphi_1'"] & \\ L \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & L' \\ & L_2 \arrow[ur, "\varphi_2'"'] & \end{tikzcd}$$

and replace sentences e_1, e_2, e with *sets of sentences* E_1, E_2, E we obtain the following form of CIP: If $\varphi_1'(E_1) \vdash \varphi_2'(E_2)$, then there exists a set E of Σ-sentences such that $E_1 \vdash \varphi_1(E)$ and $\varphi_2(E) \vdash E_2$. Now, the question of which pushout squares have CIP has a definite answer in the unsorted case: *all* of them; this is probably folklore, but also follows from a many-sorted result in [6]. On the other hand, the problem of characterizing the pushout squares which have CIP is still open for the many-sorted case.[1]

An equivalent formulation of CIP in classical logic, with a more model-theoretical flavor, is the Robinson consistency property [42], abbreviated RCP, which states that, if two theories are joint-consistent in their common

[1] Although in the context of the so-called *abstract model-theoretic logics* [3], many-sortedness has a significantly more important status than in classical logic, the issue of interpolation is still treated there only w.r.t. language inclusions.

language, then they are so in their union language. More precisely, for any theories (i.e., sets of sentences closed under deduction) T_1 and T_2 over languages L_1 and L_2 respectively, if $\{e \in T_1 \cup T_2 \mid e \text{ is a sentence in } L_1 \cap L_2\}$ has a model in $L_1 \cap L_2$, then $T_1 \cup T_2$ also has a model in $L_1 \cup L_2$. This paper builds on the generalization of RCP to the abstract level of institutions.

Some Motivation

Finding criteria as general as possible for such a significant property as RCP to hold in a logic is an interesting problem in itself, from the abstract model theory point of view. However, there are reasons why such a study might be useful in theoretical computer science too, reasons given by the tight relationship between RCP and CIP, which goes beyond classical first-order logic; indeed, inside any compact logic with enough expressive power, RCP and CIP are equivalent [49]. In fact, all our RCP results, since they will be based on conditions that make RCP and CIP equivalent, are also results regarding CIP.

CIP is a very useful and broadly studied property in mathematical logic and theoretical computer science - see especially [5, 20, 2], but also [7, 18, 21, 6] for some discussion on the usefulness of this property. Applications of CIP mostly deal with *combining and decomposing theories* and involve areas like algebraic specifications [4, 20, 48, 21], theorem proving and symbolic model checking [38, 39, 30, 31, 52], or algebraic logic [2, 45, 27].[2]

In what follows, we shall offer some motivation for the study of CIP along the lines of our generalization, in the context of structured specifications. A good methodology in specifying hardware or software systems is the *modular* approach, which prescribes building large specifications out of small and easily analyzable pieces. As argued in [20], this allows the verification of many properties *at a very early stage*, at the level of specification rather than that of implementation, thus improving reliability of the systems. The mentioned approach combines specifications stated in different languages (signatures) into larger specifications, using the notion of *language translation*, i.e., *signature morphism.* In many settings for algebraic specification [4, 20, 48, 54], two main operations on modules are considered: that of *reusing text* in a meaningful and model-consistent way, which might involve some *renaming*, and that of *hiding information.* Both these operations, fundamentally differ-

[2] In algebraic logic, CIP is studied in connection to its algebraic counterpart, the amalgamation property; note that the latter property, stated on models and embeddings in the quasi-variety attached to the considered propositional logical system, has nothing to do with the (weak) amalgamation property that we consider later on signature morphisms.

ent in nature, are carried *along* signature morphisms; hence the distinction between two classes of signature morphisms:

- the class of *hiding morphisms*, used for hiding some of the symbols, let it be $\mathcal{H}$, and
- the class of *translating morphisms* used for renaming and/or adding some symbols, let it be $\mathcal{T}$.

A very desirable property is the existence of a (sound and) *complete proof system* for reasoning about structured specifications.[3] It was proved in [10] (for the case of first-order logic) and in [7] for the general case of *institutions* [9, 24] that, in order for such a complete proof system to exist, one needs some good properties of $\mathcal{H}$ and $\mathcal{T}$ w.r.t. each other, among which the most crucial one is $(\mathcal{H}, \mathcal{D})$*-interpolation*, stating that any pushout of signature morphisms
$(\Sigma_2 \overset{\varphi_2}{\leftarrow} \Sigma \overset{\varphi_1}{\rightarrow} \Sigma_1, \Sigma_2 \overset{\varphi_2'}{\rightarrow} \Sigma' \overset{\varphi_1'}{\leftarrow} \Sigma_1)$ with $\varphi_1, \varphi_2' \in \mathcal{H}$ and $\varphi_2, \varphi_1' \in \mathcal{T}$ has CIP. It is not clear which types of morphisms are appropriate for hiding and which for translating. But of course, for expressivity reasons, one would like to allow these types to be as general as possible, while keeping the $(\mathcal{H}, \mathcal{D})$-interpolation property. Hence the problem of finding general conditions under which a pushout of signature morphisms has CIP seems to be an important one. Our paper provides such general conditions in the abstract framework of institutions, obtaining in particular the strongest syntactic condition that we are aware of from the literature for a pushout square to have CIP in many-sorted first-order logic (*FOPL*) and in its partial-operation and infinitary-conjunction variations, *PFOPL* and *IFOPL*. Applied to algebraic specification theory, our results give more flexibility to a specification language based on first-order logic such as CASL [12]: one is allowed, for instance, to use signature morphisms that are injective on sorts for hiding purposes and *arbitrary morphisms* for translation purposes, and still have a complete proof system.

The Structure of the Paper

After a preliminary section, recalling some categorical and institutional definitions and notations, in Section 3 we state CIP and different versions of RCP in institutions and show the connections between them. In Section 4, we prove an institutional form of Robinson Consistency Theorem. The framework is that of an institution with elementary diagrams which has

[3] That is, provided that one already has such a proof system for flat specifications.

sufficient expressive power: admits negations and certain quantifications; this loses sight of the equational logics, but concentrates on more expressive first-order-like logics. Section 5 is dedicated to the application of our previous results to many-sorted first-order logic and two variations, infinitary and partial. We obtain a sufficient syntactic criterion for a signature square to be a Craig interpolation square and a Robinson square - this criterion does not assume injectivity on sorts, and covers the case when *one* of the morphisms is injective on sorts. Some concluding remarks and discussion of related work end the paper.

2. Preliminaries

Categories

We assume that the reader is familiar with basic categorical notions like functor, natural transformation, colimit, comma category, etc. A standard textbook on the topic is [26]. We are going to use the terminology from there, with a few exceptions that we point out below. We use both the terms "morphism" and "arrow" to refer morphisms of a category. Composition of morphisms and functors is denoted using the symbol ";" and is considered in diagrammatic order.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. Given an object $A \in |\mathcal{C}|$, the *comma category of objects in* $\mathcal{C}$ *under* A is denoted $A/\mathcal{C}$. Recall that the objects of this category are pairs (h, B), where $B \in |\mathcal{C}|$ and $A \xrightarrow{h} B$ is a morphism in $\mathcal{C}$. Throughout the paper, we might let either $(A \xrightarrow{h} B, B)$, or (h, B), or even h, indicate objects in $A/\mathcal{C}$. A morphism in $A/\mathcal{C}$ between two objects (h, B) and (g, D) is just a morphism $B \xrightarrow{f} D$ in $\mathcal{C}$ such that $h; f = g$ in $\mathcal{C}$. Thus a morphism $A \xrightarrow{h} B$ can be seen in $A/\mathcal{C}$ both as an object and as a morphism between $(1_A, A)$ and (h, B) - this "duplicity" will often appear throughout the paper, so the reader should consider herself warned! There exists a canonical forgetful functor between $A/\mathcal{C}$ and $\mathcal{C}$, mapping each (h, B) to B and each $f : (h, B) \to (g, D)$ to $f : B \to D$. Also, if $F : \mathcal{C}' \to \mathcal{C}$ is a functor, $A \in |\mathcal{C}|$, $A' \in |\mathcal{C}'|$, and $A \stackrel{u}{\to} F(A')$ is in $\mathcal{C}$, then there exists a canonical functor $u/F : A'/\mathcal{C}' \to A/\mathcal{C}$ mapping each $(A' \stackrel{h}{\to} B, B)$ to $(u; F(h), F(B))$ and each $f : (h, B) \to (g, D)$ to $F(f) : (u; F(h), F(B)) \to (u; F(g), F(D))$. If $\mathcal{C} = \mathcal{C}'$ and F is the identity functor $1_{\mathcal{C}}$, we write $u/\mathcal{C}$ instead of u/F.

Let $\mathcal{C}$ and $\mathcal{S}$ be two categories such that $\mathcal{S}$ is small. If $D : \mathcal{S} \to \mathcal{C}$ is a functor (also called a *diagram*), then a *cocone* of D is a natural transformation $\mu : D \Longrightarrow V$ between the functor D and [the constant functor mapping all objects to V and all morphisms to 1_V]; V is an object in $\mathcal{C}$, the *vertex* of the colimit, and the components of μ are the *structural morphisms* of the colimit. A diagram defined on the ordered set of natural numbers (regarded as a category) shall be called *ω-diagram*, and a colimit of such a diagram *ω-colimit*. We sometimes identify a diagram $D : J \to \mathcal{C}$ with its image in $\mathcal{C}$, $D(J)$.

Institutions

Institutions were introduced in [9] with the original goal of providing an abstract, logic-independent framework for algebraic specifications of computer science systems. However, by isolating the essence of a logical system in the abstract *satisfaction relation*, these structures also turned out to be appropriate for the development of abstract model theory, as shown by a whole series of (old and new) papers: [49, 50, 51, 46, 47, 15, 16, 18, 17, 23, 40]. See also [34] for an up-to-date discussion on institutions as abstract logics.

An institution [9, 24] consists of:

1. a category $Sign$, whose objects are called *signatures*.
2. a functor $Sen : Sign \to Set$, providing for each signature a set whose elements are called *(Σ-)sentences*.
3. a functor $Mod : Sign \to Cat^{op}$, providing for each signature Σ a category whose objects are called *(Σ-)models* and whose arrows are called *(Σ-)morphisms*.
4. a relation $\models_\Sigma \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |Sign|$, called *(Σ-) satisfaction*, such that for each morphism $\varphi : \Sigma \to \Sigma'$ in $Sign$, the *satisfaction condition*

$$M' \models_{\Sigma'} Sen(\varphi)(e) \text{ iff } Mod(\varphi)(M') \models_\Sigma e$$

holds for all $M' \in |Mod(\Sigma')|$ and $e \in Sen(\Sigma)$. Following the usual notational conventions, we sometimes let $\_\upharpoonright_\varphi$ denote the reduct functor $Mod(\varphi)$ and let φ denote the sentence translation $Sen(\varphi)$. When $M = M'\upharpoonright_\varphi$ we say that M' is a *φ-expansion* of M, and that M is the *φ-reduct* of M'; and similarly for model morphisms.

For all the following concepts related to institutions that we recall below, the reader is referred to [24] unless some other place is explicitly indicated.

Let Σ be a signature. Then,

- for each $E \subseteq Sen(\Sigma)$, let $E^* = \{M \in |Mod(\Sigma)| \mid M \models_\Sigma e \text{ for all } e \in E\}$.

- for each class $\mathcal{M}$ of Σ-models, let $\mathcal{M}^{\,*} = \{e \in Sen(\Sigma) \mid M \models_\Sigma e$ for all $M \in \mathcal{M}\}$.

