Magnetic Coherent States

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Together with M. Măntoiu and S. Richard we have defined some families of *'magnetic'* coherent states and a Berezin magnetic quantization.

References

- A. Perelomov: *Generalized coherent states and their applications*, Springer, Berlin 1986.
- N.P. Landsman: *Mathematical topics between classical and quantum mechanics*, Springer-Verlag, New-York, 1998.
- M. Combescure, D. Robert: *Coherent states and applications in mathematical physics*, Springer, Dordrecht, Heidelberg, London, New York, 2012

References

- M.V. Karasev and T.A. Osborn, *Symplectic Areas, Quantization and Dynamics in Electromagnetic Fields*, J. Math. Phys. **43** (2002), 756–788.
- M. Măntoiu and R. Purice, *The Magnetic Weyl Calculus*, J. Math. Phys. **45**, No 4 (2004), 1394–1417.
- M. Măntoiu and R. Purice: *Strict Deformation Quantization for a Particle in a Magnetic Field*, J. Math. Phys. **46**, No.5, (2005).
- V. Iftimie, M. Măntoiu and R. Purice: *Magnetic Pseudodifferential Operators*, Publ. RIMS. **43**, 585–623, (2007).
- M. Măntoiu, R. Purice and S. Richard: *Coherent States in the Presence of a Variable Magnetic Field*, to appear in Int. J. of Geom. Meth. in Mod. Phys. **8**, No. 1 (2011), 187–202.

Plan of the talk

Introduction

States and Observables

The system S

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- the family of *observables*: $\mathfrak{O}(\mathsf{S})$

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Mathematical description:

	Classical	Quantum
$\mathfrak{P}(\mathsf{S})$	a symplectic manifold (Ξ,σ)	the projective space $\mathbb{P}(\mathscr{H})$
$\mathfrak{O}(\mathsf{S})$	the Poisson algebra $C^\infty(\Xi;\mathbb{R})$	the self adjoint operators $\mathbb{S}(\mathscr{H})$

on some complex Hilbert space $\mathscr{H}$

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Thus, a basic element will be a parameter $\hbar \in I_0$ where

$$I_0 \subset \mathbb{R}_+, \qquad 0 \notin I_0,$$

but 0 is an accumulation point for I_0.

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Thus, a basic element will be a parameter $\hbar \in I_0$ where

$$I_0 \subset \mathbb{R}_+, \qquad 0 \notin I_0,$$

but 0 is an accumulation point for I_0.
We shall always denote by $I := I_0 \cup \{0\}$.

Quantization in a magnetic field

We shall study a classical Hamiltonian system that can be described on the phase space (Ξ, σ_0) associated to a configuration space of the type $\mathcal{X} \cong \mathbb{R}^d$ for some $d \geq 2$.

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We have:

- $\mathcal{X}^*$ the dual of $\mathcal{X}$ with the duality $< \xi, x > := \xi(x)$, $\forall (x, \xi) \in \mathcal{X} \times \mathcal{X}^*$.
- $\Xi := \mathbb{T}^* \mathcal{X} \cong \mathcal{X} \times \mathcal{X}^*$, $\sigma_0\big((x, \xi), (y, \eta)\big) :=< \xi, y > - < \eta, x >$.

The magnetic field

- The magnetic field is described by a closed 2-form B on $\mathcal{X}$:

$$B(x) = \sum_{1 \le j,k \le d} B_{jk}(x) dx_j \wedge dx_k, \quad B_{jk}(x) = -B_{kj}(x), \quad dB = 0.$$

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- On $\mathcal{X} := \mathbb{R}^n$ the equations $B = dA$ have always a solution, definig a vector potential A for B.

$$A(x) = \sum_{1\leq j\leq d} A_j(x)dx_j, \qquad B_{jk}(x) = \big(\partial_j A_k\big)(x) - \big(\partial_k A_j\big)(x).$$

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- **Gauge transformations.** $B=dA=dA'$ is equivalent to the existence of Φ such that $A'=A+d\Phi$.

These equations can be considered either in $\mathcal{D}'$ or on smaller spaces like C^∞ or C^∞_{pol}.

The gauge invariant formalism

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where $\mathfrak{j}_B$ is the canonical isomorphism

$$\mathfrak{j}_B:\Xi\rightarrow\Xi^*,\quad <\mathfrak{j}_B(\mathfrak{X}),\mathfrak{Y}>:=\sigma^B(\mathfrak{X},\mathfrak{Y}).$$

The gauge invariant formalism

Using the canonical global coordinates we have:

$$\{f,g\}^B(x,\xi) :=$$

$$= \sum_{j=1}^{n} \left[\big(\partial_{\xi_j} f\big)(x,\xi)\big(\partial_{x_j} g\big)(x,\xi) - \big(\partial_{x_j} f\big)(x,\xi)\big(\partial_{\xi_j} g\big)(x,\xi) \right] +$$

$$\sum_{j,k=1}^{n} B_{jk}(x)\big(\partial_{\xi_j} f\big)(x,\xi)\big(\partial_{\xi_k} g\big)(x,\xi)$$

The quantum dynamics

The main point in passing to a *quantic description*

consists in introducing a *non-commutativity* between positions (in $\mathcal{X}$) and momenta (in $\mathcal{X}^*$).

