Low lying spectral gaps induced by slowly varying magnetic fields

Radu Purice
"Simion Stoilow" Institute of Mathematics of the Romanian Academy

work done in colaboration with Horia Cornean & Bernard Helffer

Metz, May 16-18, 2017

"Spectral Theory and Mathematical Physics"

UNIVERSITÉ DE LORRAINE

Abstract

Consider a periodic Schrödinger operator in two dimensions, perturbed by a weak magnetic field whose intensity slowly varies around a positive mean. We show in great generality that the bottom of the spectrum of the corresponding magnetic Schrödinger operator develops spectral islands separated by gaps, reminding of a Landau-level structure.

H. D. Cornean, B. Helffer, R. Purice: Low lying spectral gaps induced by slowly varying magnetic fields, **Journal of Functional Analysis 273** (1), 2017, pp. 206-282
http://dx.doi.org/10.1016/j.jfa.2017.04.002

Structure of the bottom of the spectrum of periodic Hamiltonians in slowly varying magnetic fields

1. The Problem
2. Main Steps of the Proof
3. The magnetic ΨD calculus.
4. Magnetic almost-Wannier Functions
5. The Feshbach type argument
6. The magnetic quantization of the magnetic quasi Bloch function
7. The resolvent of $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$

The Problem

The periodic Hamiltonian.

- On the configuration space $\mathcal{X} := \mathbb{R}^2$ we consider a lattice $\Gamma \subset \mathcal{X}$ generated by two linearly independent vectors $\{e_1, e_2\} \subset \mathcal{X}$.
- We also consider a smooth, Γ-periodic potential $V : \mathcal{X} \to \mathbb{R}$.
- Let us fix an *elementary cell*:

$$E := \Big\{ y = \sum_{j=1}^{2} t_j e_j \in \mathbb{R}^2 \mid -1/2 \leq t_j < 1/2 \,,\ \forall j \in \{1,2\} \Big\}.$$

- We consider the quotient group $\mathcal{X}/\Gamma$ that is canonically isomorphic to the 2-dimensional torus $\mathbb{T}$.
- Consider the differential operator $-\Delta + V$, which is essentially self-adjoint on the Schwartz set $\mathscr{S}(\mathcal{X})$.
  Denote by H^0 its self-adjoint extension in $\mathcal{H} := L^2(\mathcal{X})$ with domain the Sobolev space $\mathscr{H}^2(\mathcal{X})$.

The Bloch-Zak representation.

- The dual basis $\{e_1^*, e_2^*\} \subset \mathcal{X}^*$ is defined by $\langle e_j^*, e_k\rangle = (2\pi)\delta_{jk}$, and

$$\Gamma_* = \oplus_{j=1}^2 \mathbb{Z} e_j^*, \quad \mathbb{T}_* := \mathcal{X}^*/\Gamma_*,$$

$$E_* := \left\{\theta = \sum_{j=1}^{2} t_j e_j^* \in \mathbb{R}^2 \mid -1/2 \leq t_j < 1/2\,,\ \forall j \in \{1,2\}\right\}.$$

- The map

$$(\mathscr{V}_\Gamma \varphi)(\theta, x) := |E|^{-1/2} \sum_{\gamma\in\Gamma} e^{-i<\theta, x-\gamma>} \varphi(x-\gamma)\,, \quad \forall\, (\theta, x) \in \mathcal{X} \times E_*, \forall\, \varphi$$

(where $|E|$ is the Lebesgue measure of the elementary cell E) induces a unitary operator $\mathscr{V}_\Gamma : L^2(\mathcal{X}) \to L^2\big(E_*; L^2(\mathbb{T})\big)$.

- Its inverse is given by:

$$(\mathscr{V}_\Gamma^{-1}\psi)(x) = |E_*|^{-\frac{1}{2}} \int_{E_*} e^{i<\theta, x>} \psi(\theta, x)\, d\theta\,.$$

The Bloch-Floquet theory.

- $\hat{H}^0 := \mathscr{V}_\Gamma H^0 \mathscr{V}_\Gamma^{-1} = \int_{E_*}^{\oplus} \hat{H}^0(\theta) d\theta$
  with $\hat{H}^0(\theta) := \big(-i\nabla - \theta\big)^2 + V$ in $L^2(\mathbb{T})$.
- The map $E_* \ni \theta \mapsto \hat{H}^0(\theta)$ has an extension to $\mathcal{X}^*$ that is analytic in the norm resolvent topology and is given by
$$\hat{H}^0(\theta + \gamma^*) = e^{i<\gamma^*,\cdot>} \hat{H}^0(\theta) e^{-i<\gamma^*,\cdot>}.$$
- There exists a family of continuous functions $E_* \ni \theta \mapsto \lambda_j(\theta) \in \mathbb{R}$ with periodic continuous extensions to $\mathcal{X}^* \supset E_*$, indexed by $j \in \mathbb{N}$ such that $\lambda_j(\theta) \leq \lambda_{j+1}(\theta)$ for every $j \in \mathbb{N}$ and $\theta \in E_*$, and
$$\sigma\big(\hat{H}^0(\theta)\big) = \bigcup_{j\in\mathbb{N}} \{\lambda_j(\theta)\}.$$
- There exists an orthonormal family of measurable eigenfunctions $E_* \ni \theta \mapsto \hat{\phi}_j(\theta,\cdot) \in L^2(\mathbb{T})$, $j \in \mathbb{N}$, such that $\|\hat{\phi}_j(\theta,\cdot)\|_{L^2(\mathbb{T})} = 1$ and $\hat{H}^0(\theta)\hat{\phi}_j(\theta,\cdot) = \lambda_j(\theta)\hat{\phi}_j(\theta,\cdot)$.

The first Bloch band.

W. Kirsch, B. Simon, Comm. Math. Phys. 97 (1985)

The first Bloch eigenvalue $\lambda_0(\theta)$ is always simple in a neighborhood of $\theta = 0$ and has a nondegenerate global minimum on E_* at $\theta = 0$.