With no danger of confusion, we let $\bullet$ denote any of the two compositions $**$ of the two operators $*$. Each of the two bullets is a closure operator. When E and E' are sets of sentences of the same signature Σ, we let $E \models_\Sigma E'$ denote the fact that $E^* \subseteq E'^*$. The relation $\models_\Sigma$ between sets of sentences is called the *(Σ-)semantic consequence relation* (notice that it is written just like the satisfaction relation). If $E' = \{e'\}$, we might write $E \models_\Sigma e'$. In order to simplify notation, we usually write $\models$ instead of $\models_\Sigma$, for both the satisfaction relation and the semantic consequence relation. Two sentences e and e' are called *equivalent*, denoted $e \equiv e'$, if $\{e\}^* = \{e'\}^*$. Dually, two models M and M' are called *elementary equivalent*, denoted $M \equiv M'$, if $\{M\}^* = \{M'\}^*$. The fact that two models M and M' are isomorphic is indicated by $M \simeq M'$.

A signature morphism $\varphi : \Sigma \to \Sigma'$ is called *conservative* if every Σ-model has a φ-expansion. A *presentation* is a pair (Σ, E), where $E \subseteq Sen(\Sigma)$. A *theory* is a presentation (Σ, E) with E closed, i.e., with $E^\bullet = E$. One usually calls "presentation" or "theory" only the set E, and not the whole pair (Σ, E). A *presentation morphism* $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $\varphi(E) \subseteq E'^\bullet$. A presentation morphism between theories is called *theory morphism*. For a presentation (Σ, E), we let $Mod(\Sigma, E)$ denote the category of all Σ-models A such that $A \models E$. A presentation is called *consistent* if it has at least one model; otherwise it is called *inconsistent*.

An institution is called *compact* [20] if, for each signature Σ, the closure operator $\bullet$ on $Sen(\Sigma)$ is compact; in other words, if, for each $E \cup \{e\} \subseteq Sen(\Sigma)$ such that $E \models e$, there exists a finite subset F of E such that $F \models e$. An institution is called *semi-exact* [32] if the model functor $Mod : Sign \to Cat^{op}$ preserves pushouts. A property weaker than semi-exactness that we shall consider is the following. An institution is called *weakly model-semi-exact* if for any pushout of signature morphisms $(\Sigma_2 \stackrel{\varphi_2}{\leftarrow} \Sigma \stackrel{\varphi_1}{\rightarrow} \Sigma_1, \Sigma_2 \stackrel{\varphi_2'}{\rightarrow} \Sigma' \stackrel{\varphi_1'}{\leftarrow} \Sigma_1)$, for any $M_1 \in |Mod(\Sigma_1)|$, $M_2 \in |Mod(\Sigma_2)|$ such that $M_1{\upharpoonright_{\varphi_1}} = M_2{\upharpoonright_{\varphi_2}}$, there exists a model $M' \in |Mod(\Sigma')|$ such that $M'{\upharpoonright_{\varphi_1'}} = M_1$ and $M'{\upharpoonright_{\varphi_2'}} = M_2$.

The following institutional notions dealing with logical connectives and quantifiers were defined in [49]. Let $\Sigma \in |Sign|$, $e, e_1, e_2 \in Sen(\Sigma)$, $E \subseteq Sen(\Sigma)$, $e' \in Sen(\Sigma')$, and $\varphi : \Sigma \to \Sigma'$.

- a Σ-sentence $\neg e$ is a *negation* of e when $M \models \neg e$ iff $M \not\models e$ for each $M \in |Mod(\Sigma)|$;

- a Σ-sentence $e_1 \wedge e_2$ is a *conjunction* of e_1 and e_2 when $M \models e_1 \wedge e_2$ iff $[M \models e_1$ and $M \models e_2]$ for each $M \in |Mod(\Sigma)|$;
- a Σ-sentence $\bigwedge E$ is a *conjunction* of the set of sentences E when $[M \models \bigwedge E$ iff there exists $f \in E$ such that $M \models f]$ for each $M \in |Mod(\Sigma)|$;
- a Σ-sentence $(\forall\varphi)e'$ is a *universal quantification* of e' over φ when $[M \models (\forall\varphi)e'$ iff there exists $M' \in |Mod(\Sigma')|$ such that $M'\upharpoonright_\varphi = M$ and $M' \models e']$ for each $M \in |Mod(\Sigma)|$.

The signature morphisms commute with the logical connectives [49], i.e., using the above notations,

- $\varphi(\neg e)$ is a negation of $\varphi(e)$,
- $\varphi(e_1 \wedge e_2)$ is a conjunction of $\varphi(e_1)$ and $\varphi(e_2)$,
- $\varphi(\bigwedge E)$ is a conjunction of the set of sentences $\varphi(E)$.

An institution is said to *admit*:

- *negations*, if every sentence has a negation;
- *(finite) conjunctions*, if every two sentences have a conjunction;
- *arbitrary conjunctions*, if every set of sentences has a conjunction;
- *universal quantifications* over a given signature morphism $\varphi : \Sigma \to \Sigma'$ if every Σ'-sentence has a universal quantification over φ;

A theory (Σ, T) is called *complete* [49] if it is maximally consistent, i.e., T is consistent and any strict superset T' of it is inconsistent. If the institution admits negations, then a theory T is complete iff there exists a Σ-model A such that $\{A\}^* = T$. We next give two easy, but very useful lemmas.

Lemma 1. [14] (The Institution-Independent Theorem of Constants) *Let $\varphi : \Sigma \to \Sigma'$ be a signature morphism, $E \subseteq Sen(\Sigma)$, $e' \in Sen(\Sigma')$ and $(\forall\varphi)e' \in Sen(\Sigma)$ (so we assume the existence of a universal quantification of e' over φ). Then $\varphi(E) \models e'$ if and only if $E \models (\forall\varphi)e'$.*

Lemma 2. *Assume that the institution is weakly model-semi-exact and let $(\Sigma_2 \overset{\varphi_2}{\leftarrow} \Sigma \overset{\varphi_1}{\rightarrow} \Sigma_1, \Sigma_2 \overset{\varphi_2'}{\rightarrow} \Sigma' \overset{\varphi_1'}{\leftarrow} \Sigma_1)$ be a pushout of signature morphisms. Then the following hold:*

1. *If a sentence $e_1 \in Sen(\Sigma_1)$ has a universal quantification over φ_1, then $\varphi_1'(e_1)$ has a universal quantification over φ_2'.*

2. *For each sentence e_1 having a universal quantification over φ_1, it holds that $M_2 \models (\forall\varphi_2')\varphi_1'(e_1)$ iff $M_2\upharpoonright_{\varphi_2} \models (\forall\varphi_1)e_1$ iff $M_2 \models \varphi_2((\forall\varphi_1)e_1)$ for all $M_2 \in |Mod(\Sigma_2)|$.*

Proof. (1): Let $e_1 \in Sen(\Sigma_1)$ having a universal quantification over φ_1, $(\forall\varphi_1)e_1$. We claim that $\varphi_2((\forall\varphi_1)e_1)$ is a universal quantification of $\varphi_1'(e_1)$ over φ_2'. Indeed, let $M_2 \in |Mod(\Sigma_2)|$.

- Assume $M_2 \models \varphi_2((\forall\varphi_1)e_1)$. Let M' be a φ_2'-expansion of M_2. We need to show $M' \models \varphi_1'(e_1)$, that is, $M'\upharpoonright_{\varphi_1'} \models e_1$. But the last is true, because $M'\upharpoonright_{\varphi_1'}$ is a φ_1-expansion of $M_2\upharpoonright_{\varphi_2}$ and $M_2\upharpoonright_{\varphi_2} \models (\forall\varphi_1)e_1$.
- Conversely, assume that each φ_2'-expansion of M_2 satisfies $\varphi_1'(e_1)$. In order to show $M_2 \models \varphi_2((\forall\varphi_1)e_1)$, i.e., $M_2\upharpoonright_{\varphi_2} \models (\forall\varphi_1)e_1$, let M_1 be a φ_1-expansion of $M_2\upharpoonright_{\varphi_2}$. By weak model-semi-exactness, there exists $M' \in |Mod(\Sigma')|$ such that $M'\upharpoonright_{\varphi_1'} = M_1$ and $M'\upharpoonright_{\varphi_2'} = M_2$. Then $M' \models \varphi_1'(e_1)$, that is, $M_1 \models e_1$.

(2): Immediate by the proof of (1). ■

Elementary Diagrams

Diagrams are an important concept and proof tool in classical model theory [11]. They were first generalized to the institutional framework in [50, 51]; there it is defined the concept of *abstract algebraic institution*, which is an institution subject to some additional natural requirements (like finite-exactness, existence of direct products of models etc.) and enriched with a system of diagrams. The reason for introducing diagrams there was making all algebras *accessible*, for specification purposes. Our proof of the Robinson Consistency Theorem will make heavy use of a more recent institutional notion of elementary diagram, defined in [16].

An institution $\mathcal{I} = (Sign, Sen, Mod, \models)$ is said to have *elementary diagrams* [16] if

1. for each signature Σ and Σ-model A there exists a signature morphism $\iota_\Sigma(A) : \Sigma \to \Sigma_A$ (called the *elementary extension* of Σ via A) and a set E_A of Σ_A-sentences (called the *elementary diagram* of A) such that $Mod(\Sigma_A, E_A)$ and $A/Mod(\Sigma)$ are isomorphic by an isomorphism $i_{\Sigma,A}$ making the following diagram commutative:

$$\begin{tikzcd} Mod(\Sigma_A, E_A) \arrow[rr, "i_{\Sigma,A}"] \arrow[dr, "-\upharpoonright_{\iota_\Sigma(A)}"'] & & A/Mod(\Sigma) \arrow[dl, "\text{forgetful}"] \\ & Mod(\Sigma) & \end{tikzcd}$$

2. ι is "functorial", i.e., for each signature morphism $\varphi : \Sigma \to \Sigma'$, each $A \in |Mod(\Sigma)|$, $A' \in |Mod(\Sigma')|$ and $h : A \to A'\upharpoonright_\varphi$ in $Mod(\Sigma)$, there exists a presentation morphism $\iota_\varphi(h) : (\Sigma_A, E_A) \to (\Sigma'_{A'}, E_{A'})$ making the following diagram commutative:

$$\begin{tikzcd}
\Sigma \arrow[r, "\iota_\Sigma(A)"] \arrow[d, "\varphi"'] & \Sigma_A \arrow[d, "\iota_\varphi(h)"] \\
\Sigma' \arrow[r, "\iota_{\Sigma'}(A')"'] & \Sigma'_{A'}
\end{tikzcd}$$

3. i is natural, i.e., for each signature morphism $\varphi : \Sigma \to \Sigma'$, each $A \in |Mod(\Sigma)|$, $A' \in |Mod(\Sigma')|$ and $h : A \to A'\upharpoonright_\varphi$ in $Mod(\Sigma)$, the following diagram is commutative:

$$\begin{tikzcd}
Mod(\Sigma_A, E_A) \arrow[r, "i_{\Sigma,A}"] & A/Mod(\Sigma) \\
Mod(\Sigma'_{A'}, E_{A'}) \arrow[r, "i_{\Sigma',A'}"] \arrow[u, "-\upharpoonright_{\iota_\varphi(h)}"] & A'/Mod(\Sigma') \arrow[u, "h/Mod(\varphi)"']
\end{tikzcd}$$

In classical model theory, Σ_A is the signature Σ enriched with all the elements of A as constants, $\iota_\Sigma(A) : \Sigma \to \Sigma_A$ is the inclusion of signatures, and E_A is a set of parameterized sentences which hold in A, depending on the considered type of arrow in the categories of models (yielding "elementary diagram" for elementary embeddings, "positive diagram" for arbitrary model homomorphisms, or "diagram" for model embeddings - see [11]). All the three ingredients $\Sigma_A, E_A, \iota_\Sigma(A)$ are also present at the abstract algebraic institutions in [50, 51], where it is also required the natural and potentially very useful fact that A_A be accessible. The important additions of the definition in [16] that we use here are the "functoriality" and naturality conditions, which postulate smooth communication between diagrams along signature morphisms, taking real advantage of the categorical structure of institutions.