The canonical commutation relations between the position observables $\{q_1, \ldots, q_n\}$ and the momenta $\{p_1, \ldots, p_n\}$ must be of the form:

$$[q_i, q_j] = 0, \ [p_i, p_j] = 0, \ [p_i, q_j] = -i\hbar\delta_{ij}, \qquad i, j = 1, \ldots, n.$$

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A way to introduce these commutation relations in a mathematical precise form is the Weyl system.

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- and two strongly continuous unitary representations:

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- satisfying the Weyl commutation relations:

$$U_\hbar(x)V(\xi) = e^{i\hbar<\xi,x>}\, V(\xi)U_\hbar(x), \qquad x \in \mathcal{X},\ \xi \in \mathcal{X}^*.$$

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The Weyl system - symplectic form

- Is given by a complex Hilbert space $\mathcal{H}$
- and a strongly continuous map

$$\Xi \ni X \mapsto W_\hbar(X) \in \mathcal{U}(\mathcal{H}),$$

- satisfying the relations

$$W_\hbar(X)W_\hbar(Y) = \exp\left\{\frac{i\hbar}{2}\sigma(X,Y)\right\} W_\hbar(X+Y), \quad W_\hbar(0) = 1.$$

(just take $W_\hbar(x,\xi) := e^{(i\hbar/2)<\xi,x>}U_\hbar(-x)V(\xi)$)

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The quantum observables

- For any test function $\phi \in \mathcal{S}(\Xi)$
- we can define the associated *quantum observable*

$$\mathfrak{Op}_\hbar(\phi) := (2\pi)^{-d} \int_\Xi [\mathcal{F}^{-1}\phi](X)\, W_\hbar(X)\, dX \ \in \mathbb{B}(\mathcal{H})$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform on $\mathcal{S}(\Xi)$.

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Then we have

- $[W_\hbar(x,\xi)f](y) = e^{-i\xi(y+(\hbar/2)x)}f(y+\hbar x)$,
- $[\mathfrak{Op}_\hbar(\phi)f](y) = (2\pi\hbar)^{-d}\int_X dz \int_{X'} d\zeta\; e^{(i/\hbar)\zeta(y-z)}\phi\left(\frac{y+z}{2},\zeta\right) f(z)$,

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and we can extend $\mathfrak{Op}_\hbar$ to a map

$$\mathfrak{Op}_\hbar : \mathcal{S}(\Xi)' \to \mathbb{B}(\mathcal{S}(\mathcal{X}); \mathcal{S}(\mathcal{X})')$$

that is an isomorphism of linear topological spaces.

The magnetic Schrödinger representation

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$$[Q_j, Q_k] = 0, \quad [\Pi_j^A, Q_k] = -i\hbar\delta_{jk}, \quad [\Pi_j^A, \Pi_k^A] = i\hbar B_{jk}(Q).$$

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representing the canonical variables in the magnetic field.

- We shall use the unitary groups associated to the above $2n$ self-adjoint operators and define the Magnetic Weyl system:

$$W_\hbar^A((x,\xi)) := e^{-i<\xi,(Q+(\hbar/2)x)>} e^{-(i/\hbar)\int_{[Q,Q+\hbar x]} A} e^{i\hbar<x,P>}$$

The *magnetic* algebra of quantum observables (1)

- For any test function $f : \Xi \to \mathbb{C}$ we define the associated magnetic Weyl operator:

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- In fact for any tempered distribution $F \in \mathcal{S}'(\Xi)$ we can define the linear operator:

$$\mathfrak{Op}^A_\hbar(F) := (2\pi)^{-d} \int_\Xi dX \hat{F}(X) W^A_\hbar(X) \in \mathbb{B}[\mathcal{S}(\mathcal{X}); \mathcal{S}'(\mathcal{X})]$$

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Observation: *Gauge covariance*

The Schrödinger representations associated to any two gauge-equivqlent vector potentials are unitarily equivqlent:

$$A' = A + d\varphi \quad \Rightarrow \quad \mathfrak{Op}^{A'}_\hbar(f) = e^{i\varphi(Q)} \mathfrak{Op}^A_\hbar(f) e^{-i\varphi(Q)}.$$

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The magnetic field B has components of class $C^\infty_{\text{pol}}(\mathcal{X})$.

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The magnetic Moyal product

The above functional calculus induces a *magnetic composition* on the complex linear space of test functions $\mathcal{S}(\Xi)$:

$$\mathfrak{Op}^A_\hbar(f \sharp^B_\hbar g) := \mathfrak{Op}^A_\hbar(f) \cdot \mathfrak{Op}^A_\hbar(g)$$

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Explicitely we have:

$$(f \sharp^B_\hbar g)(X) := (\pi\hbar)^{-2d} \int_\Xi dY \int_\Xi dZ \, e^{-(i/\hbar) \int_{\mathcal{T}_X(Y,Z)} \sigma^B} f(X-Y)\, g(X-Z)$$

where $\mathcal{T}_X(Y,Z)$ is the triangle in Ξ having vertices:

$$X-Y-Z, \quad X+Y-Z, \quad X-Y+Z.$$

The *magnetic* algebra of quantum observables (3)

By the Schwartz Kernel Theorem, any operator $T \in \mathbb{B}\big(\mathscr{S}(\mathcal{X});\mathscr{S}'(\mathcal{X})\big)$ is an integral operator with a distribution kernel $\mathfrak{K}(T) \in \mathscr{S}'(\mathcal{X} \times \mathcal{X})$.
For $F \in \mathscr{S}'(\mathcal{X} \times \mathcal{X})$ let $\mathfrak{I}(F) \in \mathbb{B}\big(\mathscr{S}(\mathcal{X});\mathscr{S}'(\mathcal{X})\big)$ be the associated operator.

