# QMATH-10

# Local smoothing with lots of trapping

Maciej Zworski
UC Berkeley

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In this talk I would like to show an example of an interaction between

# mathematical physics

and

# partial differential equations.

Let us consider one of our favourite operators:

$$e^{-it\Delta} : L^2(\mathbf{R}^n) \longrightarrow L^2(\mathbf{R}^n) \, .$$

This Schrödinger propagator, $e^{-it\Delta}$, is unitary on any Sobolev space, so regularity is not improved in propagation.

About 20 years ago, Sjölin, Vega, and also Constantin-Saut discovered that the regularity improves when we integrate in time and cut-off in space:

$$\int_0^T \|\chi \exp(-it\Delta)u\|^2_{H^{1/2}}\, dt \leq C\|u\|^2_{L^2}\,, \quad \chi \in C^\infty_{\mathrm{c}}(\mathbf{R}^n)\,.$$

In $\mathbf{R}^n$ we can take $T = \infty$ , and $\chi(x) = \langle x\rangle^{-1/2-\epsilon}$.

This much exploited effect is known as

LOCAL SMOOTHING.

$$\int_0^T \|\chi \exp(-it\Delta)u\|^2_{H^{1/2}}\, dt \leq C\|u\|^2_{L^2}\,, \quad \chi \in C^\infty_{\mathrm{c}}(\mathbf{R}^n)\,.$$

Proof: [due to Burq] The bound is equivalent to boundedness of

$$T : L^2_x \longrightarrow L^2_t H^{1/2}_x\,, \quad Tu(x,t) := \chi(x)(e^{-it\Delta}u)(x)\,,$$

or, equivalently,

$$TT^* : L^2_t H^{-1/2}_x \longrightarrow L^2_t H^{1/2}_x\,,$$

$$TT^* f(x,t) = \int_{-\infty}^{\infty} (\chi e^{-i(t-s)\Delta}\chi) f(x,s)\, ds$$

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$$= \sum_{\pm} \Big(\mathbf{1}_{\pm[0,\infty)}(\bullet)\chi e^{-i\bullet\Delta}\chi\Big) * f(\bullet, x)(t)\,.$$

We want to show that

$$\int_{-\infty}^{\infty} \|(\mathbf{1}_{\pm[0,\infty)}(\bullet)\chi e^{-i\bullet\Delta}\chi) * f(\bullet, x)(t)\|^2_{H_x^{1/2}} dt$$

$$\leq C \int_{-\infty}^{\infty} \|f(t,x)\|^2_{H_x^{-1/2}}\, dt\,,$$

or by Plancherel's Theorem,

$$\int_{-\infty}^{\infty} \|\mathcal{F}_{t\mapsto\lambda}\Big(\mathbf{1}_{\pm[0,\infty)}(t)\chi e^{-it\Delta}\chi\Big)(\lambda)\hat{f}(\lambda, x)\|^2_{H_x^{1/2}} d\lambda$$

$$\leq C \int_{-\infty}^{\infty} \|\hat{f}(\lambda, x)\|^2_{H_x^{-1/2}}\, d\lambda\,,$$

$$\int_{-\infty}^{\infty} \| \mathcal{F}_{t \mapsto \lambda} \Big( \mathbf{1}_{\pm[0,\infty)}(t) \chi e^{-it\Delta} \chi \Big)(\lambda) \widehat{f}(\lambda, x) \|^2_{H_x^{1/2}} d\lambda$$

$$\leq C \int_{-\infty}^{\infty} \| \widehat{f}(\lambda, x) \|^2_{H_x^{-1/2}} \, d\lambda$$

We recognize that

$$\mathcal{F}_{t \mapsto \lambda} \Big( \mathbf{1}_{\pm[0,\infty)}(t) \chi e^{-it\Delta} \chi \Big)(\lambda) = \chi(-\Delta - \lambda \mp i0)^{-1} \chi \, .$$

So, all we need is:

$$\chi(-\Delta - \lambda \mp i0)^{-1} \chi = \mathcal{O}(1) : H^{-1/2} \longrightarrow H^{1/2} \, ,$$

and that is the same as:

$$\chi(-\Delta - \lambda \mp i0)^{-1} \chi = \mathcal{O}\left( \frac{1}{\sqrt{\lambda}} \right) : L^2 \longrightarrow L^2 \, .$$

The estimate

$$\chi(-\Delta-\lambda\mp i0)^{-1}\chi=\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right):L^2\longrightarrow L^2\,.$$

is a quantitative version of the limiting absorbtion principle established by many authors Jensen-Mourre-Perry, Robert-Tamura, Gérard-Martinez, Wang, Robert... And in more geometric setting by Vasy-Zworski, Vodev.

We have established that local smoothing is related to standard issues of mathematical physics.

Local smoothing plays a crucial rôle in the study of nonlinear Schrödinger equations and is closely related to Strichartz estimates. Burq used results of Ikawa to obtain local smoothing estimates for some exterior problems.

# Geometric setting

Instead of $\mathbf{R}^n$ we can consider $(X, g)$, a Riemannian manifold close to $\mathbf{R}^n$ near infinity.

Local smoothing for $\Delta_g$, the Laplace-Beltrami operator means:

$$\int_0^T \|\chi \exp(-it\Delta_g)u\|^2_{H^{1/2}(X)}\, dt \leq C\|u\|^2_{L^2(X)}\,, \qquad (1)$$

Doi (1996) proved a remarkable result that (1) implies that the metric is non-trapping. Roughly, that means that all geodesics, escape to infinity. But that is natural since non-trapping assumptions are needed for

$$\chi(-\Delta - \lambda \mp i0)^{-1}\chi = \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right) : L^2 \longrightarrow L^2\,.$$

On the other hand, Burq's proof shows that

$$\chi(-\Delta_g - \lambda \mp i0)^{-1}\chi = \mathcal{O}\left(\frac{1}{\lambda^{\alpha/2}}\right) : L^2 \longrightarrow L^2 \,.$$

gives a weaker version of local smoothing:

$$\int_0^T \|\chi \exp(-it\Delta_g)u\|^2_{H^{\alpha/2}(X)}\,dt \leq C\|u\|^2_{L^2(X)}\,, \quad \chi \in C^\infty_{\mathrm{c}}(X)\,.$$

but that is sufficient for many applications – Burq, Christianson.