The above definition of elementary diagrams may seem, at a first sight, to be adding a great deal of complicated extra structure to institutions. However, it has several advantages:

- looks extremely natural and self-explanatory to anyone familiar with diagrams from classical logic;
- it is so general, that almost all meaningful institutions have elementary diagrams;

- it really provides a "method" for proving logical properties, as we exemplify in this paper.

Here are some notational conventions that we hope will make the reader's life easier. Let $\varphi : \Sigma \to \Sigma'$ be a signature morphism, $A' \in |Mod(\Sigma')|$, and $h : A \to B$ in $Mod(\Sigma)$. We write $\iota_\Sigma(h)$ instead of $\iota_{1_\Sigma}(h)$ and $\iota_\varphi(A'\upharpoonright_\varphi)$ instead of $\iota_\varphi(1_{(A'\upharpoonright_\varphi)})$. Let A be a fixed object in $Mod(\Sigma)$ and let $B, C \in |Mod(\Sigma)|$ and $f : A \to B$, $g : A \to C$, $u : B \to C$ morphisms in $Mod(\Sigma)$ such that $f; u = g$. Then (f, B) and (g, C) are objects in $A/Mod(\Sigma)$ and u is also a morphism in $A/Mod(\Sigma)$ between (f, B) and (g, C). We further establish the following notations: $B_f = i^{-1}_{\Sigma,A}(f, B)$ (and, similarly, $C_g = i^{-1}_{\Sigma,A}(g, C)$), $u_{f,g} = i^{-1}_{\Sigma,A}((f, B) \xrightarrow{u} (g, C))$. Thus, for instance, let $f : A \to B$ be a Σ-model morphism. Then $f_{1_A,f}$ is the image through $i^{-1}_{\Sigma,A}$ of the morphism $f : (1_A, A) \to (f, B)$ in $A/Mod(\Sigma)$, and has source $A_{(1_A)}$ and target B_f. We shall usually write A_A instead of $A_{(1_A)}$ and $f_{A,f}$ instead of $f_{1_A,f}$.

In [16], there are given some examples of institutions with elementary diagrams. Most institutions that were defined on "working" logical systems tend to have elementary diagrams. For the purposes of this paper, we only point out three examples, with their elementary variations.

1. *FOPL* - the institution of many-sorted first-order predicate logic (with equality). The signatures are triplets (S, F, P), where S is the set of sorts, $F = \{F_{w,s}\}_{w \in S^*, s \in S}$ is the ($S^* \times S$ -indexed) set of operation symbols, and $P = \{P_w\}_{w \in S^*}$ is the (S^*-indexed) set of relation symbols. By a slight notational abuse, we let F and P also denote and $\bigcup_{(w,s) \in S^* \times S} F_{w,s}$ and $\bigcup_{w \in S^*} P_w$ respectively. A signature morphism between (S, F, P) and (S', F', P') is a triplet $\varphi = (\varphi^{sort}, \varphi^{op}, \varphi^{rel})$, where $\varphi^{sort} : S \to S'$, $\varphi^{op} : F \to F'$, $\varphi^{rel} : P \to P'$ such that $\varphi^{op}(F_{w,s}) \subseteq F'_{\varphi^{sort}(w), \varphi^{sort}(s)}$ and $\varphi^{rel}(P_w) \subseteq P'_{\varphi^{sort}(w)}$ for all $(w, s) \in S^* \times S$. When there is no danger of confusion, we may let φ denote each of φ^{sort}, φ^{rel} and φ^{op}. Given a signature $\Sigma = (S, F, P)$, a Σ-model A is a triplet $A = (\{A_s\}_{s \in S}, \{A^w_s(\sigma)\}_{(w,s) \in S^* \times S, \sigma \in F_{w,s}}, \{A^w(R)\}_{w \in S^*, R \in P_w})$ interpreting each sort s as a set A_s, each operation symbol $\sigma \in F_{w,s}$ as a function $A^w_s(\sigma) : A_w \to A_s$ (where A_w stands for $A_{s_1} \times \ldots \times A_{s_n}$ if $w = s_1 \ldots s_n$), and each relation symbol $R \in P_w$ as a relation $A^w(R) \subseteq A_w$. When there is no danger of confusion we may let A_σ and A_R denote $A^w_s(\sigma)$ and $A^w(R)$ respectively. Morphisms between models are the usual Σ-homomorphisms, i.e., S-sorted functions that preserve the structure. The Σ-sentences are obtained from *atoms*, i.e., equality atoms $t_1 = t_2$, where $t_1, t_2 \in (T_F)_s$,[4] or relational atoms $R(t_1, \ldots, t_n)$, where $R \in P_{s_1 \ldots s_n}$ and

[4] T_F is the ground term algebra over F.

$t_i \in (T_F)_{s_i}$ for each $i \in \{1, \ldots, n\}$, by applying for a finite number of times:

- negation, conjunction, disjunction;
- universal or existential quantification over finite sets of constants.

Satisfaction is the usual first-order satisfaction and is defined using the natural interpretations of ground terms t as elements A_t in models A. The definitions of functors *Sen* and *Mod* on morphisms are the natural ones: for any signature morphism $\varphi : \Sigma \to \Sigma'$, $Sen(\varphi) : Sen(\Sigma) \to Sen(\Sigma')$ translates sentences symbol-wise, and $Mod(\varphi) : Mod(\Sigma') \to Mod(\Sigma)$ is the forgetful functor.

As shown in [16], *FOPL* has elementary diagrams in the institutional sense. However, we shall be interested in what is called "elementary diagram" according to the classical model theory terminology [11]; the latter are in fact the diagrams of a remarkable subinstitution of *FOPL*, which has the same signatures, sentences and models, but restricts the class of model morphisms to *elementary embeddings* only. We shall be more precise below.

Let $\Sigma = (S, F, P)$ be a signature in *FOPL* and let $A \xrightarrow{h} B$ be a model morphism in $Mod(\Sigma)$. Let $\Sigma_A = (S, F_A, P)$, where F_A extends F by adding, for each $s \in S$, all elements in A_s as constants of sort s; also, let A_A be the expansion of A to Σ_A which interprets each constant $a \in A_s$ as itself, for all $s \in S$. The signature Σ_B and the Σ_B-model B_B are defined similarly. Define $\iota_\Sigma(h) : Sen(\Sigma_A) \to Sen(\Sigma_B)$ to be the following: if $e \in Sen(\Sigma_A)$, then $\iota_\Sigma(h)(e)$ is obtained from e by symbol-wise translation, mapping:

- for all $s \in S$, each $a \in A_s$ into $h_s(a)$,
- each other symbol u that appears in e into u.

A morphism $A \xrightarrow{h} B$ in $Mod(\Sigma)$ is said to be an *elementary embedding* if, for each $e \in Sen(\Sigma_A)$, $A_A \models e$ iff $B_B \models \iota_\Sigma(h)(e)$. The term "embedding" is appropriate, since all the elementary embeddings are injective morphisms. It is well known, and can be easily seen, that the elementary embeddings form a broad subcategory of $Mod(\Sigma)$ and are preserved by reduct functors. Thus we have an "elementary" subinstitution of *FOPL*, denoted *ElFOPL*, which has all the structure identical to *FOPL*, just that the model morphisms are restricted to be elementary embeddings. We now define some elementary diagrams for *ElFOPL*:

Let $\Sigma = (S, F, P)$ be a *FOPL* signature and $A \in |Mod(\Sigma)|$. Then:

- $\Sigma_A = (S, F_A, P)$ and A_A were already indicated above;
- $E_A = (A_A)^* = \{e \in Sen(\Sigma_A) \mid A_A \models e\}$;

- $\Sigma \overset{\iota_\Sigma(A)}{\to} \Sigma_A$ is the signature inclusion;
- The functor $i_{\Sigma,A} : Mod(\Sigma_A, E_A) \to A/Mod(\Sigma)$ is defined: on objects, by $i_{\Sigma,A}(N') = (A \overset{h}{\to} N, N)$, where $N = N'\restriction_{\iota_\Sigma(A)}$ and, for each $s \in S$ and $a \in A_s$, $h_s(a) = N'_a$; on morphisms, by $i_{\Sigma,A}(f) = f$.

Let $\varphi : \Sigma = (S, F, P) \to \Sigma' = (S', F', P')$ be a *FOPL* signature morphism, $A \in |Mod(\Sigma)|$, $C \in |Mod(\Sigma')|$, and $h : A \to C\restriction_\varphi$ an elementary morphism in $Mod(\Sigma)$. Then the natural presentation morphism $\iota_\varphi(h) : (\Sigma_A, E_A) \to (\Sigma_C, E_C)$ from the definition of elementary diagrams is the following: if $e \in Sen(\Sigma_A)$, then $\iota_\varphi(h)(e)$ is obtained from e by symbol-wise translation, mapping:

- each $f \in F$ into $\varphi^{op}(f)$,
- each $R \in P$ into $\varphi^{rel}(R)$,
- for all $s \in S$, each $a \in A_s$ into $h_s(a) \in C_{\varphi^{sort}(s)}$,
- for all $s \in S$, each variable $x\!:\!s$ of sort s into a variable $x\!:\!\varphi^{sort}(s)$ of sort $\varphi^{sort}(s)$,
- each other symbol u that appears in e (e.g., logical connectives and quantifiers) into u.

It is routine to check that *ElFOPL*, together with the above structure, is an institution with elementary diagrams.

2. *PFOPL* - the institution of partial first-order predicate logic, an extension of *FOPL* whose signatures $\Sigma = (S, F, F', P)$ contain, besides relation and (total) operation symbols (in F and P), *partial* operation symbols too, in F'. Models of course interpret the symbols in F' as partial operations of appropriate ranks. Σ-model morphisms $h : A \to B$ are S-sorted functions which commute with the total operations and relations in the usual way, and with the partial operations $\sigma \in F'_{s_1 \dots s_n, s}$ in the following way: for each $(a_1, \dots, a_n) \in A_{s_1 \dots s_n}$, if $A_\sigma(a_1, \dots, a_n)$ is defined, then so is $B_\sigma(h_{s_1}(a_1), \dots, h_{s_n}(a_n))$, and in this case the latter is equal to $h_s(A_\sigma(a_1, \dots, a_n))$. Signature morphisms are allowed to map partial operation symbols to total operation symbols, but not *vice versa*. There exist three kinds of atoms: *relational* atoms just like at *FOPL*, *undefinedness* atoms $t\uparrow$, and *(strong) equality* atoms $t = t'$. A relational atom $R(t_1, \dots, t_n)$ holds in a model A when all terms t_i are defined and their interpretations A_{t_i} stay in relation A_R. The undefinedness $t\uparrow$ of a term t holds in a model A when the corresponding interpretation A_t of the term is undefined. The equality $t = t'$

holds when both terms are undefined or both terms are defined and equal. The sentences are obtained from atoms just like in the case of *FOPL*. Partial algebras (i.e., *PFOPL*-models over signatures with no relation symbols) and their applications were extensively studied in [41] and [8].

3. *IFOPL* - the institution of infinitary first-order logic, an infinitary extension of *FOPL*, which allows conjunctions on arbitrary sets of sentences. This logical system is known under the name $L_{\infty,\omega}$ [29, 28][5] and plays an important role in categorical logic.

