Peirl's substitution at the bottom of the spectrum in the absence of Wannier functions

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Abstract

Consider a periodic Schrödinger operator in two dimensions, perturbed by a weak magnetic field whose intensity slowly varies in space. We show in great generality that the bottom of the spectrum of the corresponding magnetic Schrödinger operator develops spectral islands separated by gaps, reminding of a Landau-level structure.
In the spectral analysis of periodic pseudodifferential operators a very important ingredient is the existence of so-called *Wannier basis* for a spectral island of the operator. Nevertheless, in most generic situations the existence of such Wannier basis is either difficult to prove or even false.
In our work we provide a general method to obtain significant spectral information without Wannier basis, using some kind of *deformed Wannier basis* that are easy to construct in very general situations.

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Let us consider a periodic Scrödinger operator in dimension $d = 2$. More precisely:

- Suppose that in the 2-dimensional configuration space $\mathcal{X} \cong \mathbb{R}^2$ we are given a 2-dimensional regular lattice $\Gamma \cong \mathbb{Z}^2 \subset \mathcal{X}$.
- We consider the following differential operator:

$$H^0 = \sum_{1 \leq j \leq d} \big(-i\partial_j - A_j^\Gamma(x) \big)^2 + V_\Gamma(x)$$

  where
  - $A_j^\Gamma \in BC^\infty(\mathcal{X})$ and $A_j^\Gamma(x+\gamma) = A_j^\Gamma(x)$, $\forall (x,\gamma) \in \mathcal{X} \times \Gamma$,
  - $V_\Gamma \in BC^\infty(\mathcal{X})$ and $V_\Gamma(x+\gamma) = V_\Gamma(x)$, $\forall (x,\gamma) \in \mathcal{X} \times \Gamma$.

- We shall perturb it by a weak magnetic field of the form

$$B_{\epsilon,\kappa}(x) = \epsilon B_0 + \kappa\epsilon B(\epsilon x)$$

  with B_0 constant and B having components of class $BC^\infty(\mathcal{X})$.
- We denote by $H^{\epsilon,\kappa}$ the perturbed Hamiltonian.

Peierls substitution for non-smooth or topologically non-trivial Bloch projections

Notations & Reminder of Bloch-Floquet Theory

Notations:

- $\mathcal{X}^*$ the dual of $\mathcal{X}$ with generic elements $\xi, \eta, \zeta, \ldots$ and duality map $< \cdot, \cdot >: \mathcal{X}^* \times \mathcal{X} \to \mathbb{R}$
- $\Xi := \mathcal{X} \times \mathcal{X}^*$ the phase space with the canonical symplectic form $\sigma\big((x,\xi),(y,\eta)\big) :=< \xi, y > - < \eta, x >$.
- $\forall (F,z) \in \mathscr{S}'(\mathcal{X}) \times \mathcal{X}$, $\big(\mathcal{T}_z F\big)(\phi) := F(\phi \circ \tau_z)$, $\forall \phi \in \mathscr{S}(\mathcal{X})$, $\tau_z(x) := x + z \; \forall x \in \mathcal{X}$.
- Similar notations for translations on $\mathcal{X}^*$ and Ξ.
- $\mathcal{H} := L^2(\mathcal{X})$.
- H^0 and $H^{\epsilon,\kappa}$ are the unique self-adjoint extensions in $\mathcal{H}$ of the given periodic differential operators.

Then $\{\mathcal{T}_\gamma\}_{\gamma\in\Gamma}$ induces a unitary representation of $\mathbb{Z}^2$ on $\mathcal{H}$ that commutes with the self-adjoint operator H^0.
We may decompose this representation with respect to the family of irreducible representations of $\mathbb{Z}^d$ that are indexed by its dual group $\mathcal{X}^*/\Gamma_*$ where Γ^* is the dual lattice:

$$\Gamma_* \ := \ \{\xi \in \mathcal{X}^* \ \mid <\xi, \gamma>\in 2\pi\mathbb{Z}, \forall\gamma\in\Gamma\}$$

in order to obtain the *Bloch-Floquet representation*.

The non-trivial group extension structure.

We have
$\mathcal{X}/\Gamma \cong \mathcal{X}^*/\Gamma_* \cong [\mathbb{S}^1]^2 =: \mathbb{T} \cong \{\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2,\ |z_1| = |z_2| = 1\}$.

and the short exact sequence of topological groups
$0 \hookrightarrow \mathbb{Z}^2 \overset{\mathfrak{j}}{\hookrightarrow} \mathbb{R}^2 \overset{\mathfrak{e}^2}{\rightarrowtail} \mathbb{T} \rightarrowtail 1,$
with

- $\mathfrak{j} : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ the embedding map
- $\mathfrak{e}(t) := \exp(2\pi i t) \in \mathbb{S}^1 := \{z \in \mathbb{C},\ |z| = 1\}$ for any $t \in \mathbb{R}$
- $\mathfrak{s}(z) := (1/2\pi i)\ln(z) \in [-1/2, 1/2) \subset \mathbb{R}$
- and $\mathfrak{e}^2 := (\mathfrak{e}, \mathfrak{e}) : \mathbb{R}^2 \to \mathbb{T}$, $\mathfrak{s}^2 := (\mathfrak{s}, \mathfrak{s}) : \mathbb{T} \to \mathbb{R}^2$.

The Bloch-Floquet representation.

The Bloch-Floquet representation.

We have the following unitary equivalence:

$$\mathscr{V}_\Gamma : L^2(\mathcal{X}) \xrightarrow{\sim} \mathscr{G} :$$

$$\big(\mathscr{V}_\Gamma f\big)(\mathbf{z},\xi) := \sum_{\gamma\in\Gamma} e^{-i<\xi,\gamma+\mathfrak{s}^2(\mathbf{z})>} f(\gamma+\mathfrak{s}^2(\mathbf{z})), \quad \forall(\mathbf{z},\xi)\in\mathbb{T}_*\times\mathcal{X}^*.$$

The Bloch-Floquet Theorem.

- The operator $\widetilde{H}^0 := \mathscr{V}_\Gamma H^0 \mathscr{V}_\Gamma^{-1}$ decomposes in the Bloch-Floquet representation, defining a family of operators indexed by $\mathcal{X}^*$:
$$\big\{\widetilde{H}^0(\gamma^* + \tilde{\xi}) = U^\dagger(\gamma^*)\widetilde{H}^0(\tilde{\xi})U^\dagger(\gamma^*)^{-1}\big\}_{(\gamma^*,\tilde{\xi})\in\Gamma_*\times E_*}.$$
- $\widetilde{H}^0(\tilde{\xi})$ is the unique self-adjoint extension in $L^2(\mathbb{T})$ of the differential operator
$$\sum_{1\leq j\leq d}\big(-i\partial_j + \tilde{\xi}_j - A_j^\Gamma(x)\big)^2 + V_\Gamma(x), \quad \tilde{\xi}\in E_*$$
having compact resolvent and defining an *analytic family of type A in the sense of Kato*.
- There exists a family of continuous functions $\mathbb{T}_* \ni \theta \mapsto \lambda_j(\mathbf{z}) \in \mathbb{R}$ indexed by $j \in \mathbb{N}$, called *the Bloch eigenvalues*, such that
$$\lambda_j(\mathbf{z}) \leq \lambda_{j+1}(\mathbf{z}), \quad \forall (j,\mathbf{z}) \in \mathbb{N}\times\mathbb{T}_*$$
$$\sigma\big(\widetilde{H}^0(\mathfrak{s}^2(\mathbf{z}))\big) = \bigcup_{j\in\mathbb{N}}\{\lambda_j(\mathbf{z})\}.$$

The Bloch Projections.

- **For each fixed $\xi \in \mathcal{X}^*$** we can define *the Riesz spectral projections* :

$$\pi_n(\xi) := \frac{i}{2\pi} \int_{\mathscr{C}_n} \left(\mathfrak{z} - \widetilde{H}^0(\xi)\right)^{-1} d\mathfrak{z}$$

for $\mathscr{C}_n$ a closed contour isolating $\lambda_n(\mathfrak{e}^2(\xi))$ from the rest of the spectrum of $\widetilde{H}^0(\xi)$.

- **The Bloch eigenvalues** are smooth functions

$$\mathcal{E}_* \supset W \ni \mathbf{z} \mapsto \lambda_n(\mathbf{z}) \in \mathbb{R}$$

on any open subset W on which they have constant multiplicity.

- **The Bloch projections** $\pi_n(\xi)$ have discontinuities at points where the multiplicity of the corresponding eigenvalue $\lambda_n(\mathfrak{e}^2(\xi))$ changes.

Onsager-Peierls substitution for an isolated Bloch band.

The isolated Bloch band.