This is is useful when there is trapping! Simplest case:

Theorem 1.(Nonnenmacher-Zworski 2007) Suppose that $(X, g)$ is a surface Euclidean outside of a compact set and that the geodesic flow is hyperbolic on the trapped set.

If the dimension of the trapped set (inside of the three dimensional $S^*X$) is less than two then

$$\chi(-\Delta_g - \lambda \mp i0)^{-1}\chi = \mathcal{O}\left(\frac{\log \lambda}{\sqrt{\lambda}}\right) : L^2 \longrightarrow L^2 \,.$$

and consequently,

$$\int_0^T \|\chi \exp(-it\Delta_g)u\|^2_{H^{1/2-\epsilon}(X)}\, dt \leq C\|u\|^2_{L^2(X)}\,, \quad \chi \in C^\infty_{\mathrm{c}}(X)\,.$$

Remarks.

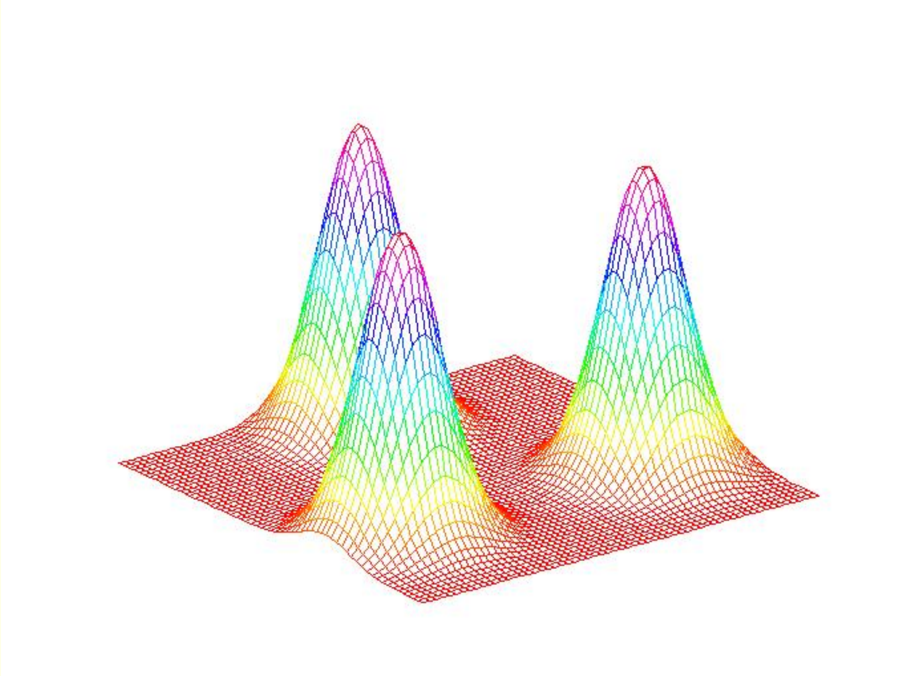
1. The resolvent bound is probably optimal. Examples of Colin de Verdière-Parisse, Christianson, Alexandrova-Bony-Ramond, give

$$\|\chi(-\Delta_g - \lambda \mp i0)^{-1}\chi\|_{L^2 \to L^2} \geq \left( \frac{\sqrt{\log \lambda}}{\sqrt{\lambda}} \right)$$

2. Resolvent estimates are closely related to having a gap between quantum resonances and the real axis.

Hence the tools needed to prove Theorem 1 are closely related to the tools needed to understand the behaviour of resonances in chaotic scattering.

# Quantum mechanical perspective



Classically we consider

$$H = \xi^2 + V(x)\,, \quad x \in \mathbf{R}^2$$

( $n = 2$ for simplicity only) and on the quantum level,

$$\widehat{H} = -h^2\Delta + V(x)\,.$$

The resonances of $\widehat{H}$ are defined as the poles of the meromorphic continuation of the resolvent:

$$(\widehat{H} - z)^{-1} : C^\infty_{\mathrm{c}} \longrightarrow C^\infty \,.$$

So it is not surprising that a presence of resonances near the real axis will destroy good bounds on the resolvent.

The nontrapping bounds correspond, after rescaling to

$$\chi(\widehat{H} - E \pm i0)^{-1}\chi = \mathcal{O}\left(\frac{1}{h}\right) : L^2 \ \longrightarrow \ L^2 \,,$$

and the bounds in Theorem 1 to

$$\chi(\widehat{H} - E \pm i0)^{-1}\chi = \mathcal{O}\left(\frac{\log(1/h)}{h}\right) : L^2 \ \longrightarrow \ L^2,,$$

We assume that the flow is hyperbolic on the trapped set:

$$K_E = \Gamma_E^+ \cap \Gamma_E^-$$

where $\Gamma_E^\pm =$

$$\{(x,\xi) \ : \ \xi^2 + V(x) = E \,, \ \ (x(t),\xi(t)) \not\to \infty \,, \ \ t \to \mp\infty\} \,,$$

and the flow is defined by Newton 1687

$$x'(t) = 2\xi(t) \,, \ \ \xi'(t) = -\nabla V(x(t)) \,,$$

$$x(0) = x \,, \ \ \xi(0) = \xi \,.$$

Where do the resonant state live?

In phase space they live on $\Gamma_E^+$.