The corresponding "elementary" subinstitutions of *IFOPL* and *PFOPL*, denoted *ElIFOPL* and *ElPFOPL*, as well as their diagrams, are defined similarly to the case of *FOPL*. For *ElIFOPL*, the definitions are identical, while for *PFOPL* they have to be incremented in the obvious way to consider the partial operation symbols too.

Notice that models in the above institutions are not required to have non-empty carriers on sorts. There are subtle issues in algebraic specifications (like the unconditional existence of free models) that plead for this approach, which departs from the (unsorted) algebraic tradition of assuming non-emptiness of carrier sets. However, it seems to be a habit taking the non-emptiness assumption when considering Craig interpolation, sometimes even within algebraic specification frameworks [44, 43, 6]. In what follows, we shall take the trouble of distinguishing between the two apparently very similar situations, and shall point out some differences w.r.t. RCP and CIP (see Corollaries 9 and 10) that give a technical explanation for the above mentioned habit. Let *ElFOPL'*, *ElIFOPL'*, *ElPFOPL'*, *FOPL'*, *IFOPL'*, *PFOPL'* denote the variations of *ElFOPL*, *ElIFOPL*, *ElPFOPL*, *FOPL*, *IFOPL*, *PFOPL* with the additional requirement that models have non-empty carriers on all sorts. Many relevant properties of the original institutions, like semi-exactness (hence weak model-semi-exactness), compactness etc., hold for their non-empty-carrier versions too. Also, for our future discussions about elementary chains, the non-emptiness assumption is irrelevant (see also the proof of Corollary 9). The only moment when important technical differences will come into the picture is occasioned by quantifications over signature morphisms.

[5] Actually, the mentioned books allow a more general form of signature, with infinitary operation- and relation- symbols too. The results of this paper cover the cases of such signatures too, as an interested reader could easily check.

3. Institutional Formulation of the Robinson Consistency Property

We fix an institution $\mathcal{I}$. Next, we state some logical properties regarding language translation, following a generalization originating in [49], on *arbitrary* squares of signature morphisms rather than inclusion squares.

DEFINITION 3. Let $\mathcal{S}$ be a commutative signature square

$$\begin{tikzcd} & \Sigma_1 \arrow[dr, "\varphi'_1"] & \\ \Sigma \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & \Sigma' \\ & \Sigma_2 \arrow[ur, "\varphi'_2"'] & \end{tikzcd}$$

$\mathcal{S}$ is said to be:

1. a *weak amalgamation square (w.a. square)*, if every two models $A_1 \in |Mod(\Sigma_1)|$ and $A_2 \in |Mod(\Sigma_2)|$ having the same reduct (i.e., such that $A_1 \upharpoonright_{\varphi_1} = A_2 \upharpoonright_{\varphi_2}$), have a common expansion (i.e., there exists $A' \in |Mod(\Sigma')|$ such that $A' \upharpoonright_{\varphi'_1} = A_1$ and $A' \upharpoonright_{\varphi'_2} = A_2$);
2. a *Craig interpolation square (CI square)*, if for every $E_1 \subseteq Sen(\Sigma_1)$ and $E_2 \subseteq Sen(\Sigma_2)$ such that $\varphi'_1(E_1) \models \varphi'_2(E_2)$, there exists $E \subseteq Sen(\Sigma)$ such that $E_1 \models \varphi_1(E)$ and $\varphi_2(E) \models E_2$.

Note that if the institution is weakly model-semi-exact, then any pushout of signatures is a weak amalgamation square. The CI property from above was defined [49] on arbitrary *pushout squares*. However we shall prefer to work, in the style of [18], under the slightly more general hypothesis of w.a. square. We next provide three candidates for the notion of Robinson square, two of them already defined in the literature:

DEFINITION 4. A commutative square as in the figure of Definition 3 is said to be:

1. a 1-*Robinson square*, if for every consistent theories $T_2 \subseteq Sen(\Sigma_2)$, $T_1 \subseteq Sen(\Sigma_1)$ and complete theory $T \subseteq Sen(\Sigma)$ such that φ_1, φ_2 are theory morphisms, it holds that $\varphi'_1(T_1) \cup \varphi'_2(T_2)$ is consistent;
2. a 2-*Robinson square*, if for every two models $A_1 \in |Mod(\Sigma_1)|$ and $A_2 \in |Mod(\Sigma_2)|$ such that $A_1 \upharpoonright_{\varphi_1} \equiv A_2 \upharpoonright_{\varphi_2}$, there exists $A' \in |Mod(\Sigma')|$ such that $A' \upharpoonright_{\varphi'_1} \equiv A_1$ and $A' \upharpoonright_{\varphi'_2} \equiv A_2$;

3. a 3-*Robinson square*, if for every two consistent theories $T_1 \subseteq Sen(\Sigma_1)$ and $T_2 \subseteq Sen(\Sigma_2)$ such that $\varphi_1^{-1}(T_1) \cup \varphi_2^{-1}(T_2)$ is consistent, it holds that $\varphi'_1(T_1) \cup \varphi'_2(T_2)$ is consistent;

Remark 5. *1. The converses of the [*2 *and* 3*]-Robinson properties in Definition 4 are always true:*

- *if for two models $A_1 \in |Mod(\Sigma_1)|$ and $A_2 \in |Mod(\Sigma_2)|$ there exists $A' \in |Mod(\Sigma')|$ such that $A'\upharpoonright_{\varphi'_1} \equiv A_1$ and $A'\upharpoonright_{\varphi'_2} \equiv A_2$, then $A_1\upharpoonright_{\varphi_1} \equiv A_2\upharpoonright_{\varphi_2}$;*
- *if $\varphi'_1(T_1) \cup \varphi'_2(T_2)$ is consistent, then so is $\varphi_1^{-1}(T_1) \cup \varphi_2^{-1}(T_2)$.*

2. If the institution is compact and admits negations and finite conjunctions, then the definition of 3*-Robinson square can be rewritten as follows (with the notations of Definition 4.(3)): if $\varphi'_1(T_1) \cup \varphi'_2(T_2)$ is not consistent then there exists $e \in \varphi_1^{-1}(T_1)$ such that $\neg e \in \varphi_2^{-1}(T_2)$.*

The 1-Robinson property was defined in [49] following a variant of the corresponding classical property in unsorted first-order logic. The 3-Robinson property follows the other equally used classical definition [53]. On the other hand, the differently looking 2-Robinson property, introduced in [47] for *preinstitutions* following an idea from [36, 37], was also called, for obvious reasons, the *elementary amalgamation* property [18]. In many institutions, the three Robinson properties, as well as the CI-property, are all equivalent.

Proposition 6. *Assume that $\mathcal{I}$ has negations and finite conjunctions and is compact. Then the following are equivalent for a commutative square $\mathcal{S}$:*

1. $\mathcal{S}$ is a 1*-Robinson square;*

2. $\mathcal{S}$ is a 2*-Robinson square;*

3. $\mathcal{S}$ is a 3*-Robinson square;*

4. $\mathcal{S}$ is a CI square.

Proof. Let $\mathcal{S}$ be a commutative square as in the figure of Definition 3.
(1) implies (2): Take $T = \{A_1\upharpoonright_{\varphi_1}\}^* = \{A_2\upharpoonright_{\varphi_2}\}^*$, $T_i = \{A_i\}^*$, $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. If $e \in T$, then $A_i\upharpoonright_{\varphi_i} \models e$, thus $A_i \models \varphi_i(e)$, thus $\varphi_i(e) \in T_i$. Hence φ_i is a theory morphism. Moreover, T, T_1, T_2 are complete, thus there exists a Σ-model $A' \models \varphi'_1(T_1) \cup \varphi'_2(T_2)$. But $A\upharpoonright_{\varphi'_i} \models T_i$, hence $A\upharpoonright_{\varphi'_i} \equiv A_i$.
(2) implies (1): Since T is complete, there exists a Σ-model A such that $\{A\}^* = T$. Let also $A_i \models T_i$, $i \in \{1, 2\}$. Since φ_i is a theory morphism, $A_i\upharpoonright_{\varphi_i} \models T$, thus $A_i\upharpoonright \varphi_i \equiv A$, $i \in \{1, 2\}$. Then there exists a Σ-model A' such that $A\upharpoonright \varphi'_i \equiv A_i$, hence $A\upharpoonright_{\varphi'_i} \models T_i$, hence $A \models \varphi'_i(T_i)$, $i \in \{1, 2\}$. Thus $\varphi'_1(T_1) \cup \varphi'_2(T_2)$ is consistent.

(1) equivalent to (4): Was proved in [49], Corollary 3.1.
(3) implies (4): First notice that in Definition 4.(3) the property of being a 3-Robinson square can be equivalently expressed not assuming T_1 and T_2 to be theories (but just sets of sentences), and considering $\varphi_1^{-1}(T_1^\bullet) \cup \varphi_2^{-1}(T_2^\bullet)$ instead of $\varphi_1^{-1}(T_1) \cup \varphi_2^{-1}(T_2)$.

Let now $E_1 \subseteq Sen(\Sigma_1)$ and $E_2 \subseteq Sen(\Sigma_2)$ such that $\varphi_1'(E_1) \models \varphi_2'(E_2)$. Fix $e_2 \in E_2$. We have $\varphi_1'(E_1) \models \varphi_2'(e_2)$, so $\varphi_1'(E_1) \cup \{\varphi_2'(\neg e_2)\}$ is inconsistent. Applying the 3-Robinson square property we obtain that $\varphi_1^{-1}(E_1^\bullet) \cup \varphi_2^{-1}(\{\neg e_2\}^\bullet)$ is also inconsistent, which implies, by compactness and finite conjunctions, the existence of a sentence $e \in Sen(\Sigma)$ such that $\varphi_1^{-1}(E_1^\bullet) \models e$ and $\varphi_2^{-1}(\{\neg e_2\}^\bullet) \models \neg e$. But $\varphi_1^{-1}(E_1^\bullet)$ and $\varphi_1^{-1}(\{e_2\}^\bullet)$ are closed, so $e \in \varphi_1^{-1}(E_1^\bullet)$ and $\neg e \in \varphi_2^{-1}(\{\neg e_2\}^\bullet)$, i.e. $E_1 \models \varphi_1(e)$ and $\neg e_2 \models \varphi_2(\neg e)$, the last equality being equivalent to $\neg e_2 \models \neg\varphi_2(e)$, and further to $\varphi_2(e) \models e_2$. Thus, for any $e_2 \in E_2$ we found an $e \in Sen(\Sigma)$ such that $E_1 \models \varphi_1(e)$ and $\varphi_2(e) \models e_2$. Let $E \subseteq Sen(\Sigma)$ be the set of all such e, for each $e_2 \in E_2$. Then $E_1 \models \varphi_1(E)$ and $\varphi_2(E) \models E_2$.
(4) implies (3): Let $T_1 \subseteq Sen(\Sigma_1)$ and $T_2 \subseteq Sen(\Sigma_2)$ be two theories such that $\varphi_1'(T_1) \cup \varphi_2'(T_2)$ is inconsistent. Using finite conjunctions and compactness, we find $\gamma_2 \in T_2$, such that $\varphi_1'(T_1) \cup \{\varphi_2'(\gamma_2)\}$ is inconsistent. Since $\mathcal{I}$ has negations, it follows that $\varphi_1'(T_1) \models \neg\varphi_2'(\gamma_2)$, that is, $\varphi_1'(T_1) \models \varphi_2'(\neg\gamma_2)$. By Craig interpolation, there exists $E \subseteq Sen(\Sigma)$ such that $T_1 \models \varphi_1(E)$ and $\varphi_2(E) \models \neg\gamma_2$. Hence, by compactness and finite conjunctions, $\varphi_2(e) \models \neg\gamma_2$, for some $e \in E^\bullet$; this means $\gamma_2 \models \neg\varphi_2(e)$, i.e., $\gamma_2 \models \varphi_2(\neg e)$. Furthermore, $\varphi_1(E^\bullet) \subseteq \varphi_1(E)^\bullet \subseteq T_1^\bullet = T_1$ so $T_1 \models \varphi_1(e)$. We have obtained $T_1 \models \varphi_1(e)$ and $T_2 \models \varphi_2(\neg e)$. Since T_1 and T_2 are theories, it holds that $e \in \varphi_1^{-1}(T_1)$ and $\neg e \in \varphi_2^{-1}(T_2)$, making $\varphi_1^{-1}(T_1) \cup \varphi_2^{-1}(T_2)$ inconsistent. ■

Since we shall only deal with institutions satisfying the hypotheses in Proposition 6, we can safely say *Robinson square* instead of *i-Robinson square*. However, we are going to use the property of 3-Robinson square.