- Up to a shift in energy we may take this minimum to be equal to zero.
- Because H^0 has a real symbol, we have $\overline{\hat{H}^0(\theta)} = \hat{H}^0(-\theta)$.
- Since $\lambda_0(\cdot)$ is simple, it must be an even function $\lambda_0(\theta) = \lambda_0(-\theta)$.

Non-crossing Hypothesis.

H.1: Non-crossing condition with a gap.

$$\sup(\lambda_0) < \inf(\lambda_1).$$

or

H.2: Non-crossing condition with range overlapping and no gap.

The eigenvalue $\lambda_0(\theta)$ remains simple for all $\theta \in \mathbb{T}_*$, but $\sup(\lambda_0) \geq \inf(\lambda_1)$.

The magnetic field.

We consider a 2-parameter family of magnetic fields

$$B_{\epsilon,\kappa}(x) := \epsilon B_0 + \kappa\epsilon B(\epsilon x),$$

indexed by $(\epsilon,\kappa) \in [0,1] \times [0,1]$.

- $B_0 > 0$ is constant.
- $B : \mathcal{X} \to \mathbb{R}$ is smooth and bounded together with all its derivatives.

We choose some smooth *vector potentials* $A^0 : \mathcal{X} \to \mathcal{X}$ and $A : \mathcal{X} \to \mathcal{X}$ such that:

$$B_0 = \partial_1 A^0_2 - \partial_2 A^0_1 \,, B = \partial_1 A_2 - \partial_2 A_1 \,,$$

$$A^{\epsilon,\kappa}(x) := \epsilon A^0(x) + \kappa A(\epsilon x) \,,\ B_{\epsilon,\kappa} = \partial_1 A^{\epsilon,\kappa}_2 - \partial_2 A^{\epsilon,\kappa}_1 \,.$$

The vector potential A^0 is always in the *transverse gauge*, i.e.

$$A^0(x) = (B_0/2)\big(-x_2, x_1\big).$$

The magnetic periodic Hamiltonian.

We consider the following magnetic Schrödinger operator:

$$H^{\epsilon,\kappa} := (-i\partial_{x_1} - A_1^{\epsilon,\kappa})^2 + (-i\partial_{x_2} - A_2^{\epsilon,\kappa})^2 + V,$$

essentially self-adjoint on $\mathscr{S}(\mathcal{X})$.

When $\kappa = \epsilon = 0$ we recover the periodic Schrödinger Hamiltonian without magnetic field H^0.

The main result, for $B_{\epsilon,\kappa}(x) = \epsilon B_0 + \kappa\epsilon B(\epsilon x)$.

Theorem (H. Cornean, B. Helffer, R.P.: JFA (2017).)

Consider either Hypothesis H.1 or Hypothesis H.2. Fix an integer $N > 1$*. Then there exist some constants* $C_0, C_1, C_2 > 0$ *, and* $\epsilon_0, \kappa_0 \in (0,1)$ *, such that for any* $\kappa \in (0, \kappa_0]$ *and* $\epsilon \in (0, \epsilon_0]$ *there exist* $a_0 < b_0 < a_1 < \cdots < a_N < b_N$ *with* $a_0 = \inf\{\sigma(H^{\epsilon,\kappa})\}$ *so that:*

$$\sigma(H^{\epsilon,\kappa}) \cap [a_0, b_N] \subset \bigcup_{k=0}^{N} [a_k, b_k]\,, \quad \dim\big(\mathrm{Ran} E_{[a_k,b_k]}(H^{\epsilon,\kappa})\big) = +\infty\,,$$

$$b_k - a_k \leq C_0\, \kappa\epsilon + C_1\, \epsilon^{4/3}\,,\ 0 \leq k \leq N\,,$$

$$a_{k+1} - b_k \geq C_2^{-1} \epsilon\,,\ 0 \leq k \leq N-1\,.$$

Moreover, given any compact set $K \subset \mathbb{R}$*, there exists* $C > 0$ *, such that, for* $(\kappa, \epsilon) \in [0,1] \times [0,1]$ *, we have (here* dist_H *means Hausdorff distance):*

$$\mathrm{dist}_H\big(\sigma(H^{\epsilon,\kappa}) \cap K, \sigma(H^{\epsilon,0}) \cap K\big) \leq C\sqrt{\kappa\epsilon}.$$

The existing result, for $B_\epsilon(x) = \epsilon B_0$.

Theorem (B. Helffer, J. Sjöstrand, LNP 345 (1989).)

Suppose fixed some $E > 0$ *small enough. Under Hypothesis H.1,*
$\forall L \in \mathbb{N}^*$,
there exist $\epsilon_0 > 0$ *and* $C > 0$, *such that for* $\epsilon \in (0, \epsilon_0]$ *there exist* $N(\epsilon)$ *and* $a_0 < b_0 < \ldots < a_N < b_N$ *such that:*

- $a_0 = \inf\{\sigma(H^{\epsilon,0})\}$,
- $\sigma(H^{\epsilon,0}) \cap (-\infty, E) \subset \bigcup_{k=0}^{N}[a_k, b_k]$,
- $|b_k - a_k| \leq C\,\epsilon^L$ *for* $0 \leq k \leq N(\epsilon)$,
- $a_{k+1} - b_k \geq \epsilon/C$ *for* $0 \leq k \leq N(\epsilon) - 1$.

a_k *is determined by a Bohr-Sommerfeld rule* $a_k = f((2k+1)\epsilon, \epsilon)$,
where $t \mapsto f(t, \epsilon)$ *has a complete expansion in powers of* ϵ,
$f(0, 0) = \inf \lambda_0$ *and* $\partial_t f(0, 0) \neq 0$

Main Steps of the Proof

Step 1: Construction of an effective magnetic matrix. (A)

1. Being assumed to be isolated, we can associate with $\lambda_0(\theta)$ an orthonormal projection π_0 commuting with H^0.
   This might **not** be a spectral projection for H^0, unless there is a gap between the first band and the rest (Hypothesis H.1).
2. Results by Nenciu, Cornean-Helffer-Nenciu and Fiorenza-Monaco-Panati show that in both cases the range of π_0 has a basis consisting of exponentially localized Wannier functions.
3. When ϵ and κ are small enough, we can construct an orthogonal system of exponentially localized *magnetic almost Wannier functions* starting from the unperturbed Wannier basis of π_0; the corresponding orthogonal projection $\pi_0^{\epsilon,\kappa}$ is **almost invariant** for $H^{\epsilon,\kappa}$.
   In the case with a gap (H.1), $\pi_0^{\epsilon,\kappa}$ is a spectral projection for $H^{\epsilon,\kappa}$.