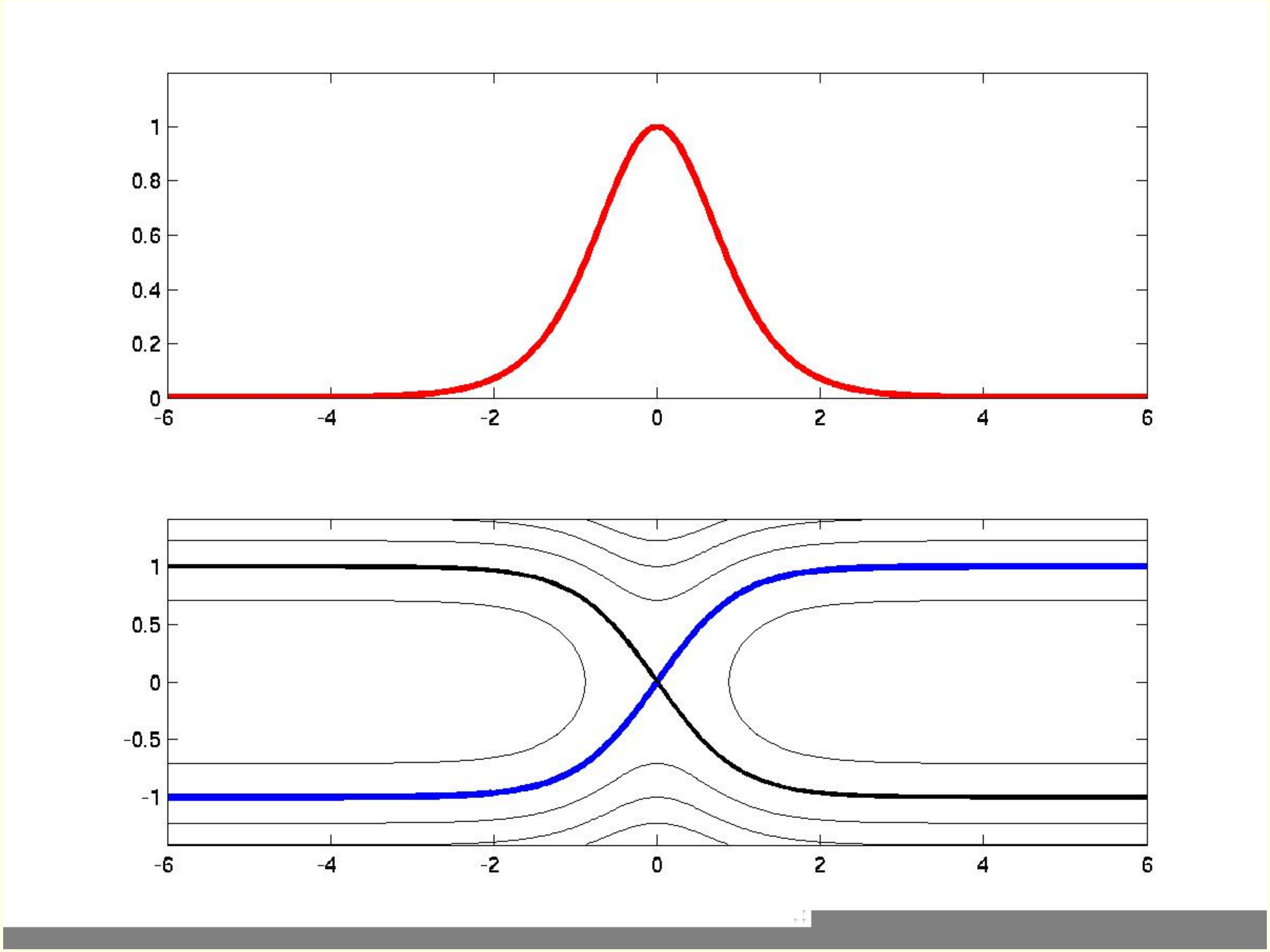
Theorem 2.(Nonnenmacher-Rubin 2006) Let $u(h_k)$ be resonant states corresponding to $z(h_k)$. with $\operatorname{Re} z(h_k) = E + o(1)$ and $\operatorname{Im} z(h) \geq -Ch$. Let $\mu$ be a semiclassical measure associated to $u(h_k)$: Then

$$\mathrm{supp}\mu \subset \Gamma_E^+ \,,$$

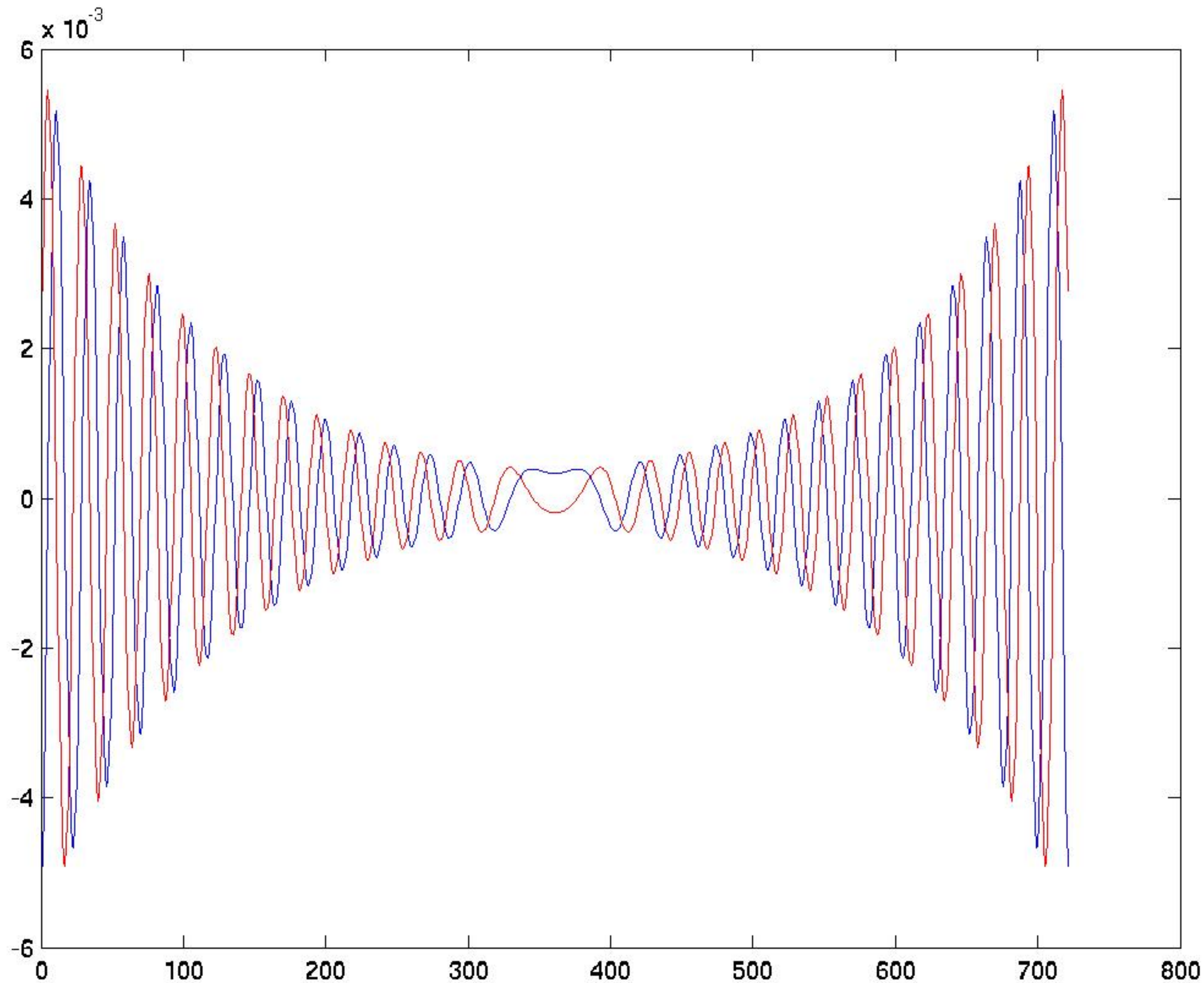
$$\exists\, \lambda > 0 \,, \quad \lim_{k \to \infty} \operatorname{Im} z(h_k)/h_k = -\lambda/2 \,,$$

$$\mathcal{L}\mu = \lambda\mu \,,$$

where $\mathcal{L}$ is the Lie derivative along the flow.