We introduce a final technical concept. The following notion of *lifting isomorphisms* generalizes a similar one in [18], from signature morphisms, to signature squares. The intuition is that φ_1 and φ_2 *together* lift isomorphisms. Notice that the below definition does not use the morphisms φ_1' and φ_2'; we keep the "square" terminology just for uniformity.

Definition 7. A commutative square as in the figure of Definition 3 is said to *lift isomorphisms* if, for each $A_1 \in |Mod(\Sigma_1)|$ and $A_2 \in |Mod(\Sigma_2)|$ such that $A_1\upharpoonright_{\varphi_1}$ is isomorphic to $A_2\upharpoonright_{\varphi_2}$, there exist $B_1 \in |Mod(\Sigma_1)|$ and $B_2 \in |Mod(\Sigma_2)|$ such that:

- B_1 is isomorphic to A_1;
- B_2 is isomorphic to A_2;
- $B_1\upharpoonright_{\varphi_1}= B_2\upharpoonright_{\varphi_2}$.

4. The Robinson Consistency Theorem

THEOREM 8. (The Consistency Theorem) *We assume that the institution $\mathcal{I}$:*

- *has all the model morphisms preserving satisfaction, i.e., for each signature Σ'' and $A \to B$ in $Mod(\Sigma'')$, it holds that $\{A\}^* \subseteq \{B\}^*$,*
- *has elementary diagrams,*
- *has pushouts of signatures and is weakly model-semi-exact,*
- *has ω-colimits of models preserved by the reduct functors,*
- *admits (finite) conjunctions and negations,*
- *is compact.*[6]

Then any w.a. square (and in particular any pushout square) as in the figure of Definition 3, which lifts isomorphisms and, in addition, has the property:

- *the institution admits universal quantifications over morphisms of the forms $\iota_\Sigma(h)$ and $\iota_\Sigma(A)$ for each Σ-model morphism $A \xrightarrow{h} B$* [7] *(with the notations of elementary diagrams introduced in Section 2),*

is a Robinson square (hence a CI square).

PROOF. Let $\mathcal{S}$ be a w.a. square as in the figure of Definition 3 and $T_1 \subseteq Sen(\Sigma_1)$, $T_2 \subseteq Sen(\Sigma_2)$ be two theories. Denote $\Gamma_1 = \varphi_1^{-1}(T_1)$ and $\Gamma_2 = \varphi_2^{-1}(T_2)$. Γ_1 and Γ_2 are also theories. We assume that $\Gamma_1 \cup \Gamma_2$ is consistent and want to prove that $\varphi_1'(T_1) \cup \varphi_2'(T_2)$ is consistent. It suffices to find two models $M_1 \models T_1$ and $M_2 \models T_2$ such that $M_1\upharpoonright_{\varphi_1}= M_2\upharpoonright_{\varphi_2}$ (and then apply weak amalgamation to find the desired model M' of $\varphi_2'(T_1) \cup \varphi_2'(T_2)$). We first construct inductively two chains of models, as indicated below.

(1) We find a model $A_1 \models T_1$ such that $A_1\upharpoonright_{\varphi_1}\models \Gamma_2$. If such a model didn't exist, then $T_1 \cup \varphi_1(\Gamma_2)$ would be inconsistent, so, by compactness and the existence of finite conjunctions, $T_1 \cup \{\varphi_1(\gamma_2)\}$ would be inconsistent, for some $\gamma_2 \in \Gamma_2$. By the existence of negations, this would imply $T_1 \models \neg\varphi_1(\gamma_2)$, that is, $T_1 \models \varphi_1(\neg\gamma_2)$, so $\neg\gamma_2 \in \Gamma_1$, making $\Gamma_1 \cup \Gamma_2$ inconsistent, a contradiction.

[6] In particular, any institution which admits arbitrary conjunctions, such as *IFOPL*, is compact.

[7] Notice that this last condition is a local one, involving the fixed signature Σ of the considered square.

(2) We find $A_2 \models T_2$ and $A_1{\upharpoonright}_{\varphi_1} \xrightarrow{h} A_2{\upharpoonright}_{\varphi_2}$ in $Mod(\Sigma)$. Using the elementary diagrams, it suffices to find $B \models E_{(A_1{\upharpoonright}_{\varphi_1})}$ and $A_2 \models T_2$ such that $B{\upharpoonright}_{\iota_\Sigma(A_1{\upharpoonright}_{\varphi_1})} = A_2{\upharpoonright}_{\varphi_2}$. Moreover, it suffices to consider the pushout of signatures

$$\begin{tikzcd}
 & \Sigma_{(A_1{\upharpoonright}_{\varphi_1})} \arrow[dr, "u"] & \\
\Sigma \arrow[ur, "\iota_\Sigma(A_1{\upharpoonright}_{\varphi_1})"] \arrow[dr, "\varphi_2"'] & & \Sigma' \\
 & \Sigma_2 \arrow[ur, "v"'] &
\end{tikzcd}$$

and find, in $Mod(\Sigma')$, a model of $u(E_{(A_1{\upharpoonright}_{\varphi_1})}) \cup v(T_2)$.[8] If such a model didn't exist, making $u(E_{(A_1{\upharpoonright}_{\varphi_1})}) \cup v(T_2)$ inconsistent, then, using negations, finite conjunctions and compactness, we would find $e \in (E_{(A_1{\upharpoonright}_{\varphi_1})})^\bullet$ such that $v(T_2) \models u(\neg e)$. By Lemma 1, we would have $T_2 \models (\forall v)u(\neg e)$ (notice that the sentence $(\forall v)u(\neg e)$ exists in our institution according to Lemma 2.(1)). Furthermore, by Lemma 2.(2), $\varphi_2((\forall \iota_\Sigma(A_1{\upharpoonright}_{\varphi_1}))\neg e) \equiv (\forall v)u(\neg e)$, thus $\varphi_2((\forall \iota_\Sigma(A_1{\upharpoonright}_{\varphi_1}))\neg e) \in T_2^\bullet = T_2$, which means $(\forall \iota_\Sigma(A_1{\upharpoonright}_{\varphi_1}))\neg e \in \Gamma_2$. But $A_1{\upharpoonright}_{\varphi_1} \models \Gamma_2$, so $A_1{\upharpoonright}_{\varphi_1} \models (\forall \iota_\Sigma(A_1{\upharpoonright}_{\varphi_1}))\neg e$, contradicting the fact that $(A_1{\upharpoonright}_{\varphi_1})_{(A_1{\upharpoonright}_{\varphi_1})} \models e$.

(3) We find $B_1 \in |Mod(\Sigma_1)|$, $A_1 \xrightarrow{g} B_1$ in $Mod(\Sigma_1)$, and $A_2{\upharpoonright}_{\varphi_2} \xrightarrow{f} B_1{\upharpoonright}_{\varphi_1}$ in $Mod(\Sigma)$ such that $h; f = g{\upharpoonright}_{\varphi_1}$. It suffices to find $D_1 \in |Mod(\Sigma_{1_{A_1}}, E_{A_1})|$ and $D_2 \in |Mod(\Sigma_{(A_2{\upharpoonright}_{\varphi_2})}, E_{(A_2{\upharpoonright}_{\varphi_2})})|$ such that $D_1{\upharpoonright}_{\iota_{\varphi_1}(A_1{\upharpoonright}_{\varphi_1})} = D_2{\upharpoonright}_{\iota_\Sigma(h)}$. Indeed, let us first assume that we found such models D_1 and D_2. Then g would be $i^{-1}_{\Sigma_1, A_1}(D_1)$ and f would be $i^{-1}_{\Sigma,(A_2{\upharpoonright}_{\varphi_2})}(D_2)$. In order to prove that $h; f = g{\upharpoonright}_{\varphi_1}$, we apply the "functoriality" of ι and obtain that the below diagram is commutative:

$$\begin{tikzcd}
 & \Sigma_{1_{A_1}} & & \\
\Sigma_1 \arrow[ur, "\iota_{\Sigma_1}(A_1)"] & & \Sigma_{(A_1{\upharpoonright}_{\varphi_1})} \arrow[ul, "\iota_{\varphi_1}(A_1{\upharpoonright}_{\varphi_1})"'] \arrow[dr, "\iota_\Sigma(h)"] & \\
 & \Sigma \arrow[ul, "\varphi_1"] \arrow[ur, "\iota_\Sigma(A_1{\upharpoonright}_{\varphi_1})"'] \arrow[rr, "\iota_\Sigma(A_2{\upharpoonright}_{\varphi_2})"'] & & \Sigma_{(A_2{\upharpoonright}_{\varphi_2})}
\end{tikzcd}$$

[8] We actually applied here the converse of Robinson Consistency Property for the theories $E_{(A_1{\upharpoonright}_{\varphi_1})}$ and T_2; see Remark 5(1).

We now apply the naturality of i to get that the below diagram is commutative:

$$\begin{tikzcd}
 & Mod(\Sigma 1_{A_1}, E_{A_1}) \arrow[dl, "i_{\Sigma 1, A_1}"'] \arrow[dr, "-\upharpoonright_{\iota_{\varphi_1}(A_1\upharpoonright_{\varphi_1})}"] & & Mod(\Sigma_{(A_2\upharpoonright_{\varphi_2})}, E_{(A_2\upharpoonright_{\varphi_2})}) \arrow[dl, "-\upharpoonright_{\iota_\Sigma(h)}"'] \arrow[ddl, "i_{\Sigma,(A_2\upharpoonright_{\varphi_2})}"] \\
A_1/Mod(\Sigma_1) \arrow[dr, "-\upharpoonright_{\varphi_1}"'] & & Mod(\Sigma_{(A_1\upharpoonright_{\varphi_1})}, E_{(A_1\upharpoonright_{\varphi_1})}) \arrow[dl, "i_{\Sigma,(A_1\upharpoonright_{\varphi_1})}"] & \\
 & A_1\upharpoonright_{\varphi_1}/Mod(\Sigma) & A_2\upharpoonright_{\varphi_2}/Mod(\Sigma) \arrow[l, "h/Mod(\Sigma)"] &
\end{tikzcd}$$

Then, since $D_1\upharpoonright_{\iota_{\varphi_1}(A_1\upharpoonright_{\varphi_1})} = D_2\upharpoonright_{\iota_\Sigma(h)}$, we have $i_{\Sigma,(A_1\upharpoonright_{\varphi_1})}(D_1\upharpoonright_{\iota_{\varphi_1}(A_1\upharpoonright_{\varphi_1})})$ $= i_{\Sigma,(A_1\upharpoonright_{\varphi_1})}(D_2\upharpoonright_{\iota_\Sigma(h)})$, so $(i_{\Sigma_1,A_1}(D_1))\upharpoonright_{\varphi_1} = h; i_{\Sigma,(A_2\upharpoonright_{\varphi_2})}(D_2)$, that is, $g\upharpoonright_{\varphi_1} = h; f$.