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- We consider the change of variables:

$$\mathfrak{m}_\hbar : \mathscr{S}'(\Xi) \to \mathscr{S}'(\Xi), \quad \big(\mathfrak{m}_\hbar\big)(x, \xi) := F(x, \hbar\xi),$$

$$\Theta : \mathscr{S}'(\mathcal{X} \times \mathcal{X}) \to \mathscr{S}'(\mathcal{X} \times \mathcal{X}), \quad \big(\Theta(F)\big)(x, y) := F\left(\frac{x+y}{2}, y - x\right).$$

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- The inverse Fourier transform in the second variable:

$$\mathfrak{F} := \mathbb{1} \otimes \mathcal{F}^- : \mathscr{S}'(\Xi) \to \mathscr{S}'(\mathcal{X} \times \mathcal{X})$$

- and the operator $\mathfrak{e}^A_\hbar$ of multiplication on $\mathscr{S}'(\mathcal{X} \times \mathcal{X})$ with the C^∞ function $e^{-(i/\hbar)\int_{[x,y]} A}$ (we choose the components of A in C^∞_{pol}).

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Then, for $F \in \mathscr{S}'(\Xi)$ we have
$\mathfrak{Op}^A_\hbar(F) = \mathfrak{I}\big(\mathfrak{e}^A_\hbar \circ \Theta \circ \mathfrak{F} \circ \mathfrak{m}_\hbar(F)\big)$, all the applications being bijective.

Strict deformation quantization

Let us recall:

Definition

A Poisson algebra is a triple $(\mathscr{A}, \circ, \{\cdot, \cdot\})$, where $\mathscr{A}$ is a real vector space, $\circ$, $\{\cdot, \cdot\}$ are bilinear maps : $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ such that $\circ$ is associative and commutative, $\{\cdot, \cdot\}$ is antisymmetric and for each $\varphi \in \mathscr{A}$, $\{\varphi, \cdot\}$ is a derivation both with respect to $\circ$ and to $\{\cdot, \cdot\}$. Thus, aside bilinearity, the two maps satisfy for all $\varphi, \psi, \rho \in \mathscr{A}$:

(i) $\psi \circ \varphi = \varphi \circ \psi, \qquad (\psi \circ \varphi) \circ \rho = \psi \circ (\varphi \circ \rho)$,

(ii) $\{\psi, \varphi\} = -\{\varphi, \psi\}$,

(iii) $\{\varphi, \psi \circ \rho\} = \psi \circ \{\varphi, \rho\} + \{\varphi, \psi\} \circ \rho \quad \text{(Leibnitz rule)}$,

(iv) $\{\varphi, \{\psi, \rho\}\} = \{\{\varphi, \psi\}, \rho\} + \{\psi, \{\varphi, \rho\}\} \quad \text{(Jacobi's identity)}$.

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Let us recall:

Definition

A Poisson algebra is a triple $(\mathscr{A}, \circ, \{\cdot,\cdot\})$, where $\mathscr{A}$ is a real vector space, $\circ$, $\{\cdot,\cdot\}$ are bilinear maps : $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ such that $\circ$ is associative and commutative, $\{\cdot,\cdot\}$ is antisymmetric and for each $\varphi \in \mathscr{A}$, $\{\varphi,\cdot\}$ is a derivation both with respect to $\circ$ and to $\{\cdot,\cdot\}$. Thus, aside bilinearity, the two maps satisfy for all $\varphi, \psi, \rho \in \mathscr{A}$:

(i) $\psi \circ \varphi = \varphi \circ \psi, \qquad (\psi \circ \varphi) \circ \rho = \psi \circ (\varphi \circ \rho)$,

(ii) $\{\psi, \varphi\} = -\{\varphi, \psi\}$,

(iii) $\{\varphi, \psi \circ \rho\} = \psi \circ \{\varphi, \rho\} + \{\varphi, \psi\} \circ \rho \quad \text{(Leibnitz rule)}$,

(iv) $\{\varphi, \{\psi, \rho\}\} = \{\{\varphi, \psi\}, \rho\} + \{\psi, \{\varphi, \rho\}\} \quad \text{(Jacobi's identity)}$.

Let $\mathscr{A}_0$ be a Poisson algebra which is densely contained in the self-adjoint part $\mathfrak{C}^0_{\mathbb{R}}$ of an abelian C^*-algebra $\mathfrak{C}^0$.

Strict deformation quantization

Definition

A strict quantization of the Poisson algebra $(\mathscr{A}_0, \circ, \{\cdot,\cdot\})$ is a family of maps $\left(\mathfrak{Q}^\hbar : \mathscr{A}_0 \to \mathfrak{C}^\hbar_{\mathbb{R}}\right)_{\hbar \in I}$, where

(i) $\forall \hbar \in I_0$, $\mathfrak{C}^\hbar$ is a C^*-algebra, with product $\sharp^\hbar$ and norm $\|\cdot\|_\hbar$.