- Suppose we have a Bloch eigenvalue $\lambda_n : \mathbb{T}_* \rightarrow \mathbb{R}$ that has constant multiplicity 1 on $\mathbb{T}_*$.
- It follows that the graph of λ_n does not intersect any graph of a different Bloch eigenvalue.
- Thus:
  - $\mathbb{T}_* \ni \mathfrak{z} \mapsto \lambda_n(\theta\mathfrak{z}) \in \mathbb{R}$ is smooth (and evidently periodic)
  - the family $\{\pi_n(\xi)\}_{\xi \in \mathcal{X}^*}$ defines a smooth rank 1 sub-bundle $\pi_n\mathcal{G} \rightarrow \mathbb{T}_*$.
- Is it possible to find a smooth section $\widetilde{\psi} : \mathbb{T}_* \rightarrow \pi_n\mathcal{G}$ of unit vectors?
- This is equivalent with the triviality of the sub-bundle $\pi_n\mathcal{G} \rightarrow \mathbb{T}_*$.
- This is the case for Hamiltonians commuting with complex conjugation, for d=2,3. (G. Panati)

The Wannier functions.

Suppose:

- $\lambda_n : \mathbb{T}_* \rightarrow \mathbb{R}$ has constant multiplicity 1 on $\mathbb{T}_*$.
- there exists a smooth section $\widetilde{\psi} : \mathbb{T}_* \rightarrow \pi_n \mathscr{G}$ of unit vectors.

Define:

$$\Psi(\gamma + \tilde{x}) := \big(\mathscr{V}_\Gamma^{-1}\widetilde{\psi}\big)(\gamma + \tilde{x}) = (2\pi)^{-d} \int_{\mathbb{T}_*} \mathbf{z}^{(\gamma+\tilde{x})}\widetilde{\psi}(\mathbf{z}, \tilde{x})\, d\mathbf{z}.$$

Then:

- $\Psi \in \mathscr{S}(\mathscr{X})$.
- $\{\mathscr{T}_\gamma \Psi\}_{\gamma, \in \Gamma}$ is an orthonormal basis for $\Pi_n \mathscr{H}$ with $\Pi_n := \mathscr{V}_\Gamma^{-1} \pi_n \mathscr{V}_\Gamma$.
- $\big\langle \mathscr{T}_\alpha \Psi\, ,\, H^0 \mathscr{T}_\beta \Psi \big\rangle_{\mathscr{H}} = \hat{\lambda}_n(\alpha - \beta) := (2\pi)^{-1} \int\limits_{\mathbb{T}_*} \mathbf{z}^{(\alpha-\beta)} \lambda_n(\mathbf{z})\, d\mathbf{z}$.

The Onsager-Peierls substitution.

- Let $I_0 := \lambda_n(\mathbb{T}_*) \subset \mathbb{R}$.
- Then $I_0 \subset \sigma(H^0)$ and $d_H\big(I_0, \sigma(H^0) \setminus I_0\big) > 0$.
- Thus for $\epsilon > 0$ and $\kappa > 0$ small enough, there exist an interval $I_{\epsilon,\kappa} \subset \mathbb{R}$ such that
  - $d_H\big(I_0, I_{\epsilon,\kappa}\big) = o(\epsilon)$
  - $I_{\epsilon,\kappa} \subset \sigma(H^{\epsilon,\kappa})$
  - $d_H\big(I_{\epsilon,\kappa}, \sigma(H^{\epsilon,\kappa}) \setminus I_{\epsilon,\kappa}\big) > 0$

The Onsager-Peierls conjecture: if we denote by $E_{\epsilon,\kappa} := E_{I_{\epsilon,\kappa}}\big(H^{\epsilon,\kappa}\big)$ the spectral projection of $H^{\epsilon,\kappa}$ on the interval $I_{\epsilon,\kappa}$ and by $A^{\epsilon,\kappa}$ a vector potential for $B_{\epsilon,\kappa}$ we have that

$$\boxed{E_{I_{\epsilon,\kappa}}\big(H^{\epsilon,\kappa}\big) H^{\epsilon,\kappa} E_{I_{\epsilon,\kappa}}\big(H^{\epsilon,\kappa}\big) = \lambda_n\big(-i\nabla - A^{\epsilon,\kappa}(Q)\big) + o(\epsilon).}$$

(Here $d_H(M_1, M_2)$ is the Hausdorff distance between two subsets of $\mathbb{R}$).

The Onsager-Peierls substitution

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The main result.

The problem.

$$H^{\epsilon,\kappa} := (-i\partial_{x_1} - A_1^\Gamma - A_1^{\epsilon,\kappa})^2 + (-i\partial_{x_2} - A_2^\Gamma - A_2^{\epsilon,\kappa})^2 + V_\Gamma\,.$$
$$A^{\epsilon,\kappa} = \epsilon A_0(x) + \kappa A(\epsilon x),\; B_{\epsilon,\kappa}(x) = \epsilon B_0 + \kappa\epsilon B(\epsilon x),$$
$$A_0(x) := (1/2)B_0 \llcorner x,\;\; A_\epsilon(x) := A(\epsilon x),\;\; B_\epsilon(x) := B(\epsilon x),$$
$$dA_0 = B_0,\;\; dA_\epsilon = \epsilon B_\epsilon.$$

We parametrise $\mathbf{z} \in \mathbb{S}^2$ by $\theta = \mathfrak{s}^2(\mathbf{z}) \in E_* = [-1/2, 1/2)^2 \subset \mathbb{R}^2$.

Hypothesis

The Bloch eigenvalue $\lambda_0 : \mathbb{T}_* \to \mathbb{R}$ has a unique non-degenerate global minimum value realized for $\theta_0 \in \mathring{E}_*$ and $\lambda_0(\theta_0) = 0\,.$

Consequence: There exists $\widetilde{b} > 0$ such that:

- For every $0 < b \leq \widetilde{b}$ the set $\Sigma_b := \lambda_0^{-1}([0, b)) \subset E_*$ is diffeomorphic to the open unit disc in $\mathbb{R}^2$, has a smooth boundary and contains $\theta_0\,.$
- The function λ_0 is smooth on Σ_b and its Hessian matrix is positive.
- For θ outside of Σ_b we have $\widetilde{H}^0(\theta) \geq b\,.$

The magnetic quantization.

We recall that given a magnetic field B with components of class $BC^\infty(\mathcal{X})$ we may replace the usual *Weyl quantization* by a *magnetic quantization* defined by an associated vector potential A (chosen to have components of class $C^\infty_{\text{pol}}(\mathcal{X})$:

$$\mathscr{S}(\Xi) \ni f \mapsto \mathfrak{Op}^A(f) \in \mathcal{L}\big(\mathscr{S}'(\mathcal{X}); \mathscr{S}(\mathcal{X})\big),$$

$$\big(\mathfrak{Op}^A(f)\phi\big)(x) := (2\pi)^{-d} \int_{\mathcal{X}} dy \int_{\mathcal{X}^*} d\xi \, e^{i<\xi, x-y>} e^{-i\int_{[x,y]} A} f\big((x+y)/2, \xi\big)\phi(y), \quad \forall \phi \in \mathscr{S}(\mathcal{X}).$$

We also recall that there is also a *magnetic Moyal product* associated to this magnetic quantization:

$$\mathfrak{Op}^A(f)\mathfrak{Op}^A(g) \;=\; \mathfrak{Op}^A(f \sharp^B g),$$

$$\big(f\sharp^B g\big)(X) \;=\; \pi^{-2d} \int_{\Xi} dY \int_{\Xi} dZ \, e^{-2i\sigma(X-Y, X-Z)} e^{-iF^B(x,y,z)} f(Y) g(Z)$$

$\forall (f,g) \in [\mathscr{S}(\Xi)]^2$, where $F^B(x,y,z)$ is the integral of the 2-form B over the triangle with vertices $x-y+z, y-z+x, z-x+y$.

The main result.

Let $H^{\epsilon,\kappa}$ $((\epsilon,\kappa)\in[0,1]\times[0,1])$ be as above with H^0 satisfying the Hypothesis above and let $\widetilde{b}$ be as in the remark above.

Then $\exists b\in(0,\widetilde{b})$ and a smooth real function $\lambdabar^{\epsilon}\colon\mathbb{T}\to\mathbb{R}$ such that:

1. $\forall k\in\mathbb{N}$, $\exists C_k>0$ such that $\forall\theta\in\Sigma_b$ and $|\alpha|=k$

$$\big|\big(\partial^{\alpha}\lambdabar^{\epsilon}\big)(\theta)-\big(\partial^{\alpha}\lambda_0\big)(\theta)\big|\leq C_k\,\epsilon\,.$$

2. $\lambdabar^{\epsilon}(\theta)\geq b/2$ outside Σ_b if ϵ is small enough.