Step 1: Construction of an effective magnetic matrix. (B)

1. Using a *Feshbach-type argument*, we prove that the bottom of the spectrum of $H^{\epsilon,\kappa}$ is at a Hausdorff distance of order ϵ^2 from the spectrum of the reduced operator $\pi_0^{\epsilon,\kappa} H^{\epsilon,\kappa} \pi_0^{\epsilon,\kappa}$.
2. In the basis of magnetic almost Wannier functions, the reduced operator $\pi_0^{\epsilon,\kappa} H^{\epsilon,\kappa} \pi_0^{\epsilon,\kappa}$ defines an *effective magnetic matrix* acting on $\ell^2(\Gamma)$.

Conclusion 1.

If the *effective magnetic matrix* has spectral gaps of order ϵ, the same holds true for the bottom of the spectrum of $H^{\epsilon,\kappa}$.

Step 2: Replacing the magnetic matrix with a magnetic pseudodifferential operator with periodic symbol.

1. For $\kappa = 0$, i.e. for a constant magnetic field ϵB_0, we define a periodic *magnetic Bloch band function* λ^ϵ which is a perturbation of order ϵ of the first Bloch eigenvalue λ_0.
2. We define the *magnetic quantization* $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$ of this *magnetic Bloch band function* considered as a periodic symbol, in the magnetic field $B_{\epsilon,\kappa}$.
3. It turns out that the spectrum of $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$ is located at a Hausdorff distance of order $\kappa\epsilon$ from the spectrum of the effective operator $\pi_0^{\epsilon,\kappa} H^{\epsilon,\kappa} \pi_0^{\epsilon,\kappa}$.

Conclusion 2.

Hence if $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$ has gaps of order ϵ (provided that κ is smaller than some constant independent of ϵ), the same is true for the bottom of the spectrum of $H^{\epsilon,\kappa}$.

Step 3: Spectral analysis of $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$.

1. We compare the bottom of the spectrum of $\mathfrak{Op}^{A^{\epsilon,\kappa}}(\lambda^\epsilon)$ with the bottom of the spectrum of an unbounded quadratic symbol defined using the Hessian of λ^ϵ near its simple, isolated minimum; this is achieved by proving that the magnetic quantization of an explicitly defined symbol is in fact a *quasi-resolvent* for the magnetic quantization of λ^ϵ.
2. An important technical component is the development of a magnetic Moyal calculus for symbols with *weak spatial variation* that replaces the Moyal calculus for a constant field as appearing in the previous papers by Helffer and his coworkers.

The magnetic pseudodifferential calculus

The magnetic pseudodifferential calculus

- Let us denote by $\mathcal{X}^*$ the dual of $\mathcal{X}$ (*the momentum space*) with $\langle\cdot,\cdot\rangle : \mathcal{X}^* \times \mathcal{X} \to \mathbb{R}$ denoting the duality map.
- Let $\Xi := \mathcal{X} \times \mathcal{X}^*$ be *the phase space* with the canonical symplectic form
$$\sigma(X, Y) := \langle \xi, y\rangle - \langle \eta, x\rangle,$$
for $X := (x, \xi) \in \Xi$ and $Y := (y, \eta) \in \Xi^*$.

We consider the spaces $BC(\mathcal{V})$ of bounded continuous functions on any finite dimensional real vector space $\mathcal{V}$ with the $\|\cdot\|_\infty$ norm.

We shall denote by $C^\infty(\mathcal{V})$ the space of smooth functions on $\mathcal{V}$ and by $C^\infty_{\text{pol}}(\mathcal{V})$ (resp. by $BC^\infty(\mathcal{V})$) its subspace of smooth functions that are polynomially bounded together with all their derivatives, (resp. smooth and bounded together with all their derivatives), endowed with the usual locally convex topologies.

The magnetic pseudodifferential calculus

Given a vector potential A with components of class $C^\infty_{\text{pol}}(\mathcal{X})$, for any tempered distribution $F \in \mathscr{S}'(\Xi)$ we can associate the following linear operator (defined as oscillatory integral):

$$\mathscr{S}(\mathcal{X}) \ni u \mapsto \big(\mathfrak{Op}^A(F)u\big)(x) :=$$
$$= (2\pi)^{-2} \int_{\mathcal{X}} \int_{\mathcal{X}^*} e^{i\langle \xi, x-y\rangle} e^{-i\int_{[x,y]} A} F\left(\frac{x+y}{2}, \xi\right) u(y)\, d\xi\, dy.$$

Remark

The application $\mathfrak{Op}^A$ extends to a linear and topological isomorphism between $\mathscr{S}'(\Xi)$ and $\mathcal{L}\big(\mathscr{S}(\mathcal{X}); \mathscr{S}'(\mathcal{X})\big)$ (considered with the strong topologies).

The magnetic pseudodifferential calculus

The operator composition of the operators $\mathfrak{Op}^A(F)$ and $\mathfrak{Op}^A(G)$ induces a *magnetic Moyal product*, such that

$$\mathfrak{Op}^A(F)\,\mathfrak{Op}^A(G) = \mathfrak{Op}^A(F \sharp^B G).$$

This product depends only on the magnetic field B and is given by the following oscillating integrals:

$$\begin{aligned}(F \sharp^B G)(X) &:= \pi^{-4} \int_\Xi dY \int_\Xi dZ\, e^{-2i\sigma(Y,Z)} e^{-i \int_{T(x,y,z)} B} F(X-Y)\, G(X-Z) \\ &= \pi^{-4} \int_\Xi dY \int_\Xi dZ\, e^{-2i\sigma(X-Y,X-Z)} e^{-i \int_{\widetilde{T}(x,y,z)} B} F(Y)\, G(Z),\end{aligned}$$

where $T(x,y,z)$ is the triangle of vertices $x-y-z$, $x+y-z$, $x-y+z$ and $\widetilde{T}(x,y,z)$ the triangle in $\mathcal{X}$ of vertices $x-y+z$, $y-z+x$, $z-x+y$.