A potential with a simple trapped set.



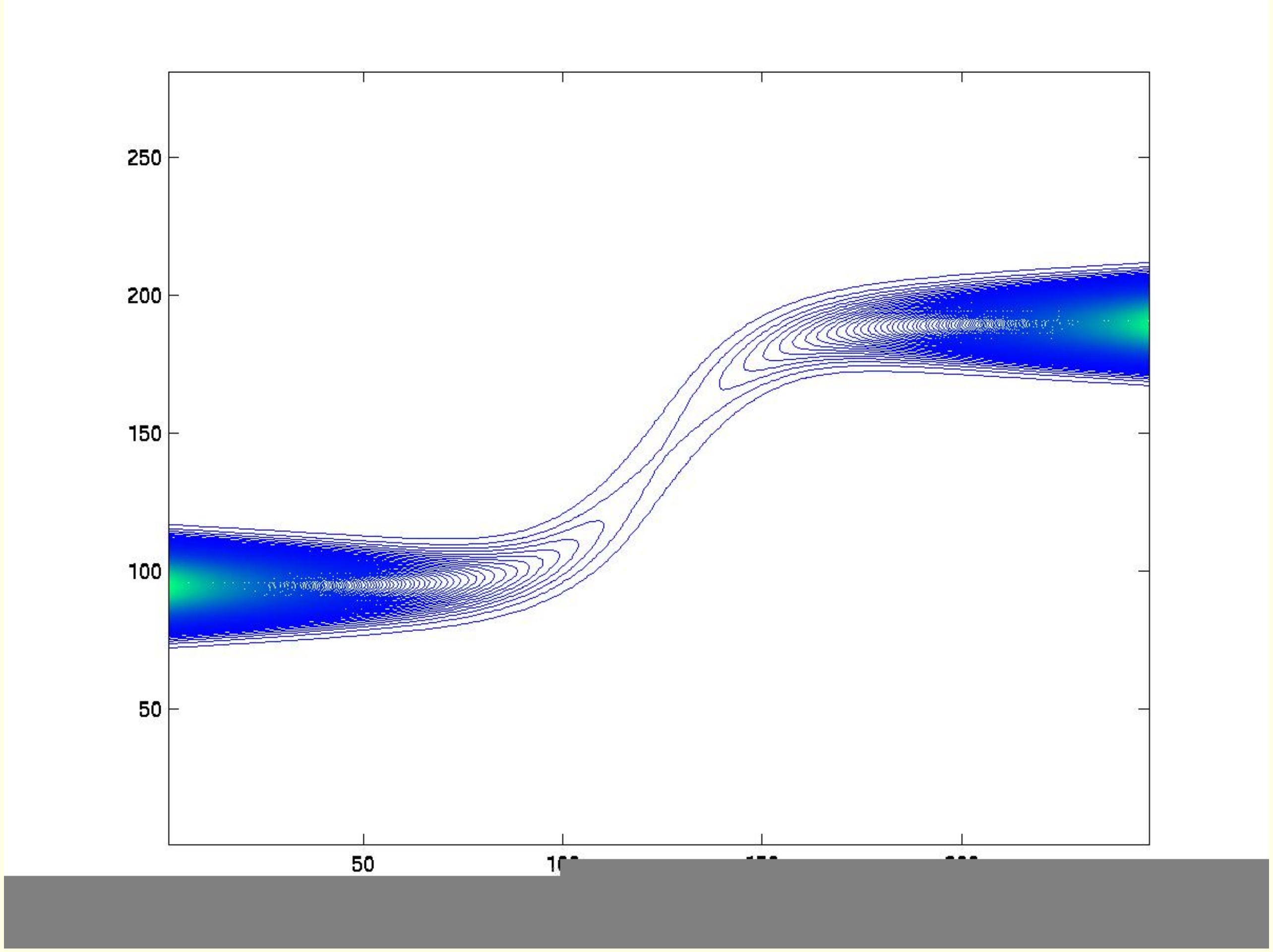

The “first” resonant function for $h = 1/16$.