Now let us come back: we need to prove the existence of two models $D_1 \in |Mod(\Sigma 1_{A_1}, E_{A_1})|$ and $D_2 \in |Mod(\Sigma_{(A_2\upharpoonright_{\varphi_2})}, E_{(A_2\upharpoonright_{\varphi_2})})|$ with a common reduct to $\Sigma_{A_1\upharpoonright_{\varphi_1}}$, or, sufficiently, with a common expansion to Σ_0, where

$$\begin{tikzcd}
 & \Sigma 1_{A_1} \arrow[dr, "u"] & \\
\Sigma_{A_1\upharpoonright_{\varphi_1}} \arrow[ur, "\iota_{\varphi_1}(A_1\upharpoonright_{\varphi_1})"] \arrow[dr, "\iota_\Sigma(h)"'] & & \Sigma_0 \\
 & \Sigma_{(A_2\upharpoonright_{\varphi_2})} \arrow[ur, "v"'] &
\end{tikzcd}$$

is a pushout of signatures. Let us assume that there are no such models, i.e., that $u(E_{A_1}) \cup v(E_{(A_2\upharpoonright_{\varphi_2})})$ is not consistent. We again invoke negations, conjunctions and compactness to find $e \in (E_{(A_2\upharpoonright_{\varphi_2})})^\bullet$ such that $u(E_{A_1}) \models v(\neg e)$. Similarly to step 2, we apply Lemma 1 to get $E_{A_1} \models (\forall u)v(\neg e)$. This implies $A1_{A_1} \models (\forall u)v(\neg e)$. By Lemma 2.(1), $(\forall u)v(\neg e) \equiv \iota_{\varphi_1}(A_1)((\forall \iota_\Sigma(h))\neg e)$, hence $A1_{A_1} \models \iota_{\varphi_1}(A_1)((\forall \iota_\Sigma(h))\neg e)$, hence $A1_{A_1}\upharpoonright_{\iota_{\varphi_1}(A_1)} \models (\forall \iota_\Sigma(h))\neg e$. Because of the naturality of ι, we have that $A1_{A_1}\upharpoonright_{\iota_{\varphi_1}(A_1)} = (A_1\upharpoonright_{\varphi_1})_{(A_1\upharpoonright_{\varphi_1})}$. We obtain $(A_1\upharpoonright_{\varphi_1})_{(A_1\upharpoonright_{\varphi_1})} \models (\forall \iota_\Sigma(h))\neg e$. Since, like any model morphism, $(A_1\upharpoonright_{\varphi_1})_{(A_1\upharpoonright_{\varphi_1})} \stackrel{h_{(A_1\upharpoonright_{\varphi_1}),h}}{\rightarrow} (A_2\upharpoonright_{\varphi_2})_h$ preserves satisfaction, we have $(A_2\upharpoonright_{\varphi_2})_h \models$

$(\forall \iota_\Sigma(h))\neg e$, contradicting the fact that $(A_2\upharpoonright_{\varphi_2})_{(A_2\upharpoonright_{\varphi_2})}$, a $\iota_\Sigma(h)$-expansion of $(A_2\upharpoonright_{\varphi_2})_h$, satisfies e (remember that $e \in (E_{(A_2\upharpoonright_{\varphi_2})})^\bullet$).

(4) We reuse the technique of step 3 in order to find $B_2 \in |Mod(\Sigma_2)|$, $A_2 \xrightarrow{s} B_2$ in $Mod(\Sigma_2)$, and $B_1\upharpoonright_{\varphi_1} \xrightarrow{p} B_2\upharpoonright_{\varphi_2}$ in $Mod(\Sigma)$ such that $f;p = s\upharpoonright_{\varphi_2}$. Applying this a countable number of times we obtain two ω-diagrams Ch_1 and Ch_2:

$$A_1^0 \xrightarrow{f_1^0} A_1^1 \xrightarrow{f_1^1} A_1^2 \xrightarrow{f_1^2} A_1^3 \dots \quad \text{in} \quad Mod(\Sigma_1)$$

$$A_2^0 \xrightarrow{f_2^0} A_2^1 \xrightarrow{f_2^1} A_2^2 \xrightarrow{f_2^2} A_2^3 \dots \quad \text{in} \quad Mod(\Sigma_2)$$

and the following infinite commutative diagram Dg in $Mod(\Sigma)$ (which is in fact an ω-diagram too):

$$\begin{tikzcd}
A_1^0\upharpoonright_{\varphi_1} \arrow[rr, "f_1^0\upharpoonright_{\varphi_1}"] \arrow[dr, "h_0"'] & & A_1^1\upharpoonright_{\varphi_1} \arrow[rr, "f_1^1\upharpoonright_{\varphi_1}"] \arrow[dr, "h_1"] & & A_1^2\upharpoonright_{\varphi_2} \arrow[dr, "h_2"] & & \\
& A_2^0\upharpoonright_{\varphi_1} \arrow[ur, "g_0"] \arrow[rr, "f_2^0\upharpoonright_{\varphi_2}"] & & A_2^1\upharpoonright_{\varphi_2} \arrow[ur, "g_1"] \arrow[rr, "f_2^1\upharpoonright_{\varphi_2}"] & & A_2^2\upharpoonright_{\varphi_2} & \dots
\end{tikzcd}$$

where $A_1^0 = A_1$, $A_2^0 = A_2$, $A_1^1 = B_1$, $A_2^1 = B_2$, $f_0^1 = g$, $f_0^2 = s$, $h_0 = h$, $g_0 = f$, $h_1 = p$, ...

Because the reduct functors preserve ω-colimits, the colimits of Ch_1 and Ch_2 in $Mod(\Sigma_1)$ and $Mod(\Sigma_2)$, with vertexes denoted N_1 and N_2, are mapped by $Mod(\varphi_1)$ and $Mod(\varphi_2)$ into colimits in $Mod(\Sigma)$ of the ω-diagrams $Mod(\varphi_1)(Ch_1)$ and $Mod(\varphi_2)(Ch_2)$. But $Mod(\varphi_1)(Ch_1)$ and $Mod(\varphi_2)(Ch_2)$ are final segments of the ω-diagram Dg, so $N_1\upharpoonright_{\varphi_1}$ and $N_2\upharpoonright_{\varphi_2}$ are, both, colimit vertexes of Dg in $Mod(\Sigma)$. Hence $N_1\upharpoonright_{\varphi_1}$ and $N_2\upharpoonright_{\varphi_2}$ are isomorphic. On the other hand, since model morphisms preserve satisfaction and $A_1^0 \models T_1$, $A_2^0 \models T_2$, it follows that $N_1 \models T_1$ and $N_2 \models T_2$. Because $\mathcal{S}$ lifts isomorphisms, we find two models M_1 and M_2 such that $M_1 \simeq N_1$ and $M_2 \simeq N_2$ (thus $M_1 \models T_1$ and $M_2 \models T_2$) and $M_1\upharpoonright_{\varphi_1} = M_2\upharpoonright_{\varphi_2}$. ■

Among the hypotheses in the above Consistency Theorem, all look quite natural, except for two of them:

- that of satisfaction preservation by the model morphisms and
- that of the square lifting isomorphisms.

While the second is just a technical assumption, the first one is rather interesting; since $\mathcal{I}$ has negations, it implies that any two models connected through a morphism are elementary equivalent. This seems like a harsh thing to ask; but this requirement is normal if the considered model morphisms are something like elementary embeddings, thus preserving satisfaction of all "parameterized sentences", in particular of all "plain" ones. Institutions tend to have "elementary" subinstitutions; and those who have, can import the Consistency Theorem from there.

COROLLARY 9. *In each of the institutions ElFOPL', ElIFOPL', ElPFOPL', FOPL', IFOPL', PFOPL', any weak amalgamation square which lifts isomorphisms is a Robinson square (hence also a CI square).*

PROOF. We first claim that the institutions *ElFOPL'*, *ElIFOPL'*, and *ElPFOPL'* satisfy the conditions in Theorem 8. Let us check these conditions for *ElFOPL'*.

The elementary diagrams for *ElFOPL* were discussed in Section 2. It is straightforward to see that the same construction works for *ElFOPL'* too. The existence of pushout of signatures and compactness are well-known for *ElFOPL*, and are immediately inherited by *ElFOPL'*. Weak model-semi-exactness holds for *FOPL* because this institution is actually semi-exact (and even exact); and since the property only refers to models, common to *FOPL* and *ElFOPL*, it follows that *ElFOPL* is also weakly model-semi-exact; moreover, it is easy to see that, given a pushout of *FOPL* signatures $(\Sigma_2 \overset{\varphi_2}{\leftarrow} \Sigma \overset{\varphi_1}{\rightarrow} \Sigma_1, \Sigma_2 \overset{\varphi_2'}{\rightarrow} \Sigma' \overset{\varphi_1'}{\leftarrow} \Sigma_1)$ and two models $M_1 \in |Mod(\Sigma_1)|$, $M_2 \in |Mod(\Sigma_2)|$ such that $M_1{\upharpoonright}_{\varphi_1} = M_2{\upharpoonright}_{\varphi_2}$, if M_1 and M_2 have non-empty carriers on all sorts, then their common expansion $M' \in |Mod(\Sigma')|$ can be chosen with non-empty carriers on all sorts too; hence *ElFOPL'* is also weakly model-semi-exact.

The next discussion about elementary chains is valid for *ElFOPL'* (as a routine generalization of the unsorted case, with the carrier non-emptiness assumption imported from there), but also for *ElFOPL*. The existence of ω-colimits of signatures follows from Tarski's Elementary Chain Theorem (ECT) [11]. Let Σ be a signature and $(h_{i,j} : A_i \to A_j)_{i,j \in I\!N}$, denoted Dg, a diagram in $Mod_{ElFOPL'}(\Sigma)$, that is, with all morphisms $h_{i,j}$ being elementary embeddings. We can take the colimit of Dg in the category of model embeddings, let it be $(h_i : A_i \to A)_{i \in I\!N}$. According to ECT, all the h_i's are elementary. Moreover, $(h_i : A_i \to A)_{i \in I\!N}$ is actually the colimit of Dg in $Mod_{ElFOPL'}(\Sigma)$ too. Indeed, let $(g_i : A_i \to B)_{i \in I\!N}$ be another cocone of Dg in $Mod_{ElFOPL'}(\Sigma)$. According to the definition of Dg, there exists a unique embedding $f : A \to B$ such that $h_i; f = g_i$ for all $i \in I\!N$; but f is also elementary, since each parameterized sentence $e(a_1, \ldots, a_n)$ with parameters in A has all the parameters a_j, $j \in \{1, \ldots, n\}$, of the forms $h_i(b_j)$ for some large enough $i \in I\!N$; hence $A_A \models e(a_1, \ldots, a_n)$ iff $A_{iA_i} \models e(b_1, \ldots, b_n)$ iff $B_B \models e(g_i(b_1), \ldots, g_i(b_n))$ iff $B_B \models e(f(a_1), \ldots, f(a_n))$. Now, reduct functors along signature morphisms preserve elementary embeddings, as one can easily check; hence, because reduct functors preserve ω-colimits of embeddings in *FOPL'*, applying again ECT, we find that reduct functors preserve ω-colimits of models even when only elementary embeddings are taken into consideration as morphisms between models.