3. $\forall N\in\mathbb{N}^*$ there exist $C_0>0$ and $(\epsilon_0,\kappa_0)\in(0,\widetilde{b}/N)\times(0,1)$, such that $\forall(\epsilon,\kappa)\in(0,\epsilon_0]\times(0,\kappa_0]$,

$$d_H\Big(\sigma\big(H^{\epsilon,\kappa}\big)\bigcap[0,N\epsilon]\,,\,\sigma\big(\mathfrak{Op}^{\epsilon,\kappa}(\widetilde{\lambdabar^{\epsilon}})\big)\bigcap[0,N\epsilon]\Big)\;\leq\;C_0\big(\kappa\epsilon+\epsilon^2\big).$$

$\widetilde{\lambdabar^{\epsilon}}:\mathcal{X}^*\to\mathbb{R}$ is the periodic extension of $\lambdabar^{\epsilon}\colon\mathbb{T}_*\to\mathbb{R}$ as a tempered distribution on Ξ constant along $\mathcal{X}\times\{0\}$.

The spectral result.

Applying to the operator $\mathfrak{Op}^{\epsilon,\kappa}(\widetilde{\lambda^{\epsilon}})$ the spectral analysis elaborated in:

Horia D. Cornean, Bernard Helffer, Radu Purice: *Low lying spectral gaps induced by slowly varying magnetic fields*, **Journal of Functional Analysis 273**, (1), (2017), pp. 206–282.

we obtain the following results concerning the spectrum of $H^{\epsilon,\kappa}$ in the neighbourhood of $\lambda_0(\theta_0) = 0$.

Consequence 1:

Under our hypothesis, for any integer $N \geq 1$, there exist positive constants C_0, C_1, C_2 , and $(\epsilon_0, \kappa_0) \in (0, \widetilde{b}/N) \times (0,1)$, such that for any $(\epsilon, \kappa) \in (0, \epsilon_0] \times (0, \kappa_0]$ there exist $a_0 < b_0 < a_1 < \cdots < a_N < b_N$, with $a_0 = \inf\{\sigma(H^{\epsilon,\kappa})\}$ so that:

$$\boxed{\sigma(H^{\epsilon,\kappa}) \cap [a_0, b_N] \subset \bigcup_{k=0}^{N} [a_k, b_k]},$$

$$b_k - a_k \leq C_0 \epsilon \big(\kappa + C_1\, \epsilon^{1/5}\big) \text{ for } 0 \leq k \leq N\,, \tag{1}$$

$$a_{k+1} - b_k \geq \frac{1}{C_2}\epsilon\,, \text{ for } 0 \leq k \leq N-1\,, \tag{2}$$

$$\dim\big(\mathrm{Ran} E_{[a_k,b_k]}(H^{\epsilon,\kappa})\big) = +\infty. \tag{3}$$

Consequence 2:

Assume $B^\Gamma = 0$. Then for any integer $N \geq 1$, there exist some constants $C_0, C_1, C_2 > 0$, and $(\epsilon_0, \kappa_0) \in (0, \widetilde{b}/N) \times (0,1)$, such that, for any $(\epsilon, \kappa) \in (0, \epsilon_0] \times (0, \kappa_0]$, there exist $a_0 < b_0 < a_1 < \cdots < a_N < b_N$, with $a_0 = \inf\{\sigma(H^{\epsilon,\kappa})\}$ so that:

$$\boxed{\sigma(H^{\epsilon,\kappa}) \cap [a_0, b_N] \subset \bigcup_{k=0}^{N} [a_k, b_k]},$$

$$b_k - a_k \leq C_0\, \epsilon\big(\kappa + C_1\, \epsilon^{1/3}\big)\,,\ 0 \leq k \leq N\,, \tag{4}$$

$$a_{k+1} - b_k \geq \frac{1}{C_2}\epsilon\,,\ 0 \leq k \leq N-1\,, \tag{5}$$

$$\dim\big(\mathrm{Ran}E_{[a_k,b_k]}(H^{\epsilon,\kappa})\big) = +\infty. \tag{6}$$

The 'quasi' Wannier functions.

The free 'quasi' Wannier function.

- Let $\mathscr{K} := L^2(\mathbb{T})$.
- Let us choose $\widetilde{\phi}_0 \in \pi_0(\theta_0)\mathscr{K}$ with $\|\widetilde{\phi}_0\|_{\mathscr{K}} = 1$.
- Let us fix some $b \in (0, \widetilde{b})$ such that $\|\pi_0(\theta) - \pi_0(\theta_0)\|_{\mathbb{B}(\mathscr{K})} < 1/2$ and use the Sz. Nagy formula to define a unitary intertwining operator $\mathscr{R}(\theta, \theta_0) : \pi_0(\theta_0)\mathscr{K} \to \pi_0(\theta)\mathscr{K}$ as a smooth function of $\theta \in \Sigma_b$.
- We define $\widetilde{\phi}(\theta) := \mathscr{R}(\theta, \theta_0)\widetilde{\phi}_0$ as a local smooth section $\Sigma_b \to \mathscr{G}|_{\Sigma_b}$ with $\|\widetilde{\phi}(\theta)\|_{\mathscr{K}} = 1$ for any $\theta \in \Sigma_b$.

Proposition A

There exists a global smooth section $\widetilde{\psi}_0 : \mathbb{T}_* \to \mathscr{G}$, such that:

- $\widetilde{\psi}_0(\theta) = \widetilde{\phi}(\theta)$ for any $\theta \in \Sigma_b$,
- $\|\widetilde{\psi}_0(\theta)\|_{\mathscr{K}} = 1$ for any $\theta \in \mathbb{T}_*$.

Proof of Proposition A. (1)

When θ is close to θ_0, the eigenvector $\widetilde{\phi}(\theta, x)$ can be chosen to be smooth as a function of $x \in \mathbb{T}$ due to elliptic regularity.
It is also smooth as a function of θ on a small neighbourhood of θ_0.

- For $j \in \{1, 2\}$ let $f_j \in C_0^\infty(\mathring{\mathcal{E}})$, with $\|f_j\|_{L^2(\mathcal{E})} = 1$ and such that $f_1(x)f_2(x) = 0$ for all x.
- Let us define $\widetilde{f}_j(\theta, x) = \mathscr{V}_\Gamma f_j$.
- We can check that $\widetilde{f}_1(\theta, x)\widetilde{f}_2(\theta, x) = 0$ for all $x \in \mathbb{T}$, thus their scalar product in $\mathscr{K}$ equals zero.
- Moreover both have norm one at any fixed θ.

Remark: $\widetilde{\phi}(\theta_0, \cdot)$ is certainly not parallel with both $\widetilde{f}_j(\theta_0, \cdot)$, j=1,2 (that are orthogonal to each other)
Thus there must exist a $j \in \{1, 2\}$ (without loss of generality take $j = 1$) such that

$$|\langle \widetilde{\phi}(\theta_0, \cdot), \widetilde{f}_1(\theta_0, \cdot)\rangle_{\mathscr{K}}| \leq 1/\sqrt{2}\,.$$

Proof of Proposition A. (2)

- Due to continuity in θ, there exists a ball $B_r(\theta_0) \subset \overline{\Sigma}_b$ where

$$|\langle \widetilde{\phi}(\theta,\cdot), \widetilde{f}_1(\theta,\cdot)\rangle_{\mathscr{K}}| \leq 3/4\,, \quad \forall \theta \in B_r(\theta_0)\,.$$

- Let $g \in C_0^\infty(\Sigma_b;[0,1])$ with support in $B_r(\theta_0)$ and $g = 1$ on $B_{r/2}(\theta_0)$.
- Define

$$\widetilde{h}(\theta,x) := g(\theta)\widetilde{\phi}(\theta,x) + (1-g(\theta))\widetilde{f}(\theta,x).$$

Then $\|\widetilde{h}(\theta,\cdot)\|^2 \geq 1/8$ and
$\widetilde{\psi}_0(\theta,x) := \widetilde{h}(\theta,x)\|\widetilde{h}(\theta,x)\|^{-1} \in \mathscr{K}$ is a smooth extension of $\widetilde{\phi}$ to a global section.

□

The free 'quasi' Wannier function.

Definition

- $\psi_0 := \mathscr{V}_\Gamma^{-1}\widetilde{\psi}_0 \in L^2(\mathcal{X})$ is *the free 'quasi' Wannier function*,
- $\forall\gamma \in \Gamma: \ \psi_\gamma := \mathcal{T}_{-\gamma}\psi_0 \in L^2(\mathcal{X})$,
  that we call the *quasi Wannier system for the energy window* $[0, b]$.
- $\pi \in \mathbb{B}(\mathcal{H})$ the projection on $\mathcal{H}_0 := \overline{\mathcal{L}in\{\psi_\gamma : \ \gamma \in \Gamma\}} \subset L^2(\mathcal{X})$.

Proposition

- $\psi_0 \in \mathscr{S}(\mathcal{X})$.
- The family $\{\psi_\gamma : \ \gamma \in \Gamma\} \subset L^2(\mathcal{X})$ is an orthonormal system.
- $\pi\mathcal{H} \subset \mathcal{D}(H^0)$ and thus the products $H^0\pi$ and πH^0 define bounded operators on $\mathcal{H} = L^2(\mathcal{X})$.
- $\pi = \mathfrak{Op}(p)$ with $p \in S^{-\infty}(\Xi)$ being Γ_*-periodic.