The resonant state thanks to David Bindel

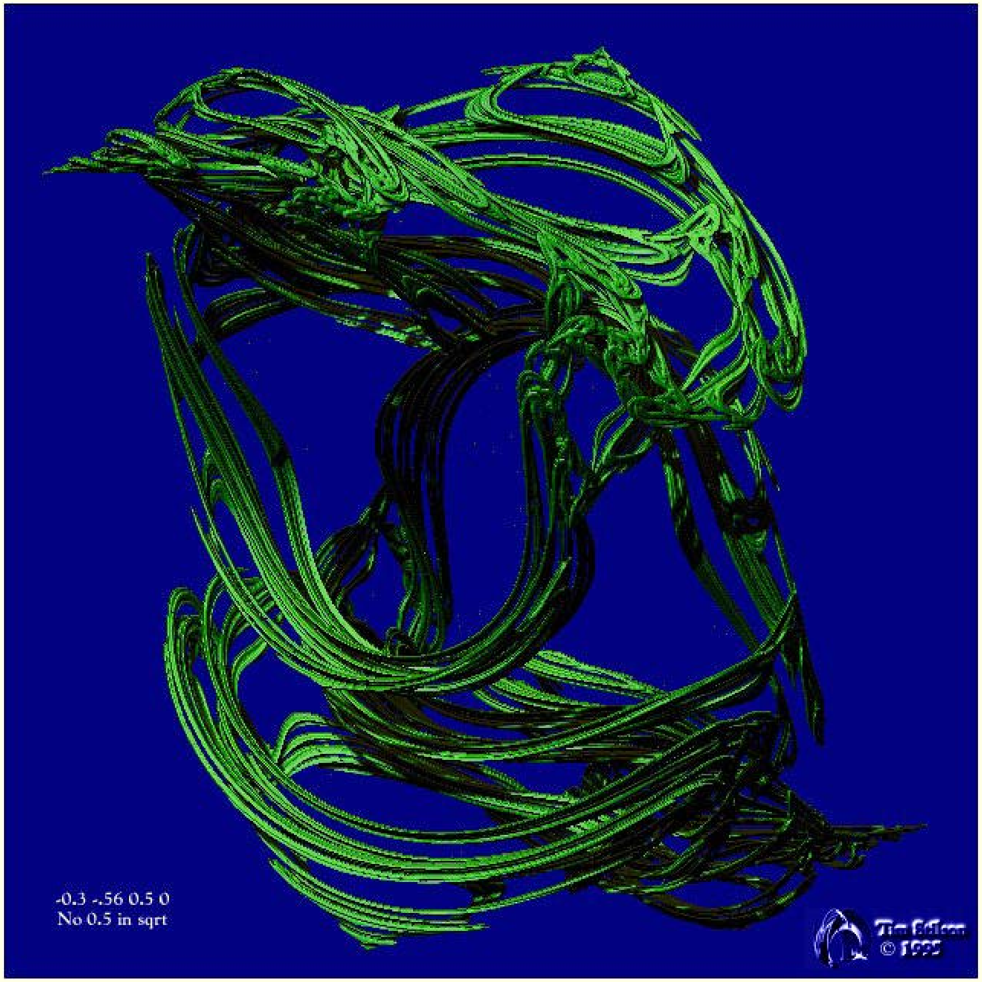
www.cims.nyu.edu/~dbindel

FBI transform thanks to Laurent Demanet

www.math.stanford.edu/~laurent



The trapped set $K_E$ lives in the three dimensional energy surface and looks similar to




This is actually a picture of a Julia set but the similarity is more than formal and similar ideas apply to zeros of Ruelle zeta functions.

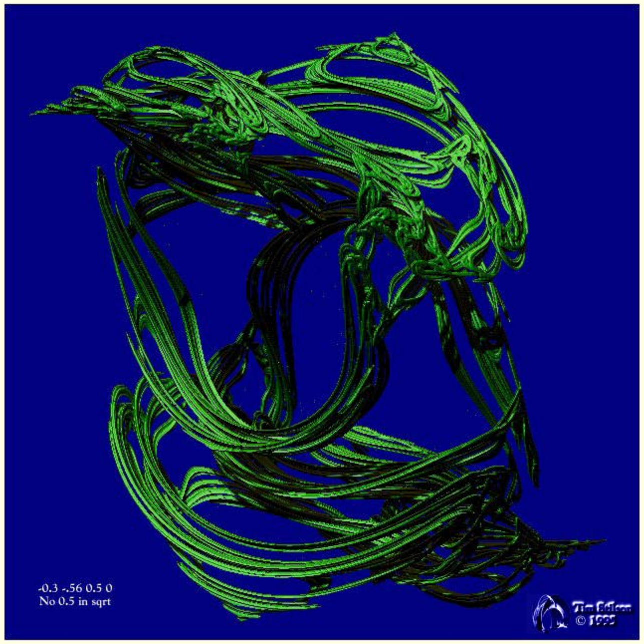
We say that the flow $\Phi^t(x,\xi) = (x(t),\xi(t))$ is hyperbolic on $K_E$, if any $\rho \in K_E$, the tangent space to $H^{-1}(E)$ at $\rho$ splits into the flow, unstable and stable directions:

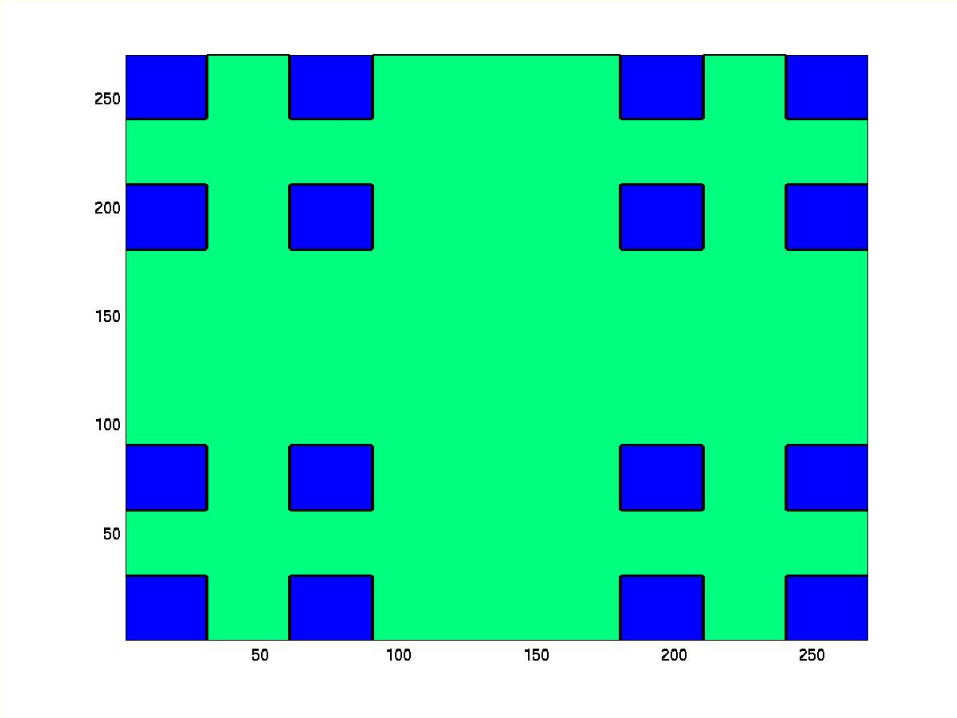
- $T_\rho(H^{-1}(E)) = \mathbf{R}\langle 2\xi(\rho), -\nabla V(x(\rho))\rangle \oplus E_\rho^+ \oplus E_\rho^-$
- $d\Phi_\rho^t(E_\rho^\pm) = E^\pm_{\Phi^t(\rho)}$
- $\exists\, \lambda > 0\,, \quad \|d\Phi_\rho^t(v)\| \leq Ce^{-\lambda|t|}\,\|v\|$

for all $v \in E_\rho^\mp$, and $\pm t \geq 0$.