The magnetic pseudodifferential calculus

Hörmander type symbols

For any $s \in \mathbb{R}$ and any $\rho \in [0,1]$, we denote by

$$S^s_\rho(\Xi) := \left\{F \in C^\infty(\Xi) \mid \nu^{s,\rho}_{n,m}(F) < +\infty\,, \forall (n,m) \in \mathbb{N} \times \mathbb{N}\right\},$$

where $$\nu^{s,\rho}_{n,m}(f) := \sup_{(x,\xi)\in\Xi} \sum_{|\alpha|\leq n} \sum_{|\beta|\leq m} \left| \langle \xi \rangle^{-s+\rho m} \left(\partial^\alpha_x \partial^\beta_\xi f \right)(x,\xi) \right|.$$

$$S^\infty_\rho(\Xi) := \bigcup_{s\in\mathbb{R}} S^s_\rho(\Xi) \text{ and } S^{-\infty}(\Xi) := \bigcap_{s\in\mathbb{R}} S^s_\rho(\Xi).$$

Remark

For symbols of class $S^0_0(\Xi)$ the associated magnetic pseudodifferential operator is bounded in $\mathcal{H}$.

The magnetic pseudodifferential calculus

For any symbol F we denote by F_B^- its inverse with respect to the magnetic Moyal product, if it exists.

Proposition

1. For any $m > 0$ and for $a > 0$ large enough (depending on m) the symbol $\mathfrak{s}_m(x, \xi) :=< \xi >^m + a$, has an inverse for the magnetic Moyal product.
2. If $F \in S^0_\rho(\Xi)$ is invertible for the magnetic Moyal product, then the inverse F_B^- also belongs to $S^0_\rho(\Xi)$.
3. For $m < 0$, if $f \in S^m_\rho(\Xi)$ is such that $1 + f$ is invertible for the magnetic Moyal product, then $(1 + f)_B^- - 1 \in S^m_\rho(\Xi)$.
4. Let $m > 0$ and $\rho \in [0, 1]$. If $G \in S^m_\rho(\Xi)$ is invertible for the magnetic Moyal product, with $\mathfrak{Op}^A\big(\mathfrak{s}_m \sharp^B G_B^-\big) \in \mathcal{L}\big(L^2(\mathcal{X})\big)$, then $G_B^- \in S^{-m}_\rho(\Xi)$.

The magnetic pseudodifferential calculus

Definition

A symbol F in $S^s_\rho(\Xi)$ is called *elliptic* if there exist two positive constants R and C such that

$$|F(x,\xi)| \geq C\,\langle\xi\rangle^s,$$

for any $(x,\xi) \in \Xi$ with $|\xi| \geq R$.

Remark

For any real elliptic symbol $h \in S^m_1(\Xi)_\Gamma$ (with $m > 0$) and for any A in $C^\infty_{\text{pol}}(\mathcal{X},\mathbb{R}^2)$, the operator $\mathfrak{Op}^A(h)$ has a closure H^A in $L^2(\mathcal{X})$ that is self-adjoint on a domain $\mathcal{H}^m_A$ (a *magnetic Sobolev space*) and lower semibounded. Thus we can define its resolvent $(H^A - \mathfrak{z})^{-1}$ for any $\mathfrak{z} \notin \sigma(H^A)$ and it exists a symbol $r^B_{\mathfrak{z}}(h) \in S^{-m}_1(\Xi)$ such that

$$(H^A - \mathfrak{z})^{-1} \;=\; \mathfrak{Op}^A(r^B_{\mathfrak{z}}(h)).$$

Weak magnetic fields

Let $B_\epsilon := \epsilon\, B^0_\epsilon$, with $B^0_\epsilon \in BC^\infty(\mathcal{X})$ uniformly for $\epsilon \in [0, \epsilon_0]$.
Let H^ϵ be the self-adjoint extension of $\mathfrak{Op}^\epsilon(h)$ for an elliptic real symbol $h \in S^m_1(\Xi)$ with $m > 0$. For $\mathrm{z} \in \rho(H^\epsilon)$, let $r^\epsilon_{\mathrm{z}}(h) \in S^{-m}_1(\Xi)$ denote the symbol of $(H^\epsilon - \mathrm{z})^{-1}$.

Proposition

For any compact subset K of $\mathbb{C} \setminus \sigma(H)$, there exists $\epsilon_0 > 0$ such that:

1. $K \subset \mathbb{C} \setminus \sigma(H^\epsilon)$, for $\epsilon \in [0, \epsilon_0]$.
2. The following expansion is convergent in $\mathcal{L}(\mathcal{H})$ uniformly with respect to $(\epsilon, \mathrm{z}) \in [0, \epsilon_0] \times K$:
$$r^\epsilon_{\mathrm{z}}(h) = \sum_{n \in \mathbb{N}} \epsilon^n r_n(h; \epsilon, \mathrm{z}),\ r_0(h; \epsilon, \mathrm{z}) = r^0_{\mathrm{z}}(h),\ r_n(h; \epsilon, \mathrm{z}) \in S_1^{-(m+2n)}(\Xi).$$
3. The map $K \ni \mathrm{z} \mapsto r^\epsilon_{\mathrm{z}}(h) \in S^{-m}_1(\Xi)$ is a $S^{-m}_1(\Xi)$-valued analytic function, uniformly in $\epsilon \in [0, \epsilon_0]$.