The magnetic 'quasi' Wannier system.

1. $\boxed{\mathring{\phi}^{\epsilon}_{\gamma}(x) := \Lambda^{\epsilon}(x,\gamma)\psi_0(x-\gamma)}$, $\quad \Lambda^{\epsilon}(x,y) := e^{-i\epsilon\int[x,y]A_0}$,

   $\mathbb{G}^{\epsilon}_{\alpha\beta} := \langle\mathring{\phi}^{\epsilon}_{\alpha}\,,\,\mathring{\phi}^{\epsilon}_{\beta}\rangle_{\mathcal{H}}\,,\ \mathbb{F}^{\epsilon} := \big(\mathbb{G}^{\epsilon}\big)^{-1/2} \in \mathbb{B}\big(\ell^2(\Gamma)\big)$,

2. $\boxed{\phi^{\epsilon}_{\gamma}(x) := \mathbb{F}^{\epsilon}_{\alpha\gamma}\mathring{\phi}^{\epsilon}_{\alpha}}$, $\pi^{\epsilon} := \sum_{\gamma\in\Gamma}|\phi^{\epsilon}_{\gamma}\rangle\langle\phi^{\epsilon}_{\gamma}|$,

3. $\boxed{\mathring{\phi}^{\epsilon,\kappa}_{\gamma}(x) := \Lambda^{\epsilon,\kappa}(x,\gamma)\psi^{\epsilon}_0(x-\gamma)}$ $\quad \Lambda^{\epsilon,\kappa}(x,y) := e^{-i\int_{[x,y]}(\epsilon A_0+\kappa A_{\epsilon})}$,

   $\mathbb{G}^{\epsilon,\kappa}_{\alpha\beta} := \langle\mathring{\phi}^{\epsilon,\kappa}_{\alpha}\,,\,\mathring{\phi}^{\epsilon,\kappa}_{\beta}\rangle_{\mathcal{H}}\,,\ \ \mathbb{F}^{\epsilon,\kappa} := \big(\mathbb{G}^{\epsilon,\kappa}\big)^{-1/2} \in \mathbb{B}\big(\ell^2(\Gamma)\big)$,

4. $\boxed{\phi^{\epsilon,\kappa}_{\gamma} := \sum_{\alpha\in\Gamma}\mathbb{F}^{\epsilon,\kappa}_{\alpha\gamma}\mathring{\phi}^{\epsilon,\kappa}_{\alpha}}$, $\quad \pi^{\epsilon,\kappa} := \sum_{\gamma\in\Gamma}|\phi^{\epsilon,\kappa}_{\gamma}\rangle\langle\phi^{\epsilon,\kappa}_{\gamma}|$.

5. $\boxed{\psi^{\epsilon}_0(x) := \sum_{\alpha\in\Gamma}\mathbb{F}^{\epsilon}_{\alpha\,0}\Omega^{\epsilon}(\alpha,0,x)\psi_0(x-\alpha)}$, $\widetilde{\Lambda}^{\epsilon,\kappa}(x,y) := e^{-i\kappa\int_{[x,y]}A_{\epsilon}}$.

Remark: $\phi^{\epsilon}_{\gamma}(x) = \Lambda^{\epsilon}(x,\gamma)\psi^{\epsilon}_0(x-\gamma), \quad \mathring{\phi}^{\epsilon,\kappa}_{\gamma}(x) = \widetilde{\Lambda}^{\epsilon,\kappa}(x,\gamma)\phi^{\epsilon}_{\gamma}(x)$.

The magnetic 'quasi' band. Properties.

There exists $\epsilon_0 > 0$ such that:

1. $\forall m \in \mathbb{N},\ \exists C_m > 0$ such that:

$$< \alpha - \beta >^m \left|\mathbb{F}^{\epsilon}_{\alpha\beta} - \delta_{\alpha\beta}\right| \leq C_m\, \epsilon\,,\ \forall(\alpha,\beta) \in \Gamma^2\,, \forall \epsilon \in [0,\epsilon_0].$$

2. $\forall m \in \mathbb{N}$ and $\forall a \in \mathbb{N}^2$, there exists $C_{m,a} > 0$ such that

$$\sup_{x\in\mathcal{X}} < x >^m \left|\big(\partial^a \psi_0^\epsilon\big)(x) - \big(\partial^a \psi_0\big)(x)\right| \leq C_{m,a}\epsilon\,, \quad \forall \epsilon \in [0,\epsilon_0]\,.$$

3. $\forall m \in \mathbb{N}$, there exists $C_m > 0$ such that

$$\sup_{(\alpha,\beta)\in\Gamma^2} < \alpha - \beta >^m \left|\mathbb{F}^{\epsilon,\kappa}_{\alpha,\beta} - \delta_{\alpha\beta}\right| \leq C_m\, \kappa\epsilon\,,\ \forall(\epsilon,\kappa) \in [0,\epsilon_0]\times[0,1]\,.$$

4. There exists $\epsilon_0 > 0$ such that for any $(\epsilon,\kappa) \in [0,\epsilon_0]\times[0,1]$ we have

$$\pi^{\epsilon,\kappa}\mathcal{H} \subset \mathcal{D}(H^{\epsilon,\kappa})\,,$$

while $H^{\epsilon,\kappa}\pi^{\epsilon,\kappa} \in \mathbb{B}(\mathcal{H})$ and $\pi^{\epsilon,\kappa}H^{\epsilon,\kappa}$ has a bounded closure.

The magnetic 'quasi' band. Properties of the symbols.

We prove that:

- $\pi^{\epsilon} = \mathfrak{Op}^{\epsilon}(p_{\epsilon})$ with the Γ-periodic symbol $p_{\epsilon} \in S^{-\infty}(\Xi)$.
- $\pi^{\epsilon,\kappa} = \mathfrak{Op}^{\epsilon,\kappa}(p_{\epsilon,\kappa})$ with the symbol $p_{\epsilon,\kappa} \in S^{-\infty}(\Xi)$.

Proposition

There exists $\epsilon_0 > 0$ such that for any seminorm ν on $S^{-\infty}(\Xi)$, there exists $C_{\nu} > 0$ such that

$$\nu(p^{\epsilon} - p) \leq C_{\nu}\, \epsilon \text{ and } \nu(p^{\epsilon,\kappa} - p^{\epsilon}) \leq C_{\nu}\, \kappa\epsilon\,, \ \forall(\epsilon,\kappa) \in [0,\epsilon_0] \times [0,1]\,.$$

Note that the commutator $\left[H^{\epsilon,\kappa}, \pi^{\epsilon,\kappa}\right]$ is not small, due to the arbitrary deformation which was made in constructing the quasi Wannier function.

Proof of the main result.

Step I: Reduction to the energy band subspace.

First, let us compare the real perturbed Hamiltonian $H^{\epsilon,\kappa}$
with
the magnetic 'band' Hamiltonian $\pi^{\epsilon,\kappa}H^{\epsilon,\kappa}\pi^{\epsilon,\kappa}$.

We compare the bottoms of their spectra by using a variant of the *Feshbach-Schur argument*.
The difficulty comes from the fact that
the norm of the bounded operator $\left[\pi^{\epsilon,\kappa}, H^{\epsilon,\kappa}\right]$ is not 'small'!
We use *the resolvent equation* and consider *the energy window around the minimum of λ_0 as small parameter!*

From our Hypothesis we have deduced that:

$$\left[\pi^{\epsilon,\kappa}\right]^{\perp} H^{\epsilon,\kappa} \left[\pi^{\epsilon,\kappa}\right]^{\perp} \geq b\left[\pi^{\epsilon,\kappa}\right]^{\perp}, \quad b > 0.$$

Let us define

- $R_{\perp}^{\epsilon,\kappa} := \left(\left[\pi^{\epsilon,\kappa}\right]^{\perp} H^{\epsilon,\kappa} \left[\pi^{\epsilon,\kappa}\right]^{\perp}\right)^{-1} \in \mathbb{B}\left(\left[\pi^{\epsilon,\kappa}\right]^{\perp}\mathcal{H}\right)$
- $Y_{\epsilon,\kappa} := \pi^{\epsilon,\kappa} + \pi^{\epsilon,\kappa} H^{\epsilon,\kappa} \left[\pi^{\epsilon,\kappa}\right]^{\perp} \left[R_{\perp}^{\epsilon,\kappa}\right]^{2} \left[\pi^{\epsilon,\kappa}\right]^{\perp} H^{\epsilon,\kappa} \pi^{\epsilon,\kappa} \geq \pi^{\epsilon,\kappa}$,
- *The 'dressed' band Hamiltonian*: $\widetilde{H}^{\circ}_{\epsilon,\kappa} :=$ $Y_{\epsilon,\kappa}^{-1/2}\left(\pi^{\epsilon,\kappa} H^{\epsilon,\kappa} H \pi^{\epsilon,\kappa} - \pi^{\epsilon,\kappa} H^{\epsilon,\kappa} \left[\pi^{\epsilon,\kappa}\right]^{\perp} R_{\perp}^{\epsilon,\kappa} \left[\pi^{\epsilon,\kappa}\right]^{\perp} H^{\epsilon,\kappa} \pi^{\epsilon,\kappa}\right) Y_{\epsilon,\kappa}^{-1/2} \in \mathbb{B}\left(\pi^{\epsilon,\kappa}\mathcal{H}\right)$.