We shall finally check the existence of quantifications over $\iota_\Sigma(h)$, where $\Sigma = (S, F, P)$ is a signature and $A \xrightarrow{h} B$ is an elementary embedding. This time, our discussion is valid only for *ElFOPL'*. Let e be a sentence in $Sen(\Sigma_B)$. In order to show that $(\forall \iota_\Sigma(h))e$ is (equivalent to) a first-order sentence, let Σ' be the signature which

- includes the image $\iota_\Sigma(h)(\Sigma_A)$ of $\iota_\Sigma(h)$ (which is a copy of Σ_A included in Σ_B)
- and contains, for each $s \in S$, as extra constants of sort s all the elements in B_s that are not in the image of h_s and appear in e.

Since e is finitary, the extra constants are in finite number, and thus, if we consider the natural injective signature morphisms $\Sigma_A \xrightarrow{j} \Sigma' \xrightarrow{u} \Sigma_B$, where $j; u = \iota_\Sigma(h)$, we have the following:
(1) u is an inclusion of signatures, thus $Sen(\Sigma') \subseteq Sen(\Sigma_B)$, and $e \in Sen(\Sigma')$; moreover, like any signature inclusion, u is conservative (remember that all models are assumed to have non-empty carriers on each sort);
(2) $(\forall j)e$ is (equivalent to) a first-order sentence, because j is an injective signature morphism adding only a finite number of constants, all of which appearing in e;
(3) $(\forall \iota_\Sigma(h))e$ is equivalent to $(\forall j)e$. Indeed, "$(\forall j)e$ implies $(\forall \iota_\Sigma(h))e$" obviously holds. Conversely, assume $M \models (\forall \iota_\Sigma(h))e$ and let M' be a j-expansion of M. By the conservativeness of u, there exists M'' a u-expansion of M'. M'' is also a $\iota_\Sigma(h)$-expansion of M and $M'' \models e$. Thus $M'' \models u(e)$, i.e., $M' \models e$. Hence $M \models (\forall j)e$.

A very similar argument as the one above, but simpler, can be used to show the existence of quantifications over signature morphisms of the form $\iota_\Sigma(A)$.

The above arguments can be easily adapted to *ElIFOPL'* and *ElPFOPL'*. (For a proof of the Elementary Chain Theorem which can be adapted to *IFOPL'*, see [25], and for an institutional proof, which covers the cases of *IFOPL'* and *PFOPL'*, see [23].)

The conclusion of Theorem 8 involves only items (signature morphisms) which are the same in *FOPL'*, *IFOPL'*, and *PFOPL'* as in their "elementary" subinstitutions - so *FOPL'*, *IFOPL'*, and *PFOPL'* enjoy this property too. ■

Corollary 10. *Let $\mathcal{I}$ be one of the institutions FOPL, IFOPL, PFOPL, ElFOPL, ElIFOPL, ElPFOPL, and let $\mathcal{S}$ be a w.a. square in $\mathcal{I}$ as in the figure of Definition 3. Then $\mathcal{S}$ is a Robinson square if one of the following conditions holds:*

1. *$\mathcal{I}$ is one of FOPL, ElFOPL and the set $\{s \in S \mid T_{F_s} = \emptyset\}$ is finite,*[9] *where $\Sigma = (S, F, P)$.*
2. *$\mathcal{I}$ is one of PFOPL, ElPFOPL and the set $\{s \in S \mid T_{F_s} = \emptyset\}$ is finite, F being the set of* total *operation symbols of Σ;*
3. *$\mathcal{I}$ is one of IFOPL, ElIFOPL.*

Proof. The only delicate issue, different from the situation in Corollary 9, is in each case the existence of universal quantifications over $\iota_\Sigma(h)$ and $\iota_\Sigma(A)$. (1): Recall statements (1)-(3) from the proof of the fact that the institution *FOPL'* admits universal quantifications over $\iota_\Sigma(h)$ in Corollary 9. Using the same notations, but working in *FOPL* instead of *FOPL'*, we get that u is still conservative because: it is injective, all items outside its image are constants, and these are on sorts where some constants already existed (since, by the elementarity of h, for each sort s, A_s is empty iff B_s is empty); the rest of the argument for $\iota_\Sigma(h)$ is just like at Corollary 9.

The only problem left is the existence of universal quantification over $\iota_\Sigma(A)$. Let $e \in Sen(\Sigma_A)$. Similarly as before, we factor $\iota_\Sigma(A)$ as $u'; u$, where: $u' : \Sigma \to \Sigma'$ and $u : \Sigma' \to \Sigma_A$ are inclusions of signatures, and Σ' has only finitely many constants outside the image of u'. Then $(\forall u')e$ is equivalent to a first-order sentence, denoted e'. Define $\overline{S} = \{s \in S \mid A_s \neq \emptyset \text{ and } (T_F)_s = \emptyset\}$. $(\forall \iota_\Sigma(A))e$ is then equivalent to the first-order sentence $[\bigvee_{s \in \overline{S}} \neg(\exists x : s)x = x] \vee e'$, denoted e''. Indeed, let M be a Σ-model. We have two cases:
Case 1: There exists $s \in \overline{S}$ such that $M_s = \emptyset$. Then $M \models e''$ and, since M does not have any $\iota_\Sigma(A)$-expansion, M vacuously satisfies $(\forall \iota_\Sigma(A))e$.
Case 2: For each $s \in \overline{S}$, $M_s \neq \emptyset$. Assume first that $M \models e''$; then $M \not\models [\bigvee_{s \in \overline{S}} \neg(\exists x : s)x = x]$, so $M \models (\forall u')e$; let M'' be a $\iota_\Sigma(A)$-expansion of M; then $M'' \models e$ because $M''\!\upharpoonright_u$, as a u'-expansion of M, satisfies e. Conversely, assume that $M \models (\forall \iota_\Sigma(A))e$ and let M' be a u'-expansion of M; because

[9] This covers the cases of S being finite and of T_{F_s} being non-empty for each $s \in S$.

$M'_s \neq \emptyset$ for each $s \in S$, M' has a u-expansion M''; but $M'' \models e$, so $M' \models e$; thus, $M' \models (\forall u')e$, which implies $M \models e''$.
(2): Similar to (1).
(3): Identical to (1). Note that here we do not need finiteness of $\overline{S}$ in order to take the disjunction $\bigvee_{s \in \overline{S}} \neg(\exists x : s)x = x$. ■

5. A Syntactic Criterion for *FOPL* Robinson Consistency

We are going to use Corollaries 9 and 10 in order to prove a very general syntactic criterion for a *FOPL* or *FOPL'* signature square to be a Robinson square. By a "syntactic criterion" we mean one which uses only the structure of signature and signature morphisms, not involving the semantic concept of a model. Since in practice one usually deals with finite signatures, a syntactic criterion is easily checkable in an automatic fashion.

Let us consider either a *FOPL*-, or a *FOPL'*-, weak amalgamation square $\mathcal{S}$ as in the figure of Definition 3, with $\Sigma = (S, F, P)$, $\Sigma_1 = (S_1, F_1, P_1)$, $\Sigma_2 = (S_2, F_2, P_2)$, $\Sigma' = (S', F', P')$. If we take it to be a *FOPL* square, we also assume that $\{s \in S \mid T_{F\,s} = \emptyset\}$ is finite.

Without loss of generality, we assume that, within each signature, the sets of operation and relation symbols of different ranks are disjoint. That is, for each $w, w' \in S^*$ and $s, s' \in S$

- $(w, s) \neq (w', s')$ implies $F_{w,s} \cap F'_{w',s'} = \emptyset$;
- $w \neq w'$ implies $P_w \cap P'_{w'} = \emptyset$;
- $F_{w,s} \cap P_{w'} = \emptyset$;
- and similarly for Σ_1, Σ_2, Σ'.

We let φ_1 denote the extension of $S \xrightarrow{\varphi_1} S_1$ to $S^* \to S_1^*$; also we let $\varphi_1(w, s)$ denote the pair $(\varphi_1(w), \varphi_1(s))$ for each $(w, s) \in S^* \times S$; and similarly for φ_2.

Proposition 11. *$\mathcal{S}$ is a Robinson square (and also a Craig square) if the following four conditions hold:*
(C_1) For each $w, w' \in S^*$, $s, s' \in S$, $\sigma \in F_{w,s}$, $\sigma' \in F_{w',s'}$ such that $(w, s) \neq (w', s')$,
$[\varphi_1(w, s) = \varphi_1(w', s')$ and $\varphi_1(\sigma) = \varphi_1(\sigma')]$ implies $[\varphi_2(w, s) = \varphi_2(w', s')$ and $\varphi_2(\sigma) = \varphi_2(\sigma')]$.
(C_2) For each $w, w' \in S^*$, $s, s' \in S$, $\sigma \in F_{w,s}$ such that $(w, s) \neq (w', s')$,
$\varphi_1(w, s) = \varphi_1(w', s')$ implies the existence of $\sigma' \in F_{w',s'}$ such that $\varphi_1(\sigma) = \varphi_1(\sigma')$.
(C'_1) For each $w, w' \in S^*$, $R \in P_w$, $R' \in P_{w'}$ such that $w \neq w'$,

$[\varphi_1(w) = \varphi_1(w')$ and $\varphi_1(R) = \varphi_1(R')]$ implies $[\varphi_2(w) = \varphi_2(w')$ and $\varphi_2(R) = \varphi_2(R')]$.
(C_2') For each $w, w' \in S^*$, $R \in P_w$, such that $w \neq w'$,
$\varphi_1(w) = \varphi_1(w')$ implies the existence of $R' \in P_{w'}$ such that $\varphi_1(R) = \varphi_1(R')$.

Proof. By Corollaries 9 and 10, it is sufficient to prove that $\mathcal{S}$ lifts isomorphisms, that is: if $A_1 \in |Mod(\Sigma_1)|$, $D_2 \in |Mod(\Sigma_2)|$ such that $A_1\restriction_{\varphi_1} \simeq D_2\restriction_{\varphi_2}$, then there exist $B_1 \in |Mod(\Sigma_1)|$, $B_2 \in |Mod(\Sigma_2)|$ such that $A_1 \simeq B_1$, $D_2 \simeq B_2$, and $B_1\restriction_{\varphi_1} = B_2\restriction_{\varphi_2}$.

We are going to construct two models B_1 and B_2 as above. In our construction and throughout the proof, we shall totally ignore the relational part of the signatures, concentrating on operations. On the relational part, the situation is perfectly similar, using conditions (C_1') and (C_2') instead of (C_1) and (C_2).

We first take B_2 to be isomorphic to D_2 such that $card(B_{2s}) = card(B_{2s'})$ implies $B_{2s} = B_{2s'}$ for all $s, s' \in S_1$. Denote $A = A_1\restriction_{\varphi_1}$ and $B = B_2\restriction_{\varphi_2}$. Since $A \simeq D$ and $B_2 \simeq D_2$, we have $A \simeq B$. Let $A \xrightarrow{g} B$ be an isomorphism between A and B. By the construction of B_2, whenever $s, s' \in S$ with $\varphi_1(s) = \varphi_1(s')$, we have $B_s = B_{s'}$ (because, if $\varphi_1(s) = \varphi_1(s')$, then $card(B_s) = card(A_s) = card(A_{\varphi_1(s)}) = card(A_{s'}) = card(B_{s'})$, hence $card(B_{2\varphi_2(s)}) = card(B_{2\varphi_2(s')})$, hence $B_{2\varphi_2(s)} = B_{2\varphi_2(s')}$, hence $B_s = B_{s'}$).

We now define B_1.