Slowly varying symbols

Definition

For any $(m,\rho) \in \mathbb{R} \times [0,1]$ and for some $\epsilon_0 > 0$, we denote by $S^m_\rho(\Xi)^\bullet$ the families of symbols $\{F^\epsilon\}_{\epsilon \in [0,\epsilon_0]}$ satisfying the following properties:

1. $F^\epsilon \in S^m_\rho(\Xi)\,, \forall \epsilon \in [0,\epsilon_0]$;
2. $\exists \lim\limits_{\epsilon \searrow 0} F^\epsilon := F^0 \in S^m_\rho(\Xi)$ in the topology of $S^m_\rho(\Xi)$;
3. $\forall (\alpha,\beta) \in \mathbb{N}^2 \times \mathbb{N}^2,\ \exists C_{\alpha\beta} > 0$ such that
$$\sup_{\epsilon \in (0,\epsilon_0]} \epsilon^{-|\alpha|} \left\| \partial_x^\alpha \partial_\xi^\beta F^\epsilon \right\|_\infty \leq C_{\alpha\beta}.$$

The following seminorms indexed by $(p,q) \in \mathbb{N}^2$

$$F^\bullet \mapsto \tilde{\nu}^{m,\rho}_{p,q}(F^\bullet) := \sup_{\epsilon \in [0,\epsilon_0]} \epsilon^{-p} \sum_{|\alpha|=p} \sum_{|\beta|=q} \sup_{(x,\xi)\in\Xi} < \xi >^{-(m-q\rho)} \left| (\partial_x^\alpha \partial_\xi^\beta F^\epsilon)(x,\xi) \right|$$

define the topology on $S^m_\rho(\Xi)^\bullet$.

Slowly varying symbols

Remark

Defining $\widetilde{F}^\epsilon(x,\xi) := F^\epsilon(\epsilon^{-1}x,\xi)$, it is easy to see that $\{F^\epsilon\}_{\epsilon\in[0,\epsilon_0]} \subset S^m_\rho(\Xi)$ belongs to $S^m_\rho(\Xi)^\bullet$ if and only if it is of the form $F^\epsilon(x,\xi) = \widetilde{F}^\epsilon(\epsilon x,\xi)$ for some bounded family $\{\widetilde{F}^\epsilon\}_{\epsilon\in[0,\epsilon_0]} \subset S^m_\rho(\Xi)$ verifying the condition

$$\exists \lim_{\epsilon\searrow 0} \widetilde{F}^\epsilon(0,\cdot) := F^0 \in S^m_\rho(\Xi) \bigcap C^\infty_{\text{pol}}(\mathcal{X}^*).$$

Slowly varying symbols

Suppose $B_{\epsilon,\kappa}(x) = \epsilon B_0 + \kappa\epsilon B(\epsilon x)$.

Proposition

If $f^\bullet \in S^m_\rho(\Xi)^\bullet$ and $g^\bullet \in S^p_\rho(\Xi)^\bullet$, then $\{f^\epsilon \sharp^{B_{\epsilon,\kappa}} g^\epsilon\}_{\epsilon\in[0,\epsilon_0]}$ belongs to $S^{m+p}_\rho(\Xi)^\bullet$ uniformly with respect to $\kappa \in [0,1]$.

Proposition

If $f^\bullet \in S^m_\rho(\Xi)^\bullet$ and if the inverse $(f^\epsilon)^- \equiv (f^\epsilon)^-_{B_{\epsilon,\kappa}} \in S^{-m}_\rho(\Xi)$ exists for every $\epsilon \in [0,\epsilon_0]$, then $\{(f^\epsilon)^-\}_{\epsilon\in[0,\epsilon_0]} \in S^{-m}_\rho(\Xi)^\bullet$.

Magnetic almost-Wannier Functions

The Wannier functions.

Under any Hypothesis H.1 or H.2,

- the analyticity (in rezolvent norm) of the application $\mathcal{X}^* \ni \theta \to \hat{H}^0(\theta)$,
- the contour integral formula for the spectral projection,

allow one to define a L^2-normalized eigenfunction for the eigenvalue λ_0 as an analytic function $\mathcal{X}^* \ni \theta \to \hat{\phi}_0(\theta, \cdot) \in L^2(\mathbb{T})$ such that

$$\hat{\phi}_0(\theta + \gamma^*, x) = e^{i<\gamma^*,x>}\hat{\phi}_0(\theta, x),$$

$$\hat{H}^0(\theta)\,\hat{\phi}_0(\theta, \cdot) \;=\; \lambda_0(\theta)\,\hat{\phi}_0(\theta, \cdot).$$

Then the principal *Wannier function* ϕ_0 is defined by:

$$\phi_0(x) = \left[\mathscr{V}_\Gamma^{-1}\hat{\phi}_0\right](x) \;=\; |E_*|^{-\frac{1}{2}} \int_{E_*} e^{i<\theta,x>}\hat{\phi}_0(\theta, x)\, d\theta.$$

The Wannier functions.

ϕ_0 has rapid decay

$\exists C > 0$ such that $\forall \alpha \in \mathbb{N}^2$, $\exists C_\alpha > 0$ such that

$$|\partial_x^\alpha \phi_0(x)| \leq C_\alpha \exp(-|x|/C)\,,\ \forall x \in \mathbb{R}^2\,.$$

We shall also consider the associated orthogonal projections

$$\hat{\pi}_0(\theta) := |\hat{\phi}_0(\theta,\cdot) >< \hat{\phi}_0(\theta,\cdot)|\,,\quad \pi_0 := \mathscr{V}_\Gamma^{-1}\left(\int_{E_*}^{\oplus} \hat{\pi}_0(\theta)d\theta\right)\mathscr{V}_\Gamma.$$

Remark

The family $\{\phi_\gamma := \tau_{-\gamma}\phi_0\}_{\gamma\in\Gamma}$ is an orthonormal basis for $\pi_0\mathcal{H}$.

Remark

Under Hypothesis H.1, π_0 is the spectral projection associated to λ_0.

The magnetic almost-Wannier functions.

Let us consider first the constant magnetic field $B_\epsilon := \epsilon B_0$.