Result of Step I:

If *the 'dressed' band Hamiltonian* $\widetilde{H}^{\circ}_{\epsilon,\kappa}$ has N spectral gaps in the compact interval $I \subset \mathbb{R}$, then the *perturbed Hamiltonian* $H^{\epsilon,\kappa}$ has N spectral gaps in the compact interval $I \subset \mathbb{R}$.

The abstract reduction argument.

We consider the following situation:

- H is a positive self-adjoint operator,
- Π is an orthogonal projection such that $H\Pi$ (and thus ΠH) is bounded,
- $\exists \beta > 0$ such that $\Pi^{\perp} H \Pi^{\perp} \geq 2\beta\Pi^{\perp}$.

This implies that $\Pi^{\perp}(H - E)\Pi^{\perp}$ is invertible in $\Pi^{\perp}\mathcal{H}$ for $E \in [0, 2\beta)$ and we denote by $R_{\perp}(E) \in \mathbb{B}(\Pi^{\perp}\mathcal{H})$ its inverse.
The spectral theorem gives: $\sup_{E\in[0,\beta]} \|R_{\perp}(E)\| \leq \beta^{-1}$.

We do not suppose that $[H, \Pi] \in \mathbb{B}(\mathcal{H})$ is small!
Instead we suppose $E > 0$ small (of the order ϵ).
and use the resolvent equation:

$$R_{\perp}(E) = R_{\perp}(0) + R_{\perp}(0)^2 + E^2 R_{\perp}(0)^2 R_{\perp}(E).$$

The Feshbach - Schur argument.

In the above setting:

For $E \in [0, \beta]$

$(H - E\mathbb{1})$ is invertible in $\mathcal{H}$ *if and only if* $S(E)$ is invertible in $\Pi\mathcal{H}$ and $S(E)^{-1} = \Pi\big(H - E\mathbb{1}\big)^{-1}\Pi$

where

$$S(E) := \Pi(H - E\mathbb{1})\Pi \; - \; \Pi H R_\perp(E) H \Pi \; \in \; \mathbb{B}(\Pi\mathcal{H}).$$

Definition:

- $Y := \Pi + \Pi H \Pi^\perp R_\perp(0)^2 \Pi^\perp H \Pi.$ **Remark:** $Y \geq \Pi,$
- $\widetilde{H} := Y^{-1/2}\left[\Pi H \Pi - \Pi H \Pi^\perp R_\perp(0) \Pi^\perp H \Pi\right] Y^{-1/2} \in \mathbb{B}(\Pi\mathcal{H}).$

Remark: $S(E) = Y^{1/2}\big(\widetilde{H} - E\mathbb{1}\big)Y^{1/2} + E^2 \Pi H R_\perp(0) R_\perp(E) R_\perp(0) H \Pi.$

Proposition I: $\forall \beta' \in [0, \beta]$ we have

$$d_H\{\sigma(H) \cap [0, \beta'], \sigma(\widetilde{H}) \cap [0, \beta']\} \leq \|H\Pi\|^2 \, (\beta')^2 \beta^{-3}.$$

Proof of Proposition I.

Assume $E \in [0, \beta'] \cap \rho(\widetilde{H})$.

$S(E) = Y^{1/2}\{\mathbb{1} + Y^{-1/2}\Pi H X(E) H \Pi Y^{-1/2}(\widetilde{H} - E)^{-1})\}(\widetilde{H} - E) Y^{1/2}$.

$\|Y^{-1/2}\Pi H X(E) H \Pi Y^{-1/2}(\widetilde{H} - E)^{-1}\| \leq \frac{\beta'^2 \|H\Pi\|^2 \beta^{-3}}{\mathrm{dist}(E, \sigma(\widetilde{H}))}$, if $E \in [0, \beta']$.

Thus: $\mathrm{dist}(E, \sigma(\widetilde{H})) > \beta'^2 \|H\Pi\|^2 \beta^{-3}$ implies $E \in \rho(H)$.

$E \in \rho(H) \cap [0, \beta']$ implies $\mathrm{dist}\big(E, \sigma(\widetilde{H}) \cap [0, \beta')\big) > \beta'^2 \|H\Pi\|^2 \beta^{-3}$.

Assume $E \in [0, \beta'] \cap \rho(H)$.

Thus: $S(E)$ is invertible in $\Pi\mathscr{H}$ with $S(E)^{-1} = \Pi\big(H - E\mathbb{1}\big)^{-1}\Pi$

but $Y^{1/2}\big(\widetilde{H} - E\mathbb{1}\big) Y^{1/2} = S(E) - E^2 \Pi H R_\perp(0) R_\perp(E) R_\perp(0) H \Pi$.

Thus: $\mathrm{dist}(E, \sigma(H)) > \beta'^2 \|H\Pi\|^2 \beta^{-3}$ implies $\widetilde{H} - E\mathbb{1}$ invertible.

$E \in \rho(\widetilde{H}) \cap [0, \beta']$ implies $\mathrm{dist}\big(E, \sigma(H) \cap [0, \beta')\big) > \beta'^2 \|H\Pi\|^2 \beta^{-3}$.

Corollary 1

Let $0 < D_1 < D_2 < \beta' < \beta$ and assume $(D_1, D_2) \cap \sigma(\widetilde{H}) = \emptyset$. Then, if

$$\|H\Pi\|^2 \, (\beta')^2 \beta^{-3} < \frac{1}{2}(D_2 - D_1)$$

we have

$$(D_1 + \|H\Pi\|^2 \, (\beta')^2 \beta^{-3} \, , D_2 - \|H\Pi\|^2 \, (\beta')^2 \beta^{-3}) \cap \sigma(H) = \emptyset \, .$$

Let's consider a family of triples $(H(\eta), \Pi(\eta), \beta)$ indexed by $\eta \in [0, \epsilon_1]$.

Corollary 2

Let $0 < D_1 < D_2 < \beta' < \beta$ and assume $(D_1, D_2) \cap \sigma(\widetilde{H}(\eta)) = \emptyset$, for all $\eta \in [0, \epsilon_1]$. Then, if

$$D := \sup_{\eta \in [0, \epsilon_1]} \|H(\eta)\Pi(\eta)\|^2 \, (\beta')^2 \beta^{-3} < \frac{1}{2}(D_2 - D_1) \, ,$$

we have

$$(D_1 + D \, , D_2 - D) \cap \sigma(H(\eta)) = \emptyset \, .$$

We only have to take $2\beta = \widetilde{b}$, $D_1 = C_1 \epsilon$, $D_2 = C_2 \epsilon$, $\beta' = (C_2 + 1)\epsilon$.

Step II: Reduction to the 'mean' constant field.

A straightforward analysis shows that we can apply the same 'modified' Feshbach-Schur argument to the pair $\big(H^\epsilon,\pi^\epsilon\big)$ with the constant magnetic field ϵB_0 and define a similar *'dressed' band Hamiltonian* $\widetilde{H}^\circ_\epsilon$.

Our second step is to prove that in the fixed spectral region, the spectrum of $\widetilde{H}^\circ_{\epsilon,\kappa}$ is at a Hausdorff distance of order $\kappa\epsilon$ from the spectrum of the dressed Hamiltonian $\widetilde{H}^\circ_\epsilon$ associated to the constant magnetic field ϵB_0.
Let $\mathfrak{S}^\epsilon(T)$ be the symbol of $T\in\mathbb{B}\big(\mathscr{S}(\mathcal{X});\mathscr{S}'(\mathcal{X})\big)$ for the ϵA_0 quantization and denote by

- $h^\epsilon_\circ:=\mathfrak{S}^\epsilon\big(\pi^\epsilon H^\epsilon\pi^\epsilon\big)$
- $h^\epsilon_\bullet:=\mathfrak{S}^\epsilon\big(\pi^\epsilon H^\epsilon\big(\mathbb{1}-\pi^\epsilon\big)\big)$
- $r_\epsilon:=\mathfrak{S}^\epsilon\big((\mathbb{1}-\pi^\epsilon)R^\epsilon_\perp(\mathbb{1}-\pi^\epsilon)\big)$
- $\mathfrak{y}^\epsilon_{1/2}:=\mathfrak{S}^\epsilon\big(\pi^\epsilon\big(Y^\epsilon\big)^{-1/2}\pi^\epsilon\big)$
- $\mathfrak{z}^\epsilon:=\mathfrak{y}^\epsilon_{1/2}-p^{\epsilon,\kappa}$.