Verification of this is not easy but in our setting it is available thanks to the work of Sinai, Ikawa, Sjöstrand, and Morita.

The Poincaré section is given by a surface in $H^{-1}(E)$ transversal to the flow:




We write the Hausdorff dimension of $K_E$ as

$$\dim K_E = 2d_E + 1\,.$$

Pesin-Sadovskaya 2001

Theorem 3.(Sjöstrand-Zworski 2005)

Let $\mathcal{R}(h)$ denote the set of resonances of

$$\widehat{H} = -h^2\Delta + V(x)\,.$$

Under the assumptions of hyperbolicity near energy $E$,

$$|\mathcal{R}(h) \cap [E-h, E+h] - i[0, Mh]| = \mathcal{O}(h^{-d_E})\,.$$

This is the analogue of the counting law for eigenvalues of a closed system. Classically everything is trapped in a closed system, so $\dim K_E = 3$, $d_E = 1$ and the number of eigenvalues is asymptotic to

$$C_E h^{-1}\,.$$

If $u$ is a resonant state for

$$z = E - i\,\Gamma$$

then

$$\exp(-it\widehat{H}/h)u = e^{-itE/h - t\Gamma/h}u\,.$$

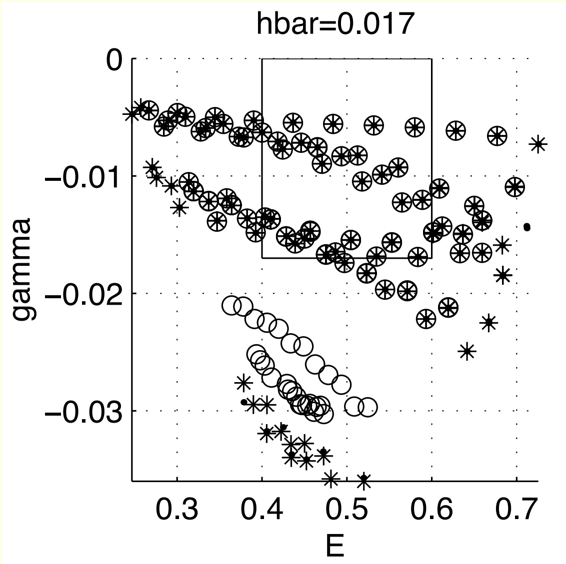
Hence states with $\Gamma \gg h$ decay too fast to be visible.

Interpretation of the imaginary part as decay rate brings us to the next theorem.

**Question:** What properties of the flow $\Phi_t$, or of $K_E$ alone, imply the existence of a gap $\gamma > 0$ such that, for $h > 0$ sufficiently small,

$$z \in \mathcal{R}(h), \quad \operatorname{Re} z \sim E \implies \operatorname{Im} z < -\gamma h ?$$

In other words, what dynamical conditions guarantee a lower bound on the quantum decay rate?



Theorem 4.(Nonnenmacher-Zworski 2006)

Suppose that the dimension $d_E$ satisfies

$$d_E < \frac{1}{2}\,.$$

(That is, the dimension of the trapped set inside the energy surface is less than two.)

Then there exists $\delta,\ \gamma > 0$ such that

$$\mathcal{R}(h) \cap \big([E_0 - \delta, E_0 + \delta] - i[0, h\gamma]\big) = \emptyset\,.$$

What is $\gamma$? It can be described using the topological pressure of the flow on $K_E$.

We can take any $\gamma$ satisfying

$$0 < \gamma < \min_{|E_0 - E| \leq \delta} (-P_E(1/2)),$$

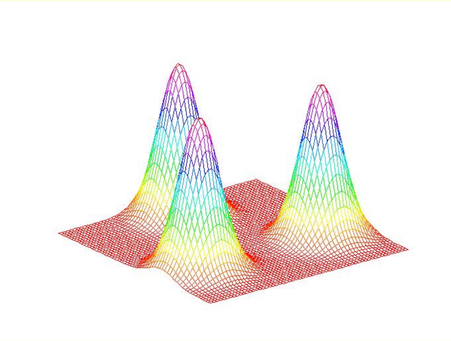
$$P_E(s) = \text{ pressure of the flow on } K_E.$$

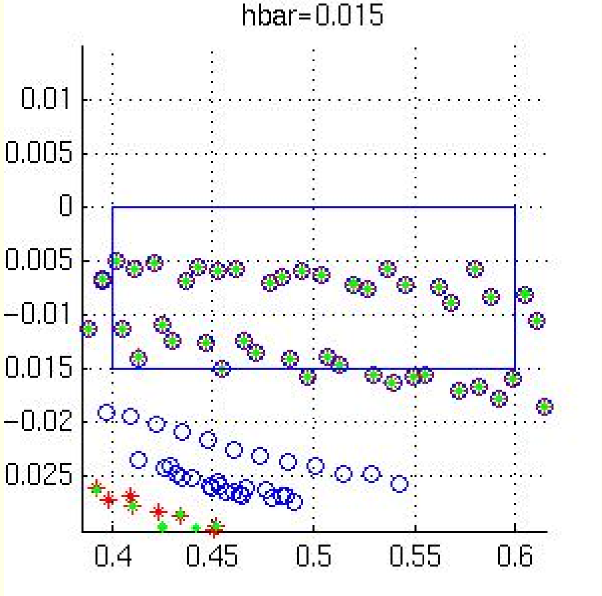
The existence of a resonance gap depends on the sign of the pressure at $s = 1/2$, $P_E(1/2)$.

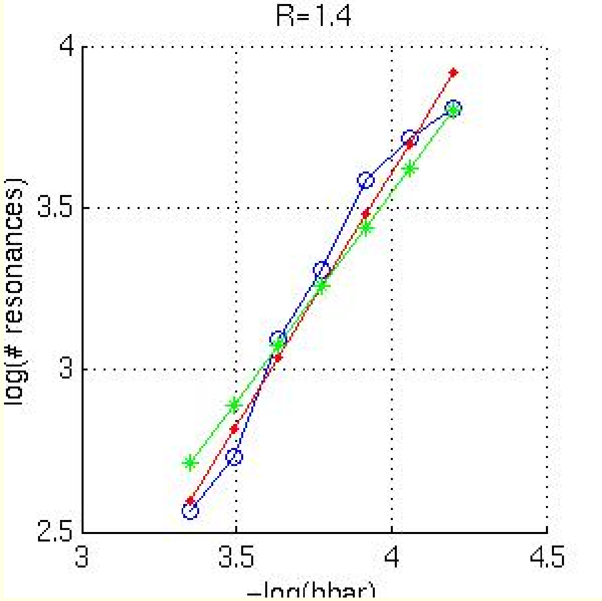
The connection between the pressure and the quantum decay rate first appeared in the physics/chemistry literature in the work of Gaspard-Rice 1989.