Definition

1. For any $\gamma \in \Gamma$ and with A^0 defined above we define:

$$\overset{\circ}{\phi}{}^\epsilon_\gamma(x) := \Lambda^\epsilon(x,\gamma)\phi_0(x-\gamma), \quad \Lambda^\epsilon(x,y) := \exp\left\{-i\,\epsilon\int_{[x,y]} A^0\right\}.$$

2. $\widetilde{\pi}_0^\epsilon$: the orthogonal projection on the closed linear span of $\{\overset{\circ}{\phi}{}^\epsilon_\gamma\}_{\gamma\in\Gamma}$.
3. $\mathbb{G}^\epsilon_{\alpha\beta} := \langle \overset{\circ}{\phi}{}^\epsilon_\alpha, \overset{\circ}{\phi}{}^\epsilon_\beta\rangle_{\mathcal{H}}$: the infinite *Gramian matrix*, indexed by $\Gamma\times\Gamma$.
4. $\mathbb{F}^\epsilon := \left(\mathbb{G}^\epsilon\right)^{-1/2}$.

The magnetic almost-Wannier functions.

Proposition

The matrix $\mathbb{G}^\epsilon$ defines a positive bounded operator on $\ell^2(\Gamma)$.
$\mathbb{F}^\epsilon$ has the following properties:

1. $\mathbb{F}^\epsilon \in \mathcal{L}(\ell^2(\Gamma)) \cap \mathcal{L}(\ell^\infty(\Gamma))$.
2. For any $m \in \mathbb{N}$, there exists $C_m > 0$ such that

$$\sup_{(\alpha,\beta)\in\Gamma\times\Gamma} < \alpha - \beta >^m \left|\mathbb{F}^\epsilon_{\alpha\beta} - 1\!\!1\right| \leq \mathrm{C_m}\,\epsilon\,,\ \forall\epsilon \in [0,\epsilon_0].$$

3. There exists a rapidly decaying function $\mathbf{F}_\epsilon : \Gamma \to \mathbb{C}$ such that for any pair $(\alpha,\beta) \in \Gamma \times \Gamma$ we have:

$$\mathbb{F}^\epsilon_{\alpha,\beta} = \Lambda^\epsilon(\beta,\alpha)\,\mathbf{F}_\epsilon(\beta - \alpha).$$

The magnetic almost-Wannier functions.

For all $\epsilon \in [0, \epsilon_0]$ we can define the following orthonormal basis of $\widetilde{\pi}_0^\epsilon \mathcal{H}$:

$$\phi_\gamma^\epsilon := \sum_{\alpha \in \Gamma} \mathbb{F}_{\alpha\gamma}^\epsilon \overset{\circ}{\phi}{}_\alpha^\epsilon, \forall \gamma \in \Gamma.$$

Proposition

With ψ_0^ϵ in $\mathscr{S}(\mathbb{R}^2)$ defined by

$$\psi_0^\epsilon(x) = \sum_{\alpha \in \Gamma} \mathbf{F}^\epsilon(\alpha)\, \Lambda^\epsilon(\alpha, x)\, \phi_0(x - \alpha),$$

we have

$$\phi_\gamma^\epsilon = \Lambda^\epsilon(\cdot, \gamma)(\tau_{-\gamma}\psi_0^\epsilon), \qquad \forall \gamma \in \Gamma.$$

The magnetic almost-Wannier functions.

Consider now $B_{\epsilon,\kappa} := \epsilon B_0 + \kappa\epsilon B(\epsilon x)$
let us choose some smooth vector potential $A(y)$ such that $dA = B$ and introduce

$$A_\epsilon(x) := A(\epsilon x) \text{ and } \widetilde{\Lambda}^{\epsilon,\kappa}(x,y) := \exp\left\{-i\kappa\int_{[x,y]} A_\epsilon\right\},$$

Definition

- $\overset{\circ}{\phi}{}^{\epsilon,\kappa}_{\gamma} = \widetilde{\Lambda}^{\epsilon,\kappa}(\cdot,\gamma)\phi^{\epsilon}_{\gamma}$.
- $\widetilde{\pi}^{\epsilon,\kappa}_0$ the orthogonal projection on the closed linear span of $\{\overset{\circ}{\phi}{}^{\epsilon,\kappa}_{\gamma}\}_{\gamma\in\Gamma}$.
- $\{\widetilde{\phi}^{\epsilon,\kappa}_{\gamma}\}_{\gamma\in\Gamma}$ the orthonormal basis of $\widetilde{\pi}^{\epsilon,\kappa}_0$ obtained from $\{\overset{\circ}{\phi}{}^{\epsilon,\kappa}_{\gamma}\}_{\gamma\in\Gamma}$ by the Gramm-Schmidt procedure.

The magnetic almost-Wannier functions.

The explicit form of the symbols of the projections $\widetilde{\pi}_0^{\epsilon,\kappa}$ and π_0 allow us to use the magnetic pseudodifferential calculus with slowly varying symbols in order to prove that:

Proposition

There exist $\epsilon_0 > 0$ and $C > 0$ such that, for any $(\epsilon, \kappa) \in [0, \epsilon_0] \times [0, 1]$, the range of $\widetilde{\pi}_0^{\epsilon,\kappa}$ belongs to the domain of $H^{\epsilon,\kappa}$ and

$$\left\| \left[H^{\epsilon,\kappa}, \widetilde{\pi}_0^{\epsilon,\kappa} \right] \right\|_{\mathcal{L}(\mathcal{H})} \leq C\,\epsilon.$$

Definition

We call *quasi-band magnetic Hamiltonian*, the operator $\widetilde{\pi}_0^{\epsilon,\kappa} H^{\epsilon,\kappa} \widetilde{\pi}_0^{\epsilon,\kappa}$ and *quasi-band magnetic matrix*, its form in the orthonormal basis $\{\widetilde{\phi}_\gamma^{\epsilon,\kappa}\}_{\gamma\in\Gamma}$.

The magnetic quasi Bloch function.