Result of Step II.

We prove that:

Proposition B.

$$\left\langle \phi^{\epsilon,\kappa}_{\alpha} \, , \, \widetilde{H}^{\circ}_{\epsilon,\kappa}\phi^{\epsilon,\kappa}_{\beta} \right\rangle_{\mathcal{H}} = \widetilde{\Lambda}^{\epsilon,\kappa}(\alpha,\beta)\left\langle \phi^{\epsilon}_{\alpha} \, , \, \mathfrak{Op}^{\epsilon}(h^{\epsilon}_{\circ})\phi^{\epsilon}_{\beta} \right\rangle_{\mathcal{H}} + \\ + \widetilde{\Lambda}^{\epsilon,\kappa}(\alpha,\beta)\left\langle \phi^{\epsilon}_{\alpha} \, , \, \mathfrak{Op}^{\epsilon}\big(\mathfrak{k}^{\epsilon}\big)\phi^{\epsilon}_{\beta} \right\rangle_{\mathcal{H}} + \mathscr{O}(\kappa\epsilon)$$

where $\mathfrak{k}^{\epsilon} := h^{\epsilon}_{\circ} \sharp^{\epsilon} \mathfrak{z}^{\epsilon} + \mathfrak{z}^{\epsilon} \sharp^{\epsilon} h^{\epsilon}_{\circ} \sharp^{\epsilon} \mathfrak{y}^{\epsilon}_{1/2} + \mathfrak{y}^{\epsilon}_{1/2} \sharp^{\epsilon} h^{\epsilon}_{\bullet} \sharp^{\epsilon} r_{\epsilon} \sharp^{\epsilon} h^{\epsilon}_{\bullet} \sharp^{\epsilon} \mathfrak{y}^{\epsilon}_{1/2}$
is the contribution of the *'dressing' factors and terms*.

Control of the rest $\mathscr{O}(\kappa\epsilon)$.

There are mainly two corrections that have to be controlled:

Phase decomposition:

- $\Lambda^{\epsilon,\kappa} = \widetilde{\Lambda}^{\epsilon,\kappa}\Lambda^{\epsilon}, \quad \widetilde{\Lambda}^{\epsilon,\kappa}(x,y) := e^{-i\kappa\int_{[x,y]}A_\epsilon}$
  coming from $A^{\epsilon,\kappa} = \epsilon A_0 + \kappa A_\epsilon$.

Orthonormation corrections:

- $$\left|\sum_{\alpha'\in\Gamma}\sum_{\beta'\in\Gamma}\overline{\mathbb{F}^{\epsilon,\kappa}_{\alpha'\alpha}}\mathbb{F}^{\epsilon,\kappa}_{\beta'\beta}\mathfrak{H}^{\epsilon,\kappa}_{\alpha',\beta'} - \mathfrak{H}^{\epsilon,\kappa}_{\alpha,\beta}\right| \leq C_m\kappa\epsilon < \alpha-\beta >^{-m}.$$

  $\mathbb{G}^{\epsilon,\kappa}_{\alpha\beta} := \langle\mathring{\phi}^{\epsilon,\kappa}_{\alpha}, \mathring{\phi}^{\epsilon,\kappa}_{\beta}\rangle_{\mathscr{H}}, \quad \mathbb{F}^{\epsilon,\kappa} := \left(\mathbb{G}^{\epsilon,\kappa}\right)^{-1/2} \in \mathbb{B}\big(\ell^2(\Gamma)\big),$
  for any uniformly bounded coefficients $\mathfrak{H}^{\epsilon,\kappa}_{\alpha,\beta}$.

Control of the rest: first estimation.

Passing from $H^{\epsilon,\kappa}$ to H^{ϵ}:

- $(-i\nabla - A^{\epsilon,\kappa}(x))^2 \widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde{\beta}) = \widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde{\beta})\Big(-i\nabla - A^{\epsilon}(x) + \kappa a_{\epsilon}(x,\tilde{\beta})\Big)^2$,

  with: $$a_{\epsilon}(x,\beta)_j = \sum_k (x-\gamma)_k \int_0^1 \epsilon B_{jk}\big(\epsilon\beta + s\epsilon(x-\beta)\big) s\, ds \ \text{ for } j=1,2\,,$$

  and using $$|a_{\epsilon}(x,\beta)| \leq C\epsilon < x-\beta > \,.$$

- Moreover we have

$$\widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde{\alpha})^{-1}\widetilde{\Lambda}^{\epsilon,\kappa}(x,\tilde{\beta}) = \widetilde{\Lambda}^{\epsilon,\kappa}(\tilde{\alpha},\tilde{\beta})\,\widetilde{\Omega}^{\epsilon,\kappa}(\tilde{\alpha},x,\tilde{\beta})\,,$$

$$|\widetilde{\Omega}^{\epsilon,\kappa}(\tilde{\alpha},x,\tilde{\beta}) - \mathbb{1}| \leq C\,\kappa\epsilon\,|x-\tilde{\alpha}|\,|x-\tilde{\beta}|\,.$$

- Finally we use the decay of $\mathcal{T}_{\alpha}\psi_0^{\epsilon}$ and $\mathcal{T}_{\beta}\psi_0^{\epsilon}$.

Control of the rest: second estimation.

For $\mathfrak{Op}^{\epsilon,\kappa}(F)$ integral operator with regular kernel $K_F(x,y)$:

- $\Big[\widetilde{\Lambda}^{\epsilon,\kappa}(\alpha',\cdot)\mathfrak{Op}^{\epsilon,\kappa}(F)\widetilde{\Lambda}^{\epsilon,\kappa}(\cdot,\beta')u\Big](x) =$
  $= \widetilde{\Lambda}^{\epsilon,\kappa}(\alpha',\beta')\widetilde{\Omega}^{\epsilon,\kappa}(\alpha',x,\beta')\int_{\mathcal{X}} dy\, \widetilde{\Omega}^{\epsilon,\kappa}(x,y,\beta')\Lambda^{\epsilon}(x,y)K_F(x,y)u(y)$
- We use
$$\left|\widetilde{\Omega}^{\epsilon,\kappa}(\alpha',x,\beta')-1\right| \leq C\kappa\epsilon\,|x-\alpha'|\,|x-\beta'|,$$
$$\left|\widetilde{\Omega}^{\epsilon,\kappa}(x,x+z,\beta')-1\right| \leq C\kappa\epsilon\,|z|\,|x-\beta'|.$$
- Finally we use the decay of $\mathcal{T}_{\alpha}\psi_0^{\epsilon}$, $\mathcal{T}_{\beta}\psi_0^{\epsilon}$, and the off-diagonal decay of the integral kernel of $\mathfrak{Op}(F)$

Step III: The constant field situation.

We have to control the 'dressing' contribution.

Let us define:

$$\mathfrak{h}^{\epsilon} := h^{\epsilon}_{\circ} + \mathfrak{k}^{\epsilon}$$

$$k^{\epsilon}(\gamma) := \Lambda^{\epsilon,\kappa}(\beta,\alpha)\left\langle \phi^{\epsilon}_{\gamma}\,,\, \mathfrak{Op}^{\epsilon}(\mathfrak{h}^{\epsilon})\phi^{\epsilon}_{0}\right\rangle_{\mathcal{H}}$$

$$\lambda^{\epsilon}(\theta) := \sum_{\gamma\in\Gamma} e^{-i<\theta,\gamma>}k^{\epsilon}(\gamma).$$

$$\mathscr{M}^{\epsilon,\kappa}(\alpha,\beta) := \Lambda^{\epsilon,\kappa}(\alpha,\beta)k^{\epsilon}(\alpha-\beta) \equiv \Lambda^{\epsilon,\kappa}(\alpha,\beta)\widehat{\lambda^{\epsilon}}_{\alpha-\beta}\,.$$

Proposition C.

The Hausdorff distance between the spectra of the operator $\mathfrak{Op}^{\epsilon,\kappa}(\lambda^{\epsilon}) \in \mathbb{B}(\mathcal{H})$ and the hermitian operator associated with the matrix $\mathscr{M}^{\epsilon,\kappa}(\alpha,\beta)$ in $\ell^2(\Gamma)$ for the canonical orthonormal basis is of order $\kappa\epsilon$.

This finishes the proof of point (3) of our main Theorem.