(ii) $\mathfrak{Q}^\hbar : \mathscr{A}_0 \to \mathfrak{C}^\hbar_{\mathbb{R}}$ is $\mathbb{R}$-linear $\forall \hbar \in I_0$ and $\mathfrak{Q}^0$ is just the inclusion map, and the following axioms are fulfilled:

(a) RIEFFEL'S CONDITION:

$I \ni \hbar \to \| \mathfrak{Q}^\hbar(\varphi) \|_\hbar \in \mathbb{R}_+$ is continuous $\forall \varphi \in \mathscr{A}_0$.

(b) VON NEUMANN CONDITION: For $\varphi, \psi \in \mathscr{A}_0$,

$\lim\limits_{\hbar \to 0} \| \frac{1}{2}\left(\varphi^\hbar \sharp^\hbar \psi^\hbar + \psi^\hbar \sharp^\hbar \varphi^\hbar\right) - \mathfrak{Q}^\hbar(\varphi \circ \psi) \|_\hbar \to 0.$

(c) DIRAC'S CONDITION: For $\varphi, \psi \in \mathscr{A}_0$,

$\lim\limits_{\hbar \to 0} \| \frac{1}{i\hbar}\left(\varphi^\hbar \sharp^\hbar \psi^\hbar - \psi^\hbar \sharp^\hbar \varphi^\hbar\right) - \mathfrak{Q}^\hbar\left(\{\varphi, \psi\}\right) \|_\hbar \to 0.$

(e) COMPLETENESS: $\mathfrak{Q}^\hbar(\mathscr{A}_0)$ is dense in $\mathfrak{C}^\hbar_{\mathbb{R}}$ for all $\hbar \in I$.

Strict deformation quantization

A strict quantization $\left(\mathfrak{Q}^\hbar : \mathscr{A}_0 \to \mathscr{C}^\hbar_{\mathbb{R}}\right)_{\hbar \in I}$ is called a strict deformation quantization if for each $\hbar$, $\mathfrak{Q}^\hbar(\mathscr{A}_0)$ is a subalgebra of $\mathscr{C}^\hbar_{\mathbb{R}}$ and $\mathfrak{Q}^\hbar$ is injective.

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Let us consider:

- $\mathscr{A}_0:=\mathscr{S}(\Xi)$ with $\circ$ the usual pointwise multiplication and $\{.,.\}=\{.,.\}_B$ the Poisson bracket associated to the 'magnetic' simplectic form on Ξ.
- $\mathfrak{C}^0:=C_\infty(\Xi)$ (continuous functions vanishing at infinity).
- $\forall\hbar>0$, $\mathfrak{C}^{\hbar}:=\mathbb{B}_\infty\big(L^2(\mathcal{X})\big)$ (the compact operators in the Schrödinger representation of the magnetic Weyl system).
- $\mathfrak{Q}^{\hbar}(\varphi):=\mathfrak{Op}^A_{\hbar}(\varphi)$ for some vector potential A associated to B.

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Theorem [MP JMP'05]

The above family is a strict deformation quantization.

Weyl coherent states

Suppose given a Weyl system $W_\hbar : \Xi \to \mathcal{U}(\mathcal{H})$ for some $\hbar \in I_0$.

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Proposition

For any two vectors $(\varphi,\psi) \in \big(\mathcal{H} \setminus \{0\}\big)^2$

- the map $\Xi \ni X \mapsto \mathcal{W}^\hbar_{\varphi,\psi}(X) := (\varphi_\hbar(X),\psi)_\mathcal{H} \in \mathbb{C}$ is of class $L^2(\Xi)$ for the measure $đ^d_\hbar X := \frac{dX^d}{(2\pi\hbar)^d}$.
- and using the canonical Riesz anti-isomorpism $\mathfrak{R} : \mathcal{H} \rightarrow \mathcal{H}^*$ the map $\mathcal{H}^* \otimes \mathcal{H} \ni \mathfrak{R}(\varphi) \otimes \psi \mapsto \mathcal{W}^\hbar_{\varphi,\psi} \in L^2\big(\Xi, \frac{dX^d}{(2\pi\hbar)^d}\big)$ is unitary.

Weyl coherent states

Let us consider the vector $\varphi \in \mathcal{H}$ to be of unit norm and let us denote its associated orthogonal projection in $\mathcal{H}$ by

$$P_\varphi \equiv |\varphi >< \varphi| \in \mathbb{P}(\mathcal{H}).$$

Thus, chosing any two vectors $(\psi_1, \psi_2) \in \mathcal{H}^2$ we have

$$(\psi_1, \psi_2)_{\mathcal{H}} = \int_{\Xi} \overline{\mathcal{W}^{\hbar}_{\varphi,\psi_1}(X)} \mathcal{W}^{\hbar}_{\varphi,\psi_2}(X) \frac{dX^d}{(2\pi\hbar)^d} = \int_{\Xi} \left(\psi_1, P_{\varphi_\hbar(X)}\psi_2\right)_{\mathcal{H}} \frac{dX^d}{(2\pi\hbar)^d}.$$

Thus $\int_{\Xi} P_{\varphi_\hbar(X)} \frac{dX^d}{(2\pi\hbar)^d} = \mathbb{1}$

in the weak operator topology on $\mathbb{B}(\mathcal{H})$.

The Projective Space

$\mathbb{P}(\mathcal{H})$ as a metric space.