1. Let the S_1-sorted set B_1 be:
   - $B_{1\varphi_1(s)} = B_s$, for each $s \in S$ (according to the above discussion, the definition of $B_{s'}$ with $\varphi_1(s) = s'$ does not depend on the choice of s)
   - $B_{1s'} = A_{1s'}$, for each $s' \in S_1 - \varphi_1(S)$
2. Fix $\theta : \varphi_1(S) \to S$ a "choice" function such that, for each $s' \in \varphi_1(S)$, $\varphi_1(\theta(s')) = s'$.
3. Let the S-sorted function $h : A_1 \to B_1$ be:
   - $h_{s'} : A_{1s'} \to B_{1s'}$, $h_{s'} = 1_{A_{1s'}}$, for each $s' \in S_1 - \varphi_1(S)$;
   - $h_{s'} : A_{1s'} \to B_{1s'}$, $h_{s'} = g_{\theta(s')}$, for each $s' \in \varphi_1(S)$ (notice that, if $s \in S$, $h_{\varphi_1(s)} = g_{\theta(\varphi_1(s))}$). h is obviously an S-sorted bijection.
4. Define a Σ_1-structure on B_1 by copying it through h from A_1: for each $w \in S_1^*$, $s \in S_1$, $\sigma \in F_{1w,s}$, $z \in B_{1w}$, let $B_{1\sigma}(z) = h_s^{-1}(A_{1\sigma}(h_w(z)))$.

Obviously, B_1 is a Σ_1-model and $A_1 \xrightarrow{h} B_1$ is a Σ_1-isomorphism. All we need to show is that $B_1\restriction_{\varphi_1} = B$.

1. On sorts: if $s \in S$, then $B_{1\varphi_1(s)} = B_s$ by definition.

2. On operations: let $w' \in S^*$, $s' \in S$, $\sigma' \in F_{w',s'}$. Let $w = \theta(\varphi_1(w'))$ and $s = \theta(\varphi_1(s'))$, where $\theta : \varphi_1(S)^* \to S^*$ is the symbol-wise extension of $\theta : \varphi_1(S) \to S$. Because $\varphi_1(w) = \varphi_1(w')$ and $\varphi_1(s) = \varphi_1(s')$, we have $B_{1\varphi_1(w')} = B_w = B_{w'}$ and $B_{1\varphi_1(s')} = B_s = B_{s'}$, so the operations $B_{1\varphi_1(\sigma')}$ and $B_{\sigma'}$ (which we want to prove equal) have the same domain and codomain. There are two cases:

   Case 1: $(w, s) = (w', s')$. Then, by definition, $B_{1\varphi_1(\sigma')}$ is the copy through $(h_{\varphi_1(w)}, h_{\varphi_1(s)})$, that is, through (g_w, g_s), of $A_{1\varphi_1(\sigma')}$; but, since g is a Σ-isomorphism, $B_{\sigma'}$ is also the copy through (g_w, g_s) of $A_{\sigma'} = A_{1\varphi_1(\sigma')}$. Hence $B_{\sigma'} = B_{1\varphi_1(\sigma')}$.

   Case 2: $(w', s') \neq (w, s)$. Since $\varphi_1(w', s') = \varphi_1(w, s)$ and $\sigma' \in F_{w',s'}$, we apply (C_2) to get $\sigma \in F_{w,s}$ such that $\varphi_1(\sigma) = \varphi_1(\sigma')$. Moreover, by (C_1), $\varphi_2(w, s) = \varphi_2(w', s')$ and $\varphi_2(\sigma) = \varphi_2(\sigma')$. Thus $A_\sigma = A_{\sigma'}$ and $B_\sigma = B_{\sigma'}$. $B_{1\varphi_1(\sigma')} : B_w \to B_s$ is, by definition the copy through (g_w, g_s) of $A_{1\varphi_1(\sigma')} = A_{\sigma'} = A_\sigma$; so $B_{1\varphi_1(\sigma')} = B_\sigma = B_{\sigma'}$. ■

REMARK 12. *1. A consequence of $(C_1) + (C_2)$ is a conditional kernel inclusion between φ_1 and φ_2: if $(w, s), (w', s') \in S^* \times S$ are such that $F_{w,s} \cup F_{w',s'} \neq \emptyset$, then $\varphi_1(w, s) = \varphi_1(w', s')$ implies $\varphi_2(w, s) = \varphi_2(w', s')$.*

2. The criterion from Proposition 11 is indeed syntactical, because the property of being a weak amalgamation square is syntactically describable: a square is a w.a. square iff it is a composition between a pushout square and a conservative signature morphism; furthermore, a signature morphism $\varphi : (S, F, P) \to (S', F', P')$ is conservative iff it is injective on [sort, operations and relation] symbols and, for each $s \in S$, $(T_F)_s = \emptyset$ iff $(T_{F'})_{\varphi(s)} = \emptyset$; and pushout squares are also syntactically describable.

COROLLARY 13. *If either φ_1 or φ_2 in $\mathcal{S}$ is injective on sorts, then $\mathcal{S}$ is a Robinson (and also Craig) square.*

PROOF. If φ_1 is injective on sorts, then all the conditions in Proposition 11 are trivially true, since it is never the case that $[(w, s) \neq (w', s')$ and $\varphi_1(w, s) = \varphi_1(w', s')]$, or $[w \neq w'$ and $\varphi_1(w) = \varphi_1(w')]$.

The case of φ_2 injective on sorts is perfectly symmetric to the previous one, thus the result follows from the symmetry of each of the two properties "w.a. square" and "Robinson square". ■

Note that Corollary 13 also has a direct proof from Corollaries 9 and 10, since $\mathcal{S}$ lifts isomorphisms whenever φ_1 or φ_2 is injective on sorts.

Example 14. Let $\mathcal{S}$ be the commutative *FOPL*-square as in the figure of Definition 3, defined as follows: $\Sigma = (\{s_1, s_2\}, \{d_1 :\to s_1, d_2 :\to s_2\})$, $\Sigma_1 = (\{s\}, \{d_1, d_2 :\to s\})$, $\Sigma_2 = (\{s\}, \{d :\to s\})$, $\Sigma' = (\{s\}, \{d :\to s\})$, all the morphisms mapping all sorts into s, φ_1 mapping d_1 and d_2 into themselves, and all the other morphisms mapping all the operation symbols into d. In [6], it is shown that $\mathcal{S}$ is not a CI square. To see this, let $E_1 = \{\neg(d_1 = d_2)\}$ and $E_2 = \{\neg(d = d)\}$. Then obviously $\varphi_1'(E_1) \models \varphi_2'(E_2)$, but E_1 and E_2 have no Σ-interpolant. Indeed, assume that there exists a set E of Σ-sentences such that $E_1 \models \varphi_1(E)$ and $\varphi_2(E) \models E_2$; let A be the Σ_1-model with $A_s = \{0, 1\}$, such that $A_{d_1} = 0$ and $A_{d_2} = 1$. Let B denote $A\upharpoonright_{\varphi_1}$. We have that $B_{s_1} = B_{s_2} = \{0, 1\}$, $B_{d_1} = 0$, $B_{d_2} = 1$. Because $A \models E_1$ and $E_1 \models \varphi_1(E)$, it holds that $B \models E$. Define the Σ-model C similarly to B, just that one takes $C_{d_1} = C_{d_2} = 0$. Now, C and B are isomorphic (notice that a and b are constants of *different* sorts in Σ), so $C \models E$; but C admits a φ_2-expansion D, and, because $\varphi_2(E) \models E_2$, $D \models E_2$, which is a contradiction, since no Σ_2-model can satisfy $\neg(d = d)$. According to Proposition 6, $\mathcal{S}$ is not a Robinson square either. The problem with this square, as depicted in [6], is that it has signature morphisms which are non-injective on sorts. In the light of Corollary 13, we can be more precise: the problem is that *none* of φ_1 and φ_2 is injective on sorts. Even more precisely, according to Corollary 10, $\mathcal{S}$ does not lift isomorphisms; in particular, it does not lift the unique isomorphism between $B = A\upharpoonright_{\varphi_1}$ and $C = D\upharpoonright_{\varphi_2}$ above.

Remark 15. *Proposition 11 (and hence Corollary 13 too) holds for IFOPL and IFOPL' with the same proof. Also, if we duplicate in Proposition 11 the conditions (C1) and (C2) to account separately for total and partial operation symbols, we obtain a similar criterion for PFOPL and PFOPL', with an almost identical proof.*

6. Related Work and Concluding Remarks

On Robinson Consistency

Robinson Consistency is broadly studied in connection with CIP, compactness, and other logical properties in a series of papers among which [36, 37, 35], in the framework offered by *abstract model-theoretic logics* [3]. An interesting phenomenon discovered there is that RCP implies compactness, which does not seem to hold for the more abstract case of institutions. No proof of RCP “from scratch” (i.e., without assuming CIP) is given there; moreover, as it is the case of all works within abstract model-theoretic logics, only the situation of language inclusions is considered. The paper [49] formulates for the first time an institutional version of RCP and proves its equivalence to CIP in compact institutions admitting negations and finite conjunctions. Another formulation of RCP, inspired by the one from [36, 37], is given in [47] in the context of preinstitutions, where there is also proved the equivalence of RCP with CIP assuming, instead of compactness and finite conjunctions, a rather strong property called *elementary expansion*. Recent work in [1] states RCP and gives some equivalent formulations for it using a syntactic notion of consistency of a theory, by not requiring the theory to have a model, but to not entail every sentence - this definition has the advantage that makes non-trivial sense for equational logics too.

Our Theorem 8 seems to be the first generalization of the Robinson Consistency Theorem to a fairly abstract logical framework. However, our result is “RCP-specific” only w.r.t. its proof technique, and not to its content, since it assumes some hypotheses under which RCP is equivalent to CIP.

On Craig Interpolation

After the original formulation and proof given in [13] for the unsorted first-order logic, several generalizations occur in the literature, among which that of [22] for many-sorted first-order logic, in the case of union and intersection of languages. However, the conclusion of studying various model-theoretic logics that extend first-order logic was that ”interpolation is indeed [a] rare [property in logical systems]” ([3], page 68). The paper [44] proves CIP for many-sorted equational logic, stating interpolation on *sets of sentences* instead of sentences. The first institutional formulation of CIP appears in [49] and uses arbitrary pushout squares of signatures. In [43], some general axiomatizability-based criteria are provided for a pullback of categories in order to satisfy a property which generalizes CIP when the categories are instanciated to classes of models over some signatures; this result covers the cases of many versions of equational logic. Another general result, proving CIP about institutions admitting Birkhoff-style axiomatizability and covering cases beyond equational logic, can be found in [18]. Moreover, [47] and [19] provide means of transporting CIP across translations of logics (institution morphisms and comorphisms). A stronger but more specialized result, concerned only with the many-sorted first-order logic and its partial-operation variant (which are the underlying logics in many specification languages, including CASL [12]), can be found in [6], where it is proved that if both starting pushout morphisms are *injective on sort names*, then CIP holds for the considered pushout square.

The present paper brings the following contributions from the CIP point of view:

1. Our Consistency Theorem 8 solves an open problem raised in [49], showing that CIP holds in institutions with additional requirements very similar to those of *abstract algebraic institutions*. This result complements the one in [18], which derives CIP from Birkhoff-like axiomatizability properties assumed on the classes of models and hence works particularly well for logics with strong axiomatizability properties; instead, our result is suitable for sufficiently expressive logics, not requiring axiomatizability, but needing appropriate machinery for the method of diagrams.
2. Our Corollary 13 (see also Remark 15) improves the result in [6] (the strongest known so far for *FOPL*), by showing that *only one* of the pushout morphisms needs to be injective on sorts in order for CIP to hold.
3. In fact, our Proposition 11, significantly more general than Corollary 13, pushes the syntactic criterion for CIP to a form which we think is close to the limit (i.e., to an "iff" criterion).
4. Finally, our interpolation results for the infinitary logical system $L_{\infty,\omega}$ seem to be new and of potential interest in categorical logic.

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