Proof of Proposition C

Let us consider the following 3 unitaries:

- $\mathscr{W}_\Gamma : L^2(\mathcal{X}) \to \ell^2(\Gamma) \otimes L^2(\mathcal{E}), \quad \big(\mathscr{W}_\Gamma u\big)(\gamma, \{x\}) := u(\gamma + \{x\})$.
- $\mathscr{U}_\epsilon \in \mathbb{U}\big(\ell^2(\Gamma) \otimes L^2(\mathcal{E})\big), \quad \big(\mathscr{U}_\epsilon \Phi\big)(\alpha, \{x\}) := \Lambda^\epsilon(\{x\}, \alpha)\Phi(\alpha, \{x\})$.
- $\mathscr{V}_{\epsilon,\kappa} \in \mathbb{U}\big(\ell^2(\Gamma) \otimes L^2(\mathcal{E})\big), \ \big(\mathscr{V}_{\epsilon,\kappa}\Phi\big)(\alpha, \{x\}) :=$
  $:= \widetilde{\Lambda}^{\epsilon,\kappa}(\alpha, \alpha + \{x\})\Phi(\alpha, \{x\})$.

and $$\mathscr{W}^{\epsilon,\kappa} := \mathscr{V}_{\epsilon,\kappa}\mathscr{U}_\epsilon\mathscr{W}_\Gamma : L^2(\mathcal{X}) \to \ell^2(\Gamma) \otimes L^2(\mathcal{E}).$$

The operator $\mathscr{W}^{\epsilon,\kappa}\mathfrak{Op}^{\epsilon,\kappa}(\lambda^\epsilon)(\mathscr{W}^{\epsilon,\kappa})^{-1}$
has the integral kernel

$$\mathfrak{K}^{\epsilon,\kappa}\big((\alpha, \{x\}), (\beta, \{y\})\big) =$$
$$\Lambda^{\epsilon,\kappa}(\alpha, \beta)\left(\widetilde{\Omega}^{\epsilon,\kappa}(\alpha, \alpha + \{x\}, \beta + \{x\})\widetilde{\Omega}^{\epsilon,\kappa}(\alpha, \beta + \{x\}, \beta)\right) k^\epsilon(\alpha - \beta)\delta_0(\{x\} - \{y\}).$$

$$\left|\widetilde{\Omega}^{\epsilon,\kappa}(\alpha, \alpha + \{x\}, \beta + \{x\}) - 1\right| \leq C\kappa\epsilon|\alpha - \beta|, \quad \left|\widetilde{\Omega}^{\epsilon,\kappa}(\alpha, \beta + \{x\}, \beta) - 1\right| \leq C\kappa\epsilon|\alpha - \beta|.$$

Step IV: Properties of $\lambda\!\!\!-^{\epsilon}$.

Proposition D.

For $b \in (0, \widetilde{b})$ as in the statement of our main Theorem there exists $\epsilon_0 > 0$ and $C > 0$ such that, for any $\theta \in \Sigma_b$ and any $\epsilon \in [0, \epsilon_0]$,

$$|\lambda\!\!\!-^{\epsilon}(\theta) - \lambda_0(\theta)| \leq C\,\epsilon.$$

This clearly implies the first 2 points of the Theorem.

Proof of Proposition D.

Step 1.

Recall that $\lambda^{\epsilon}(\theta) := \sum_{\gamma\in\Gamma} e^{-i<\theta,\gamma>}\Lambda^{\epsilon,\kappa}(\beta,\alpha)\left\langle \phi^{\epsilon}_{\gamma}\,,\,\mathfrak{Op}^{\epsilon}(\mathfrak{h}^{\epsilon})\phi^{\epsilon}_{0}\right\rangle_{\mathcal{H}}$.

Let us compute:

$$\sum_{\gamma\in\Gamma} e^{-i<\theta,\gamma>}\Lambda^{\epsilon,\kappa}(\beta,\alpha)\left\langle \phi^{\epsilon}_{\gamma}\,,\,\mathfrak{Op}^{\epsilon}(h)\phi^{\epsilon}_{0}\right\rangle_{\mathcal{H}} - \lambda_0(\theta) =$$

$$= \sum_{\gamma\in\Gamma} e^{-i<\theta,\gamma>}\left\langle \psi_{\gamma}\,,\,H^0\psi_0\right\rangle_{\mathcal{H}} - \lambda_0(\theta) + \mathscr{O}(\epsilon) =$$

$$= \sum_{n\in\mathbb{N}} \lambda_n(\theta)\left|\left\langle \hat{\psi}_0(\theta)\,,\,\hat{\phi}_n(\theta)\right\rangle_{\mathscr{F}_\omega}\right|^2 - \lambda_0(\theta) + \mathscr{O}(\epsilon) =$$

$$= \mathscr{O}(\epsilon), \quad \forall\theta\in\Sigma_b.$$

Proof of Proposition D.

Step 2.

Let us compute: $\lambda^{\epsilon}(\theta) - \langle\phi_{\gamma}^{\epsilon}, \mathfrak{Op}^{\epsilon}(h_{\circ}^{\epsilon})\phi_0^{\epsilon}\rangle_{\mathcal{H}}\Big) =$

$$= \sum_{\gamma\in\Gamma} e^{-i<\theta,\gamma>}\Big(\langle\phi_{\gamma}^{\epsilon}, \mathfrak{Op}^{\epsilon}(\mathfrak{h}^{\epsilon})\phi_0^{\epsilon}\rangle_{\mathcal{H}} - \langle\phi_{\gamma}^{\epsilon}, \mathfrak{Op}^{\epsilon}(h_{\circ}^{\epsilon})\phi_0^{\epsilon}\rangle_{\mathcal{H}}\Big) =$$

$$= \sum_{\gamma\in\Gamma} e^{-i<\theta,\gamma>}\langle\tau_{\gamma}\psi_0, Z\psi_0\rangle_{\mathcal{H}} + \mathscr{O}(\epsilon) = \widetilde{Z}(\theta) + \mathscr{O}(\epsilon) = \mathscr{O}(\epsilon),\ \forall\theta\in\Sigma_b$$

where

$$Z := H^0\big(Y^{-1/2} - \mathbb{1}\big) + \big(Y^{-1/2} - \mathbb{1}\big)H^0 + \big(Y^{-1/2} - \mathbb{1}\big)H^0\big(Y^{-1/2} - \mathbb{1}\big)$$
$$+Y^{-1/2}\pi H^0\pi^{\perp}R_{\perp}\pi^{\perp}H^0\pi Y^{-1/2} = \mathscr{V}_{\Gamma}^{-1}\left(\int_{\mathbb{T}_*}^{\oplus} d\theta\,\widetilde{Z}(\theta)\right)\mathscr{V}_{\Gamma}$$

$$Y = \pi H^0\pi^{\perp}R_{\perp}^2\pi^{\perp}H^0\pi = \mathscr{V}_{\Gamma}^{-1}\left(\int_{\mathbb{T}_*}^{\oplus} d\theta\,\widetilde{Y}(\theta)\right)\mathscr{V}_{\Gamma}$$

But: $\pi H^0\pi^{\perp} = \mathscr{V}_{\Gamma}^{-1}\left(\int_{\mathbb{T}_*}^{\oplus} d\theta\,\widetilde{K}(\theta)\right)\mathscr{V}_{\Gamma}$

and $\widetilde{K}(\theta) = |\hat{\phi}_0(\theta)\rangle\langle\hat{\phi}_0(\theta)|\Big(\sum_{n\in\mathbb{N}}\lambda_n(\theta)|\hat{\phi}_n(\theta)\rangle\langle\hat{\phi}_n(\theta)|\Big)\big(\mathbb{1} - |\hat{\phi}_0(\theta)\rangle\langle\hat{\phi}_0(\theta)|\big) = 0$ $\forall\theta\in\Sigma_b$.

The spectral gaps.
Analysis of the resolvent of $\mathfrak{Op}^{\epsilon,\kappa}(\widetilde{\lambda}^{\epsilon})$

Taylor development of λ^{ϵ} near the minimum.

Proposition

There exists $\epsilon_0 > 0$ such that $\lambda^{\epsilon}(\theta) = \lambda_0(\theta) + \epsilon\rho^{\epsilon}(\theta)$, with $\rho^{\epsilon} \in BC^{\infty}(\mathbb{T}_*)$ uniformly in $\epsilon \in [0, \epsilon_0]$ and such that $\rho^{\epsilon} - \rho^0 = \mathscr{O}(\epsilon)$.

- Thus $\lambda^{\epsilon} \in C^{\infty}(\mathcal{X}^*)$ also has an isolated non-degenerate minimum at some point $\theta^{\epsilon} \in \mathcal{X}^*$ ϵ-close to $0 \in \mathcal{X}^*$.
- On a neighbourhood of $0 \in \mathbb{T}_*$, denoting by $a^{\epsilon}_{jk} := \big(\partial^2_{jk}\lambda^{\epsilon}\big)(0)$; we have the expansions

$$\lambda^{\epsilon}(\theta) - \lambda^{\epsilon}(\theta^{\epsilon}) = \sum_{1\leq j,k\leq 2} a^{\epsilon}_{jk}(\theta_j - \theta^{\epsilon}_j)(\theta_k - \theta^{\epsilon}_k) + \mathscr{O}(|\theta - \theta^{\epsilon}|^3)$$

$$\lambda^{\epsilon}(\theta) - \lambda^{\epsilon}(\theta^{\epsilon}) = \sum_{1\leq j,k\leq 2} a^{\epsilon}_{jk}(\theta_j - \theta^{\epsilon}_j)(\theta_k - \theta^{\epsilon}_k) + \epsilon\mathscr{O}(|\theta - \theta^{\epsilon}|^3) + \mathscr{O}(|\theta - \theta^{\epsilon}|^4).$$

if λ_0 is symmetric around its minimum (as in the case $B_{\Gamma} = 0$)

The Hessian at the minimum of the modified Bloch band.