Definition

Given a complex Hilbert space $\mathcal{H}$,

$$\mathbb{P}(\mathcal{H}) := \big\{P \in \mathbb{B}_1(\mathcal{H}) \mid P^2 = P = P^*,\ \mathrm{Tr}P = 1\big\}.$$

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Given a complex Hilbert space $\mathcal{H}$,

$$\mathbb{P}(\mathcal{H}) := \left\{P \in \mathbb{B}_1(\mathcal{H}) \mid P^2 = P = P^*,\ \mathrm{Tr}P = 1\right\}.$$

- $\forall (P, Q) \in \mathbb{P}(\mathcal{H})^2$ we have that $0 \leq \mathrm{Tr}(PQ) \leq 1$.
- The following applications define equivalent metrics on $\mathbb{P}(\mathcal{H})$:

$$d_p(P, Q) := \|P - Q\|_p \equiv \left(\mathrm{Tr}|P - Q|^p\right)^{(1/p)}, \qquad 1 \leq p \leq \infty.$$

$$\widetilde{d}(P, Q) := \arccos \mathrm{Tr}(PQ).$$

- We notice that: $d_\infty(P, Q) = \sqrt{1 - \mathrm{Tr}(PQ)}$.

$\mathbb{P}(\mathcal{H})$ as a quotient space.

- Let $\mathcal{S}(\mathcal{H}) := \{\psi \in \mathcal{H} \mid \|\psi\|_{\mathcal{H}} = 1\}$.
- We have the group action $\mathbb{U}(1) \times \mathcal{S}(\mathcal{H}) \ni (\lambda, \psi) \mapsto \lambda\psi \in \mathcal{S}(\mathcal{H})$.
- Then $\mathbb{P}(\mathcal{H}) \cong \mathcal{S}(\mathcal{H})/\mathbb{U}(1)$ as metric spaces; i.e. the quotient metric $\hat{d}([\phi], [\psi]) := \inf_{\lambda \in \mathbb{U}(1)} \|\phi - \lambda\psi\|_{\mathcal{H}} = \sqrt{2\big(1 - [\mathrm{Tr}(P_\phi P_\psi)]^{1/2}\big)}$
  is equivalent with the above metrics,
  (Here $P_\phi \psi := (\phi, \psi)_{\mathcal{H}} \phi$ and we shall denote it also by $P_\phi \equiv |\phi >< \phi|$.)
- We also have $\mathbb{P}(\mathcal{H}^*) \cong \mathcal{H}/\mathbb{C}^*$
  for the natural group action $\mathcal{H}^* \times \mathbb{C}^* \ni (\psi, c) \mapsto c\psi \in \mathcal{H}^*$.

A group action on $\mathbb{P}(\mathcal{H})$.

- Let $\mathbb{U}(\mathcal{H}) := \{U \in \mathbb{B}(\mathcal{H}) \mid UU^* = U^*U = \mathbb{1}\}$ endowed with the operator multiplication and with the topology defined by the operator norm.
- We have the topological group action
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For any fixed $P \in \mathbb{P}(\mathcal{H})$ let

- $\mathcal{V}_1(P) := \{Q \in \mathbb{P}(\mathcal{H}) \mid d_\infty(P, Q) < 1\} = \{Q \in \mathbb{P}(\mathcal{H}) \mid \mathrm{Tr}(PQ) > 0\}$

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- $\mathbb{I}_P := \{U \in \mathbb{U}(\mathcal{H}) \mid UPU^* = P\}$,
$\mathfrak{p}_P : \mathbb{U}(\mathcal{H}) \to \mathbb{U}(\mathcal{H})/\mathbb{I}_P =: \mathbb{U}(\mathcal{H})_P$.

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- Let also $\mathcal{V}_{\sqrt{2}}(\mathbb{1}) := \left\{\mathrm{U} \in \mathbb{U}(\mathcal{H}) \mid \|\mathrm{U} - \mathbb{1}\|_{\mathbb{B}(\mathcal{H})} < \sqrt{2}\right\}$.

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Then we have an isometry $\mathfrak{h}_P : \mathcal{V}_1(P) \xrightarrow{\sim} \mathfrak{p}_P\big(\mathcal{V}_{\sqrt{2}}(\mathbb{1})\big) \subset \mathbb{U}(\mathcal{H})_{\mathrm{P}}$.

The manifold structure on $\mathbb{U}(\mathcal{H})$.

Differentiable curves in $\mathbb{U}(\mathcal{H})$.

- $\gamma : (-1,1) \to \mathbb{U}(\mathcal{H})$ continuous such that $\gamma(0) = \mathbb{1}$ and $\exists X_\gamma \in \mathbb{B}(\mathcal{H})$ with $\lim_{t\to 0} \left\| \frac{\gamma(t)-\mathbb{1}}{t} - X_\gamma \right\|_{\mathbb{B}(\mathcal{H})} = 0$.
- The unitarity implies that $X_\gamma^* = -X_\gamma$.
- Thus, $\mathbb{U}(\mathcal{H})$ with the operator norm topology is an infinite dimensional manifold of real Banach type having the tangent space at the identity isomorphic to the real Banach space

$$\mathbb{B}_{ah}(\mathcal{H}) := \{X \in \mathbb{B}(\mathcal{H}) \mid X^* = -X\}.$$

The manifold structure on $\mathbb{P}(\mathcal{H})$.

Let us fix $P \in \mathbb{P}(\mathcal{H})$.

- Let us transport any differentiable curve $\gamma : (-1, 1) \to \mathbb{U}(\mathcal{H})$ on $\mathbb{P}(\mathcal{H})$ by conjugation on P: $\gamma_P(t) := \gamma(t)P\gamma(t)^* \in \mathbb{P}(\mathcal{H})$.