Numerical results (Lin 2002):

Quantum resonances for the three bumps potential.

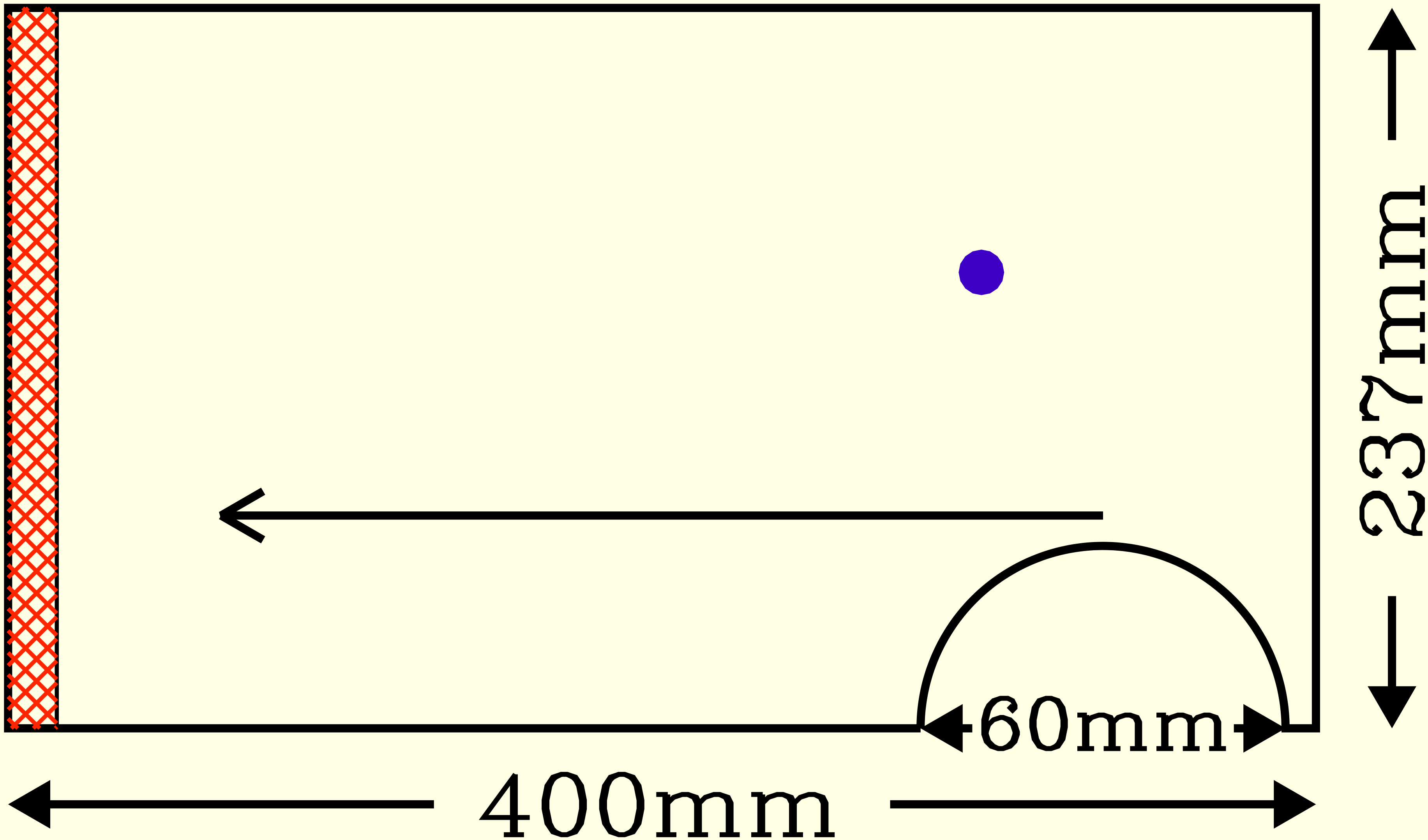





This and also some quantum map rigorous models of Nonnenmacher-Zworski 2005 suggest that Theorem 2 is optimal.

Preliminary experimental results (Kuhl-Stöckmann 2006):