- There exists $\epsilon_0 > 0$ such that, for $\epsilon \in [0, \epsilon_0]$, we can choose a local coordinate system on a neighbourhood of $\theta^\epsilon \in \mathcal{X}^*$ that diagonalizes the symmetric positive definite matrix a^ϵ and we denote by $0 < m_1^\epsilon \leq m_2^\epsilon$ its eigenvalues.
- Let $0 < m_1 \leq m_2$ be the eigenvalues of the matrix $a_{jk} = \big(\partial^2_{jk}\lambda_0\big)(0)$.
- We notice that

$$m_j^\epsilon \, = \, m_j \, + \, \epsilon\mu_j \, + \, \mathscr{O}(\epsilon^2) \text{ for } j = 1, 2,$$

with μ_j explicitly computable.

Our goal is to obtain spectral information concerning the Hamiltonian $\mathfrak{Op}^{\epsilon,\kappa}(\widetilde{\lambda^\epsilon})$ starting from the spectral information about $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$ with

$$h_{m^\epsilon}(\xi) \, := \, m_1^\epsilon \xi_1^2 \, + \, m_2^\epsilon \xi_2^2,$$

defining an elliptic symbol of class $S^2_1(\Xi)$ that does not depend on the configuration space variables.

The model Landau Hamiltonian

We compare the bottom of the spectra of the following two operators

- the magnetic Hamiltonians $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$,
- the constant field magnetic Landau operator $\mathfrak{Op}^{\epsilon,0}(h_{m^\epsilon})$.

Proposition

For any compact set M in $\mathbb{R}$, there exist $\epsilon_K > 0$, $C > 0$ and $\kappa_K \in (0,1]$, such that for any $(\epsilon,\kappa) \in [0,\epsilon_K] \times [0,\kappa_K]$, the spectrum of the operator $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$ in ϵM is contained in bands of width $C\kappa\epsilon$ centred at the points $\{(2n+1)\,\epsilon\, m^\epsilon\, B_0\}_{n\in\mathbb{N}}$.

Isolating the minimum

- We choose an even function χ in $C_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, with $\operatorname{supp} \chi \subset (-2, +2)$ and $\chi(t) = 1$ on $[-1, +1]$.
- For $\delta > 0$ we define $g_{1/\delta}(\xi) := \chi\big(h_{m^\epsilon}(\delta^{-1}\xi)\big), \quad \xi \in \mathcal{X}^*$.
- We choose δ_0 such that $B_{\sqrt{2m_1^{-1}\delta_0}}(0) \subset \mathring{E}_*$ where $B_{\sqrt{2m_1^{-1}\delta_0}}(0)$ denotes the disk centred at 0 of radius ρ and $\mathring{E}_*$ denotes the interior of E_*.
- For any $\delta \in (0, \delta_0]$ we associate $\delta^\circ := \sqrt{m_1/2m_2}\,\delta$ so that we have $g_{1/\delta^\circ} = g_{1/\delta}\, g_{1/\delta^\circ}$.

For any $\delta \in (0, \delta_0]$, $g_{1/\delta} \in C_0^\infty(E_*)$.

- We may consider it as an element of $C_0^\infty(\mathcal{X}^*)$ by extending it by 0.
- We may define its Γ_*-*periodic continuation* to $\mathcal{X}^*$:

$$\widetilde{g}_{1/\delta}(\xi) := \sum_{\gamma \in \Gamma^*} g_{1/\delta}(\xi - \gamma),$$

The ϵ-dependent cut-off

Hypothesis

We shall impose the following scaling of the cut-off parameter $\delta > 0$:

$$\epsilon = \delta^{\mu}, \quad \mu > 1.$$

Then we have the following estimation near the minimum:

$$\lambda^{\epsilon}(\xi) g_{1/\delta}(\xi) = g_{1/\delta}(\xi)\, h_{m^{\epsilon}}(\xi) + \mathcal{O}(\delta^3),$$

or in the symmetric case:

$$\lambda^{\epsilon}(\xi) g_{1/\delta}(\xi) = g_{1/\delta}(\xi)\, h_{m^{\epsilon}}(\xi) + \mathcal{O}(\epsilon\delta^3) + \mathcal{O}(\delta^4).$$

We can thus take: $1 < \mu < 3$ in the general case,
resp. $1 < \mu < 4$ in the symmetric case.

The *shift* outside the minimum

For the region outside the minima, we need the operator:

$$\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda^\epsilon + (\delta^\circ)^2 \widetilde{g}_{1/\delta^\circ}\big).$$

Proposition

There exists $\epsilon_0 > 0$ and for $(\epsilon, \kappa, \delta) \in [0, \epsilon_0] \times [0, 1] \times (0, \delta_0]$, there exist some constants $C > 0$ and $C' > 0$ such that:

$$\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda^\epsilon + (\delta^\circ)^2\, \widetilde{g}_{1/\delta^\circ}\big) \geq \big(C\,\delta^2 - C'\,\epsilon\big)\ \mathbb{1}.$$

Remark

Taking $2 < \mu$ we have that $C\,\delta^2 - C'\,\epsilon > C''\epsilon^{2/\mu} >> \epsilon$ and for $0 \leq z \leq c\epsilon$, we denote by $r_{\delta,\epsilon,\kappa}(z)$ the symbol of $\Big(\mathfrak{Op}^{\epsilon,\kappa}\big(\lambda^\epsilon + (\delta^\circ)^2\, \widetilde{g}_{1/\delta^\circ}\big) - z\mathbb{1}\Big)^{-1}$.

The *"quasi-inverse"*.

- Let us fix some compact set $K \subset \mathbb{C}$ such that:

$$K \subset \mathbb{C} \setminus \{(2n+1)m\,B_0\}_{n\in\mathbb{N}}.$$

- There exist $\epsilon_K > 0$ and $\kappa_K \in [0,1]$ such that for $(\epsilon,\kappa) \in [0,\epsilon_K] \times [0,\kappa_K]$ and for $a \in K$, the point $\epsilon a \in \mathbb{C}$ belongs to the resolvent set of $\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon})$.
- We denote by $r^{\epsilon,\kappa}(\epsilon a)$ the magnetic symbol of $\big(\mathfrak{Op}^{\epsilon,\kappa}(h_{m^\epsilon}) - \epsilon a\big)^{-1}$.

The quasi-inverse

For $a \in K$ we want to define the following symbol in $\mathscr{S}'(\mathcal{X}^*)$ as the sum of the series on the right hand side:

$$\widetilde{r}_\lambda(\epsilon a) := \sum_{\gamma^*\in\Gamma_*} \tau_{\gamma^*}\big(g_{1/\delta}\,\sharp^{\epsilon,\kappa}\,r^{\epsilon,\kappa}(\epsilon a)\big) + \big(1-\widetilde{g}_{1/\delta}\big)\,\sharp^{\epsilon,\kappa}\,r_{\delta,\epsilon,\kappa}(\epsilon a)\,, \quad \delta = \epsilon^{1/\mu}.$$

The "*quasi-inverse*".

Proposition

For K as above, there exist $C > 0$, $\kappa_0 \in (0,1]$ and $\epsilon_0 > 0$ such that for $(\kappa, \epsilon, a) \in [0, \kappa_0] \times (0, \epsilon_0] \times K$, the symbol $\widetilde{r}_\lambda(\epsilon a)$ is well defined and we have

$$\|\mathfrak{Op}^{\epsilon,\kappa}(\widetilde{r}_\lambda(\epsilon a))\| \leq C_a \epsilon^{-1},$$

and

$$\big(\lambda_\epsilon - \epsilon a\big) \sharp^{\epsilon,\kappa} \widetilde{r}_\lambda(\epsilon a) \ = \ 1 + \mathfrak{r}_{\delta,a}, \ \text{with} \quad \|\mathfrak{Op}^{\epsilon,\kappa}(\mathfrak{r}_{\delta,a})\| \leq \ C\ \delta^{\mu-2}.$$

For $N > 0$, there exist C, ϵ_0 and κ_0 such that the spectrum of $\mathfrak{Op}^{\epsilon,\kappa}(\lambda_\epsilon)$ in $[0, (2N+2)mB_0\epsilon]$ consists of spectral islands centred at $(2n+1)mB_0\epsilon$, $0 \leq n \leq N$, with a width bounded by $C\left(\epsilon\kappa + \epsilon^{1+(\mu-2)/\mu}\right)$.

We may take $\mu = 5/2$ in the general case or $\mu = 3$ in the symmetric case.

Thank you for your attention!