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- We notice that $\gamma(t) \in \mathbb{I}_P \;\Leftrightarrow\; \big[X_\gamma, P\big] = 0$. Let
  $\mathbb{H}_P := \big\{X \in \mathbb{B}_{ah}(\mathcal{H}) \mid \big[X, P\big] = 0\big\}$.

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Thus $\mathbb{T}_P\big[\mathbb{P}(\mathcal{H})\big] \simeq \mathbb{B}_{ah}(\mathcal{H})/\mathbb{H}_P$

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Hilbertian model for $\mathbb{T}_P\big[\mathbb{P}(\mathcal{H})\big]$.

- For any $\phi \in P\mathcal{H}$
- let us define $\Upsilon_\phi : \mathbb{B}_{ah} \ni X \mapsto \Upsilon_\phi X := \big(1\!\!1 - \mathrm{P}\big)\mathrm{X}\phi \in [\phi]^\perp \subset \mathcal{H}$.

Then $\Upsilon_\phi : \mathbb{B}_{ah}/\mathbb{H}_P \to \big(1\!\!1 - \mathrm{P}\big)\mathcal{H}$ is a bijective isometry.

The symplectic structure on $\mathbb{P}(\mathcal{H})$.

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- and we have the following canonical symplectic form:

$$\sigma_P^{\mathbb{P}(\mathcal{H})}(X_1, X_2) := \hat{P}\big([X_1, X_2]\big) = \mathrm{Tr}\big(P[X_1, X_2]\big) = \\ = -2\Im\big(\Upsilon_\phi X_1, \Upsilon_\phi X_2\big)_{\mathcal{H}}, \quad \text{for } \phi \in P\mathcal{H}.$$

Magnetic Coherent States

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We want these states to provide also a kind of *pure state quantization* in the sense of N.P. Landsman (that I shall briefly present further).

In fact, together with Marius Măntoiu and Serge Richard we have constructed two types of *magnetic coherent states*:

- both depending only on the magnetic field and not on the vector potential,
- both reducing to the usual *Weyl coherent states* when $B = 0$
- but each one being more adequate to certain specific features connected with the *general properties* a coherent states system is supposed to have.

Magnetic symbols of 1-d projections

Let us consider some A with $B = dA$ and the *magnetic Weyl system* with $\mathcal{H} = L^2(\mathcal{X})$.

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Let us consider $\phi \in \mathcal{S}(\mathcal{H})$, i.e. $\phi \in L^2(\mathcal{X})$ with $\int_{\mathcal{X}} |\phi(x)|^2 dx = 1$ and its associated magnetic 1-d projection $P_\phi \in \mathbb{P}(\mathcal{H})$ given by $P_\phi \psi = (\phi, \psi)_{L^2(\mathcal{X})} \phi$.

Thus $P_\phi = \mathfrak{I}(\widetilde{p}_\phi)$ where $\widetilde{p}_\phi(x, y) := \phi(x)\overline{\phi(y)}$ is a distribution kernel in $L^2(\mathcal{X} \times \mathcal{X})$. Thus its magnetic symbol $p_\phi \in L^2(\Xi)$ satisfies:

$$\mathfrak{e}^A_\hbar \circ \Theta \circ \mathfrak{F} \circ \mathfrak{m}_\hbar(p_\phi) = \widetilde{p}$$

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and thus we would obtain a symbol depending on the vector potential A. We prefer to *'rename'* the state vectors by making an A-dependent unitary transformation (*the transversal gauge*)

$$\big(\mathfrak{U}_\hbar^A \phi\big)(x) := e^{(i/\hbar) \int_{[0,x]} A} \phi(x).$$

so that $P_{\mathfrak{U}_\hbar^A \phi} = \mathfrak{I}(\widetilde{p}_\phi^A)$ with $\widetilde{p}_\phi^A(x,y) := e^{(i/\hbar) \int_{[0,x]} A} e^{(-i/\hbar) \int_{[0,y]} A} \phi(x)\overline{\phi(y)}$.

Magnetic symbols of 1-d projections

We define

the magnetic projection symbol associated to the vector $\phi \in \mathcal{S}(\mathcal{H})$ to be

$$p^B_{\phi,\hbar} = \mathfrak{m}_\hbar^{-1} \mathfrak{F}^{-1} \Theta^{-1} \left[\overline{\omega^B_\hbar} (\phi \otimes \overline{\phi}) \right]$$

with $\omega^B_\hbar(x, y) := e^{-(i/\hbar) \int_{<0,x,y>} B}$.

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We denote by $$P^A_{\phi,\hbar} := \mathfrak{Op}^A(p^B_{\phi,\hbar})$$
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the Schrödinger representation of a *gauge invariant* projection symbol.

A magnetic projection symbol

is defined as $p \in L^2(\Xi)$ such that $\overline{p} = p = p\sharp^B_\hbar p$.