Definition

We define $\mathfrak{h}^\epsilon \in \ell^2(\Gamma)$ by:

$$\mathfrak{h}^\epsilon(\gamma) := \langle \psi_0^\epsilon\,,\, \Lambda^\epsilon(x,\gamma)\tau_{-\gamma}H^\epsilon\psi_0^\epsilon\rangle_{\mathcal{H}} = \left\langle \phi_0^\epsilon\,,\, H^\epsilon\phi_\gamma^\epsilon\right\rangle_{\mathcal{H}} \text{ for } \gamma\in\Gamma,$$

and the *magnetic quasi Bloch function* λ^ϵ as its discrete Fourier transform:

$$\lambda^\epsilon : \mathbb{T}_* \to \mathbb{R}, \qquad \lambda^\epsilon(\theta) := \sum_{\gamma\in\Gamma}\mathfrak{h}^\epsilon(\gamma)e^{-i<\theta,\gamma>}.$$

Proposition

There exists $\epsilon_0 > 0$ such that, for $\epsilon \in [0,\epsilon_0]$ and $\kappa \in [0,1]$, the Hausdorff distance between the spectra of the magnetic quasi-band Hamiltonian $\widetilde{\pi}_0^{\epsilon,\kappa}H^{\epsilon,\kappa}\widetilde{\pi}_0^{\epsilon,\kappa}$ and $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$ is of order $\kappa\epsilon$.

The Feshbach type argument

The Feshbach type argument

In order to apply a Feshbach type argument we need to control the invertibility on the orthogonal complement of $\widetilde{\pi}_0^{\epsilon,\kappa}\mathcal{H}$.
Let us define

1. $\widetilde{\pi}_\perp^{\epsilon,\kappa} := \mathbb{1} - \widetilde{\pi}_0^{\epsilon,\kappa}$.
2. $m_1 := \inf_{\theta\in\mathbb{T}_*} \lambda_1(\theta)$ where λ_1 is the second Bloch eigenvalue.
3. $K^{\epsilon,\kappa} := H^{\epsilon,\kappa} + m_1\,\widetilde{\pi}_0^{\epsilon,\kappa}$.

Proposition

There exist ϵ_0 and $C > 0$ such that, for $\epsilon \in [0, \epsilon_0]$, $K^{\epsilon,\kappa} \geq m_1 - C\epsilon > 0$.

Proposition

There exists $\epsilon_0 > 0$ such that for $\epsilon \in [0, \epsilon_0]$, the Hausdorff distance between the spectra of $H^{\epsilon,\kappa}$ and $\widetilde{\pi}_0^{\epsilon,\kappa} H^{\epsilon,\kappa} \widetilde{\pi}_0^{\epsilon,\kappa}$, both restricted to the interval $[0, \frac{m_1}{2}]$, is of order ϵ^2.

The magnetic quantization of the magnetic quasi Bloch function

Properties of the magnetic quasi Bloch function

Proposition

For the magnetic quasi Bloch function λ^ϵ defined above there exists $\epsilon_0 > 0$ such that $\lambda^\epsilon(\theta) = \lambda_0(\theta) + \epsilon\rho^\epsilon(\theta)$, with $\rho^\epsilon \in BC^\infty(\mathbb{T}_*)$ uniformly in $\epsilon \in [0, \epsilon_0]$ and such that $\rho^\epsilon - \rho^0 = \mathscr{O}(\epsilon)$.

- A consequence of this Proposition is that the modified Bloch eigenvalue $\lambda^\epsilon \in C^\infty(\mathcal{X}^*)$ also has an isolated non-degenerate minimum at some point $\theta^\epsilon \in \mathcal{X}^*$ ϵ-close to $0 \in \mathcal{X}^*$.
- λ_0 being an even function we get that in a neighborhood of $0 \in \mathbb{T}_*$ we have the expansions

$$\lambda_0(\theta) = \sum_{1\leq j,k\leq 2} a_{jk}\theta_j\theta_k + \mathscr{O}(|\theta|^4), \qquad a_{jk} := \big(\partial^2_{jk}\lambda_0\big)(0);$$

$$\lambda^\epsilon(\theta)-\lambda^\epsilon(\theta^\epsilon) = \sum_{1\leq j,k\leq 2} a^\epsilon_{jk}(\theta_j-\theta^\epsilon_j)(\theta_k-\theta^\epsilon_k)+\epsilon\mathscr{O}(|\theta-\theta^\epsilon|^3)+\mathcal{O}(|\theta-\theta^\epsilon|^4).$$

The Hessian at the minimum of the magnetic quasi Bloch function

- There exists $\epsilon_0 > 0$ such that, for $\epsilon \in [0, \epsilon_0]$, we can choose a local coordinate system on a neighborhood of $\theta^\epsilon \in \mathcal{X}^*$ that diagonalizes the symmetric positive definite matrix a^ϵ and we denote by $0 < m_1^\epsilon \leq m_2^\epsilon$ its eigenvalues.
- We denote by $0 < m_1 \leq m_2$ the two eigenvalues of the matrix a_{jk}.
- We notice that

$$m_j^\epsilon = m_j + \epsilon\mu_j + \mathscr{O}(\epsilon^2) \text{ for } j = 1, 2,$$

  with μ_j explicitly computable.

Our goal is to obtain spectral information concerning the Hamiltonian $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$ starting from the spectral information about $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$ with

$$h_{m^\epsilon}(\xi) := m_1^\epsilon \xi_1^2 + m_2^\epsilon \xi_2^2,$$

defining an elliptic symbol of class $S_1^2(\Xi)$ that does not depend on the configuration space variables.

The model Landau Hamiltonian

We compare the bottom of the spectra of the following two operators

- the magnetic Hamiltonians $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$,
- the constant field magnetic Landau operator $\mathfrak{Op}^{\epsilon,0}(h_{m^\epsilon})$.

Proposition

For any compact set M in $\mathbb{R}$, there exist $\epsilon_K > 0$, $C > 0$ and $\kappa_K \in (0,1]$, such that for any $(\epsilon,\kappa) \in [0,\epsilon_K] \times [0,\kappa_K]$, the spectrum of the operator $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$ in ϵM is contained in bands of width $C\kappa\epsilon$ centered at the points $\{(2n+1)\,\epsilon\, m^\epsilon\, B_0\}_{n\in\mathbb{N}}$.