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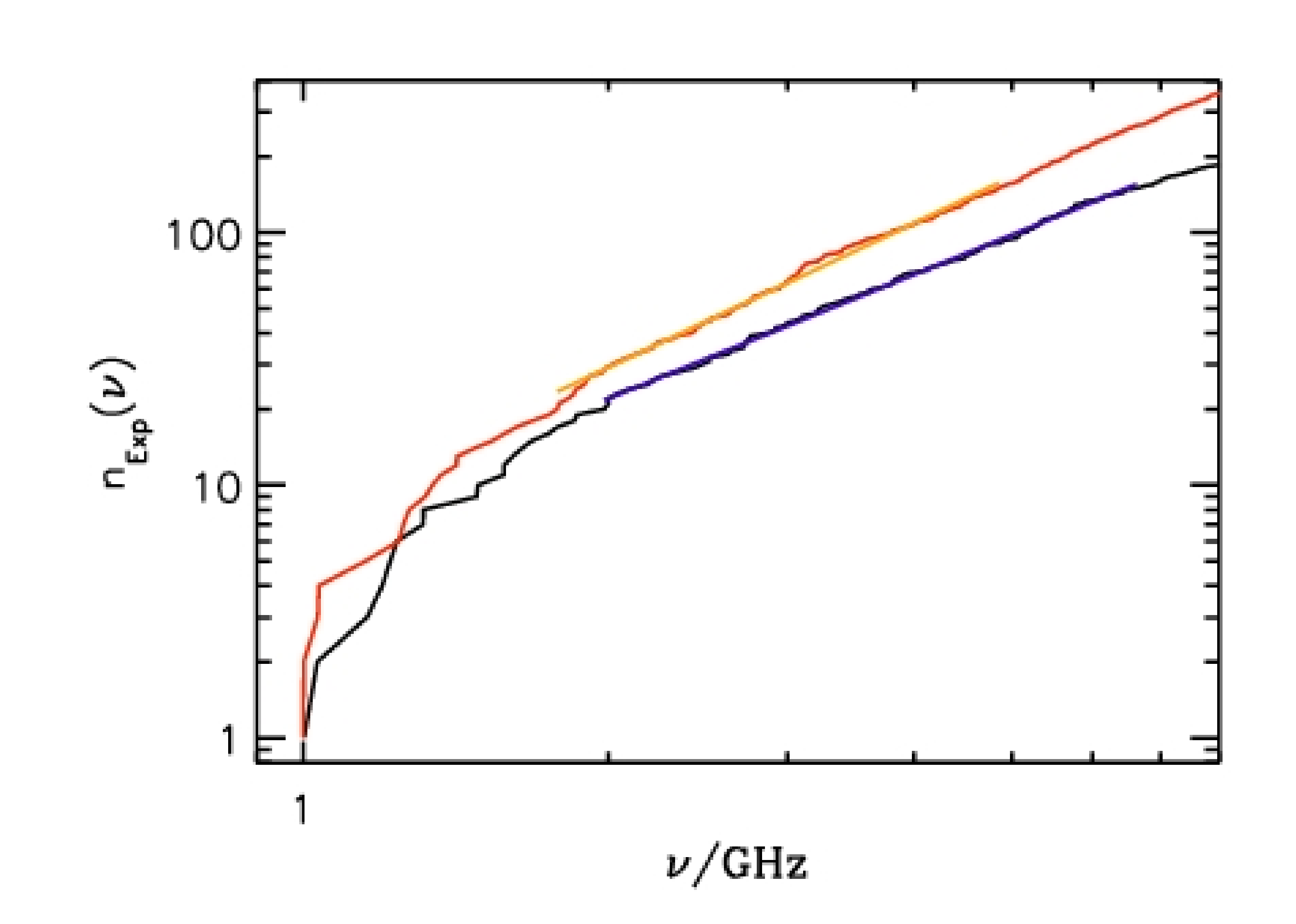

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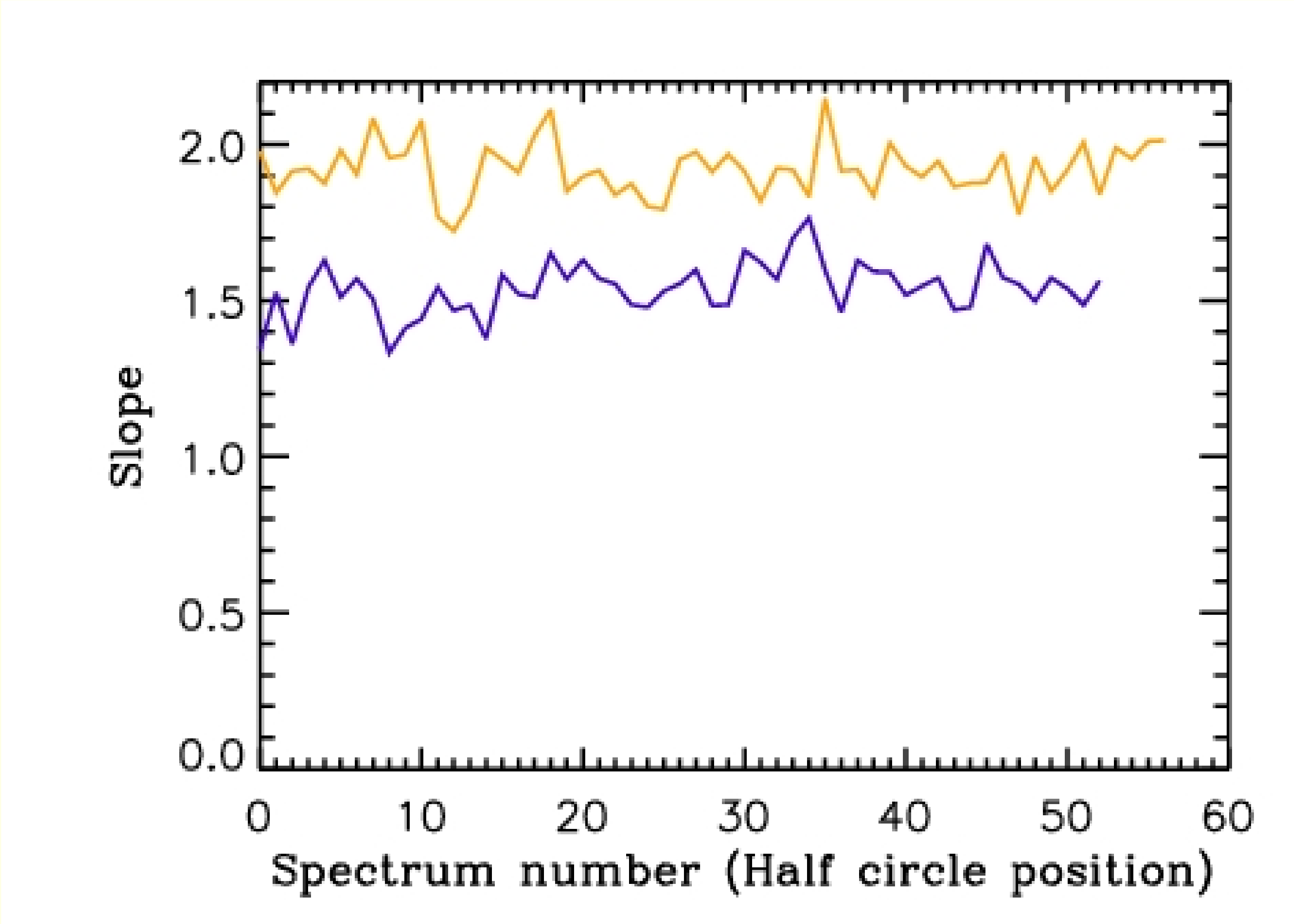

The optimality of Theorem 4 is not clear even on the heuristic or numerical grounds.

In the analogous case of scattering on convex co-compact hyperbolic surfaces the results of Dolgopyat, Naud, and Stoyanov show that the resonance free strip is larger at high energies than the strip predicted by the pressure.

That relies on delicate zeta function analysis following the work of Dolgopyat: at zero energy there exists a Patterson-Sullivan resonance with the imaginary part (width) given by the pressure but all other resonances have more negative imaginary parts.