Magnetic Coherent States

The Perelomov type magnetic coherent states

The Perelomov type Magnetic Coherent States

Definition 1

Given any $\phi \in \mathcal{S}(\mathcal{H})$ we define the following family of pure quantum states indexed by $X \in \Xi$:

$$P^A_{\phi,\hbar}(X) := W^A_\hbar(\hbar X)^{-1} P^A_{\phi,\hbar} W^A_\hbar(\hbar X)$$

Then $P^A_{\phi,\hbar}(X) = \mathfrak{Op}^A(p^B_{\phi,\hbar}(X))$ where, if we denote by $\mathfrak{e}_X(Y) := e^{-(i\sigma(X,Y)}$ $(\forall (X,Y) \in \Xi^2)$ we have

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The Perelomov type Magnetic Coherent States

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Given any $\phi \in \mathcal{S}(\mathcal{H})$ we define the following family of pure quantum states indexed by $X \in \Xi$:

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We call $\{P^A_{\phi,\hbar}(X)\}_{X\in\Xi}$ the Perelomov type magnetic coherent states.

The partition of unity property

Proposition

For any two vectors $(\phi, \psi) \in \big(\mathcal{H} \setminus \{0\}\big)^2$

- the map $\Xi \ni X \mapsto \mathcal{W}^{A,\hbar}_{\phi,\psi}(X) := (\mathfrak{U}^A_\hbar(X)\phi, \psi)_{\mathcal{H}} \in \mathbb{C}$ is of class $L^2(\Xi)$ for the measure $\bar{d}^d_\hbar X := \frac{dX^d}{(2\pi\hbar)^d}$.
- and using the canonical Riesz anti-isomorpism $\mathfrak{R} : \mathcal{H} \to \mathcal{H}^*$ the map $\mathcal{H}^* \otimes \mathcal{H} \ni \mathfrak{R}(\varphi) \otimes \psi \mapsto \mathcal{W}^{A,\hbar}_{\varphi,\psi} \in L^2\big(\Xi, \frac{dX^d}{(2\pi\hbar)^d}\big)$ is unitary.
- $\int_\Xi \frac{dX^d}{(2\pi\hbar)^d} P^A_{\phi,\hbar}(X) = 1\!\!1$ in the weak operator topology.

Magnetic Coherent States

The pure state quantization

The classical limit

We are in the following special case of a *"quantization"*:

- **Classical description:**
  The phase space $\Xi \equiv \mathcal{X} \times \mathcal{X}^*$ with simplectic form σ^B and associated Poisson bracket $\{.,.\}_B : C^\infty \times C^\infty \to C^\infty$.
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- **Quantum description:**
  The phase space $\mathbb{P}(\mathcal{H})$ (for some complex Hilbert space $\mathcal{H}$).
  The bounded observables $\mathbb{B}_h(\mathcal{H}) := \{T \in \mathbb{B}(\mathcal{H}) \mid T^* = T\}$ so that $\mathrm{Tr}(PT)$ is the mean value of $T \in \mathbb{B}_h(\mathcal{H})$ in the state $P \in \mathbb{P}(\mathcal{H})$.

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We want to connect the two descriptions when $\hbar \to 0$.
Remark: $\mathbb{P}(\mathcal{H})$ has a canonical symplectic structure with the symplectic form $\sigma^{\mathbb{P}(\mathcal{H})}$. We have to work with the symplectic form $\sigma_\hbar^{\mathbb{P}(\mathcal{H})} := \hbar\sigma^{\mathbb{P}(\mathcal{H})}$.

Transition probability structure

Let us notice that in the quantum description each state $P \in \mathbb{P}(\mathcal{H})$ is also a bounded observable (we have $\mathbb{P}(\mathcal{H}) \subset \mathbb{B}_h(\mathcal{H})$).

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The mean value of the observable state $Q \in \mathbb{P}(\mathcal{H})$ in the state $P \in \mathbb{P}(\mathcal{H})$, given by $\mathrm{Tr}(PQ) \in [0, 1]$ is called the transition probability from state P to state Q.

Pure State Quantization

Definition

We call *pure state quantization* of a symplectic space (Σ, σ) of dimension $2d$, a complex Hilbert space $\mathcal{H}$ together with a family of injective applications $\{P_\hbar : \Sigma \to \mathbb{P}(\mathcal{H})\}_{\hbar \in I_0}$ satisfying the following three axioms:

- **Axiom I:** $\int_\Xi \frac{dX^d}{(2\pi\hbar)^d} P_\hbar(X) = \mathbb{1}$ in the weak operator topology.
- **Axiom II:**
$$\lim_{\hbar\to 0} \int_\Xi \frac{dX^d}{(2\pi\hbar)^d} \mathrm{Tr}\big(P_\hbar(Z)P_\hbar(Y)\big) f(Y) = f(Z),$$
$$\forall f \in BC(\Sigma), \quad \forall Z \in \Sigma.$$
- **Axiom III:** Let us denote by $P_\hbar^* \sigma_\hbar^{\mathbb{P}(\mathcal{H})}$ the pull-back on the tangent space of Σ of the canonical symplectic form on $\mathbb{P}(\mathcal{H})$; then
$$\lim_{\hbar\to 0} P_\hbar^* \sigma_\hbar^{\mathbb{P}(\mathcal{H})} = \sigma.$$

Pure State Quantization

Theorem

Taking $\Sigma := \Xi$, a magnetic field B with components of class $BC^\infty(\mathcal{X})$, its associated symplectic form σ^B and $\mathcal{H} := L^2(\mathcal{X})$
the family of maps $\{P^A_{\phi,\hbar} : \Xi \to \mathbb{P}(\mathcal{H})\}_{\hbar \in I_0}$ satisfies Axioms I and II of a pure state quantization for any $\phi \in \mathcal{S}(\mathcal{H})$.