The resolvent of $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)$

Isolating the minimum

- We choose an even function χ in $C_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, with $\operatorname{supp} \chi \subset (-2, +2)$ and $\chi(t) = 1$ on $[-1, +1]$.
- For $\delta > 0$ we define $g_{1/\delta}(\xi) := \chi\big(h_{m^\epsilon}(\delta^{-1}\xi)\big), \quad \xi \in \mathcal{X}^*$.
- We choose δ_0 such that $D(0, \sqrt{2m_1^{-1}\delta_0}) \subset \overset{\circ}{E}_*$ where $D(0,\rho)$ denotes the disk centered at 0 of radius ρ and $\overset{\circ}{E}_*$ denotes the interior of E_*.
- For any $\delta \in (0, \delta_0]$ we associate $\delta^\circ := \sqrt{m_1/2m_2}\,\delta$ so that we have $g_{1/\delta^\circ} = g_{1/\delta}\, g_{1/\delta^\circ}$.

For any $\delta \in (0, \delta_0]$, $g_{1/\delta} \in C_0^\infty(E_*)$.

- We may consider it as an element of $C_0^\infty(\mathcal{X}^*)$ by extending it by 0.
- We may define its Γ_*-*periodic continuation* to $\mathcal{X}^*$:

$$\widetilde{g}_{1/\delta}(\xi) := \sum_{\gamma \in \Gamma^*} g_{1/\delta}(\xi - \gamma),$$

The ϵ-dependent cut-off

Hypothesis

We shall impose the following scaling of the cut-off parameter $\delta > 0$:

$$\epsilon = \delta^3.$$

Then we have the following estimation near the minimum:

$$\lambda^\epsilon(\xi) g_{1/\delta}(\xi) = g_{1/\delta}(\xi)\, h_{m^\epsilon}(\xi) + \mathcal{O}(\delta^4), \quad \text{with } \delta^4 = \epsilon^{4/3}.$$

The *shift* outside the minimum

For the region outside the minima, we need the operator:

$$\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda^\epsilon + (\delta^\circ)^2 \widetilde{g}_{1/\delta^\circ}\big).$$

Proposition

There exists $\epsilon_0 > 0$ and for $(\epsilon, \kappa, \delta) \in [0, \epsilon_0] \times [0, 1] \times (0, \delta_0]$, there exist some constants $C > 0$ and $C' > 0$ such that:

$$\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda^\epsilon + (\delta^\circ)^2\, \widetilde{g}_{1/\delta^\circ}\big) \geq \big(C\,\delta^2 - C'\,\epsilon\big)\ \mathbb{1}.$$

Remark

We have that $C\,\delta^2 - C'\,\epsilon > C''\epsilon^{2/3} >> \epsilon$ and for $0 \leq z \leq C''\epsilon^{2/3}$, we denote by $r_{\delta,\epsilon,\kappa}(z)$ the symbol of $\Big(\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda^\epsilon + (\delta^\circ)^2\, \widetilde{g}_{1/\delta^\circ}\big) - z\mathbb{1}\Big)^{-1}$.

The *"quasi-inverse"*.

- Let us fix some compact set $K \subset \mathbb{C}$ such that:

$$K \subset \mathbb{C} \setminus \{(2n+1)m\,B_0\}_{n\in\mathbb{N}}.$$

- There exist $\epsilon_K > 0$ and $\kappa_K \in [0,1]$ such that for $(\epsilon,\kappa) \in [0,\epsilon_K] \times [0,\kappa_K]$ and for $a \in K$, the point $\epsilon a \in \mathbb{C}$ belongs to the resolvent set of $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$.
- We denote by $r^{\epsilon,\kappa}(\epsilon a)$ the magnetic symbol of $\big(\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon}) - \epsilon a\big)^{-1}$.

The quasi-inverse

For $a \in K$ we want to define the following symbol in $\mathscr{S}'(\mathcal{X}^*)$ as the sum of the series on the right hand side:

$$\widetilde{r}_\lambda(\epsilon a) := \sum_{\gamma^*\in\Gamma_*} \tau_{\gamma^*}\big(g_{1/\delta}\ \sharp^{\epsilon,\kappa}\ r^{\epsilon,\kappa}(\epsilon a)\big) + \big(1 - \widetilde{g}_{1/\delta}\big)\,\sharp^{\epsilon,\kappa}\, r_{\delta,\epsilon,\kappa}(\epsilon a)\,, \quad \delta = \epsilon^{1/3}.$$

The *"quasi-inverse"*.

Proposition

For K as above, there exist $C > 0$, $\kappa_0 \in (0,1]$ and $\epsilon_0 > 0$ such that for $(\kappa, \epsilon, a) \in [0, \kappa_0] \times (0, \epsilon_0] \times K$, the symbol $\widetilde{r}_\lambda(\epsilon a)$ is well defined and we have

$$\|\mathfrak{Op}^{\epsilon,\kappa}(\widetilde{r}_\lambda(\epsilon a))\| \leq C\epsilon^{-1},$$

and

$$\big(\lambda_\epsilon - \epsilon a\big) \, \sharp^{\epsilon,\kappa} \, \widetilde{r}_\lambda(\epsilon a) \; = \; 1 + \mathfrak{r}_{\delta,a}, \;\; \text{with} \quad \|\mathfrak{Op}^{\epsilon,\kappa}(\mathfrak{r}_{\delta,a})\| \; \leq \; C \, \epsilon^{1/3}.$$

For $N > 0$, there exist C, ϵ_0 and κ_0 such that
the spectrum of $\mathfrak{Op}^{\epsilon,\kappa}(\lambda_\epsilon)$ in $[0, (2N+2)mB_0\epsilon]$ consists of spectral islands centered at $(2n+1)mB_0\epsilon$, $0 \leq n \leq N$, with a width bounded by $C\,(\epsilon\kappa + \epsilon^{4/3})$.

Thank you for your attention.