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Remark

Concerning Axiom III we have the following result:

$$\lim_{\hbar \to 0} \left[\left(P^A_{\phi,\hbar} \right)^* \sigma_\hbar^{\mathbb{P}(\mathcal{H})} \right]_X = \int_0^1 \sigma^{B(sx)} ds. \qquad (!)$$

Magnetic Coherent States

The Landsman type magnetic coherent states

The Landsman Magnetic Coherent States

Definition 1

Given any $\phi \in \mathcal{S}(\mathcal{H})$ we define the following family of pure quantum states indexed by $X \in \Xi$:

$$Q^A_{\phi,\hbar}(X) := \mathfrak{U}^A_\hbar(X) W^0_\hbar(\hbar X)^{-1} P_\phi W^0_\hbar(\hbar X) \big[\mathfrak{U}^A_\hbar(Z)\big]^{-1},$$

where

$$\big(\mathfrak{U}^A_\hbar(X)\phi\big)(y) := e^{(i/\hbar)\int_{[x,y]} A}\phi(y).$$

Then $$Q^A_{\phi,\hbar}(X) = \mathfrak{Op}^A(q^B_{\phi,\hbar}(X)) \; .$$

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We call $\{Q^A_{\phi,\hbar}(X)\}_{X\in\Xi}$ the Landsman type magnetic coherent states.

Pure State Quantization

Theorem

For $\Sigma := \Xi$, for a magnetic field B with components of class $BC^\infty(\mathcal{X})$ and for the associated symplectic form σ^B, taking $\mathcal{H} := L^2(\mathcal{X})$ and the family of maps $\{Q^A_{\phi,\hbar} : \Xi \to \mathbb{P}(\mathcal{H})\}_{\hbar \in I_0}$ is a pure state quantization for any $\phi \in \mathcal{S}(\mathcal{H})$.

Magnetic Coherent States

Comments on the classical limit

The projective representation on $\mathbb{P}(\mathcal{H})$

A basic step in defining a system of coherent states is:

- to raise the magnetic Weyl system: $W^A_\hbar : \Xi \to \mathcal{U}\big(L^2(\mathcal{X})\big)$
- to a *projective automorphism* representation on the algebra of bounded observables: $\mathcal{W}^A_\hbar : \Xi \to \mathbb{A}\mathrm{ut}\big[\mathbb{B}\big(L^2(\mathcal{X})\big)\big]$
$$\mathcal{W}^A_\hbar X \;:=\; W^A_\hbar X \big(W^A_\hbar\big)^{-1}.$$
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Remark

We have the evident equality

$$\mathbb{T}\big[P^A_{\phi,\hbar}(X)\big] \;=\; \mathbb{T}\big[\mathcal{W}^A_\hbar P_\phi\big].$$

The projective representation on $\mathbb{P}(\mathcal{H})$

Remark

The Weyl system being not a representation of the linear group Ξ we have

$$\mathcal{W}_\hbar^A(X+tZ)P \neq \mathcal{W}_\hbar^A(tZ)\mathcal{W}_\hbar^A(X)P.$$

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Thus:

- $i \left.\frac{d}{dt}\right|_{t=0} \mathcal{W}^A_\hbar(t\underline{Z})P = \zeta \cdot Q - z \cdot \Pi^A_\hbar =: \mathfrak{l}^A(Z),$
- $i \left.\frac{d}{dt}\right|_{t=0} \mathcal{W}^A_\hbar(X + t\underline{Z})P =$
  $\zeta \cdot Q - z \cdot \Pi^A_\hbar + \hbar \int_0^1 sds \sum_{j,k=1}^n z_j x_k B_{jk}\big(Q + (1-s)\hbar x\big) =: \mathcal{Z}^A_\hbar(X, \underline{Z};$

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Let us define

$$\mathbb{W}^A_{\hbar,P} : \mathbb{T}\Xi \ni (X,\underline{Z}) \mapsto \Big(\mathcal{W}^A_\hbar(X)P, (1 - \mathcal{W}^A_\hbar(X))\mathfrak{l}^A(Z)\phi_X\Big) \in \mathbb{T}\mathbb{P}(\mathcal{H}),$$

$\forall\phi_X \in \mathcal{W}^A_\hbar(X)P\mathcal{H}.$

The classical limit

Theorem

For a magnetic field with components of class $BC^\infty(\mathcal{X})$ we have

$$\lim_{\hbar\searrow 0}\sigma^{\mathbb{P}(\mathcal{H})}_{\hbar,P^A_\phi(X)}\left(\mathbb{W}^A_{\hbar,P^A_\phi}(X)Z_1,\mathbb{W}^A_{\hbar,P^A_\phi}(X)Z_2\right)=\sigma^{B(x)}_X(Z_1,Z_2),$$

Magnetic Coherent States

Magnetic Coherent States - Symbols

The symbolic calculus

- Let us notice that $\forall (f,g) \in \mathscr{S}(\Xi)^2$ we have $f \sharp_\hbar^B g \in \mathscr{S}(\mathcal{X})$.
- Let us induce the following C^*-norm on $\mathscr{S}(\Xi)$:
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The mean value of $f \in \mathfrak{C}^B_{0,\hbar}$ in the state $p^B_{\phi,\hbar}$ is

$$\int_\Xi \left[p^B_{\phi,\hbar} \sharp^B_\hbar f\right](X)\, dX.$$