QMath10 - Moeciu, September 10–15, 2007
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BOSON GAS with BCS INTERACTION

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CONTENTS

(*Based on joint works with Joe Pulé - UC Dublin, and André Verbeure - KU Leuven*)

0. Introduction, Motivation and Pair Boson Model

- The first version of Hamiltonian with BCS-Bose interaction, (**Pair Boson Hamiltonian** (PBH), or **BCS-Bose model**) was proposed by *Zubarev* and *Tserkovnikov* [Sov.Phys.Acad.Docl.1958]. Intention was to *generalize* the Bogoliubov model of the Weakly Imperfect Bose Gas by including more terms of the total interaction, without losing the possibility of having an "exact" solution.
- Two-body $v(x-y)$ interacting Bose-gas in $\Lambda \subset \mathbb{R}^\nu$:

$$H_\Lambda = \sum_{k\in\Lambda^*} \varepsilon(k) a_k^* a_k + \frac{1}{2V} \sum_{k_1,k_2,q\in\Lambda^*} \widehat{v}\,(q)\, a_{k_1+q}^* a_{k_2-q}^* a_{k_2} a_{k_1},$$

- **Truncations:** (a) WIBG "2-zero-momentum" BEC cut: $\{k_1 = k_2 = 0\}$, $\{k_1 = -q, k_2 = q\}$, $\{k_1 = 0, k_2 = q\}$,..., $\{k_{1,2} = q = 0\}$.
(b)"BCS" 1-constraint: $\{k_1 = -k_2 = p\}$ ($p = 0$ cut $\Rightarrow$ WIBG)$\Rightarrow$

$$\frac{1}{2V} \sum_{p,p'\in\Lambda^*} \widehat{v}(p-p') a_{p'}^* a_{-p'}^*\, a_{-p} a_p \text{ (If } \widehat{v}(p-p') \Rightarrow \overline{\lambda_p} \lambda_p' = \mathrm{BCS} - \mathrm{Bose})$$

1.1 BCS-Bose Hamiltonian

• Let $\Lambda \subset \mathbb{R}^\nu$ be a cube of volume $V = |\Lambda|$ centered at the origin. The kinetic energy operator for a particle of mass m enclosed into the cubic box Λ, is a self-adjoint extension of the operator $t_\Lambda = (-\Delta/2m)$ in $\mathfrak{H}_\Lambda = L^2(\Lambda)$ with **e.g., periodic boundary conditions**, i.e. with eigenvalues and eigenfunctions:

$$\varepsilon(k) = \|k\|^2/2m, \;\; f_k(x) = e^{ikx}/\sqrt{V}\,, \;\; k \in \Lambda^* := \{2\pi s/V^{1/\nu} | s \in \mathbb{Z}^\nu\}.$$

• Let $f_k \mapsto a^*(f_k)(= a_k^*)$ and $f_k \mapsto a(f_k)(= a_k)$ be **CCR representation** by the creation and annihilation operators in the **boson Fock space** $\mathfrak{F}(\mathfrak{H}_\Lambda)$: $[a^*(f_k), a(f_{k'})] = (f_k, f_{k'})_{\mathfrak{H}_\Lambda}$. Then $N_k := a_k^* a_k$ is the k-mode particle number operator and $N_\Lambda := \sum_{k\in\Lambda^*} N_k$ is the total number operator. The kinetic-energy operator $T_\Lambda := d\Gamma(t_\Lambda)$ (*Perfect Bose-gas* Hamiltonian), is

$$T_\Lambda := \sum_{k\in\Lambda^*} \varepsilon(k) a_k^* a_k.$$

• To introduce a *pairing term* in the BCS-Bose Hamiltonian we need the **operators** $A_k = A_{-k} := a_k a_{-k}$, $k \in \Lambda^*$ and

$$\tilde{Q}_\Lambda := \sum_{k \in \Lambda^*} \tilde{\lambda}(k) A_k \, .$$

• The function $\tilde{\lambda} : \mathbb{R}^\nu \mapsto \mathbb{C}$ satisfies the following conditions:

$$|\tilde{\lambda}(k)| \leq |\tilde{\lambda}(0)| = 1 \, , \;\; \tilde{\lambda}(k) = \tilde{\lambda}(-k) \;\; \text{for all} \;\; k \in \mathbb{R}^\nu \, ,$$

and there exists $\mathfrak{C} < \infty$, $\delta > 0$ such that

$$|\tilde{\lambda}(k)| \leq \frac{\mathfrak{C}}{1 + \|k\|^{\max(\nu, \nu/2+1)+\delta}}.$$

This implies that $\tilde{\lambda} \in L^1(\mathbb{R}^\nu)$ and existence of $M < \infty$ that

$$\mathfrak{m}_\Lambda := \sum_{k \in \Lambda^*} |\tilde{\lambda}(k)| \leq MV, \;\; \mathfrak{n}_\Lambda := \sum_{k \in \Lambda^*} \varepsilon(k) |\tilde{\lambda}(k)|^2 \leq MV,$$

$$\mathfrak{c}_\Lambda := \sup_{k \in \Lambda^*} \varepsilon(k) |\tilde{\lambda}(k)|^2 \leq M.$$

• The *constant* couplings u, v **Pair Boson Hamiltonian** is :

$$H_\Lambda := T_\Lambda - \frac{u}{2V}\tilde{Q}^*_\Lambda \tilde{Q}_\Lambda + \frac{v}{2V} N^2_\Lambda.$$

• Let $\varphi := \arg \tilde{\lambda}(0)$ and $\lambda(k) := \tilde{\lambda}(k) e^{-\mathrm{i}\varphi}$. Then $\lambda(0) = 1$ and

$$H_\Lambda = T_\Lambda - \frac{u}{2V} Q^*_\Lambda Q_\Lambda + \frac{v}{2V} N^2_\Lambda$$

with $Q_\Lambda := \sum_{k\in\Lambda^*} \lambda(k) A_k$, where $|\lambda(k)| \leq \lambda(0) = 1$.

• We shall assume that $v > 0$ and $\alpha := v - u > 0$. This condition ensures the (super)stability of the model.

• **REMARK:** In the case $u \leq 0$ (BCS repulsion), the second condition $\alpha > 0$ is satisfied and the **PBH** gives the same thermodynamics as the **Mean-Field** (MF) Bose-gas Hamiltonian:

$$H^{MF}_\Lambda := T_\Lambda + \frac{v}{2V} N^2_\Lambda,$$

but **not** the same **Bose-Einstein Condensation** (BEC) !!!

1.2 Thermodynamics of the BCS-Bose Hamiltonian

• The grand-canonical pressure corresponding to Hamiltonian H_Λ

$$p_\Lambda[H_\Lambda] := \frac{1}{\beta V} \ln \mathrm{Tr} \exp\{-\beta(H_\Lambda - \mu N_\Lambda)\}\,.$$

• **Theorem 1:** The limiting pressure for the PBH model with $u > 0$ (BCS attraction) has the form

$$p := \lim_\Lambda p_\Lambda[H_\Lambda] = \sup_{q\geq 0} \inf_{\rho\,:\,\sigma(q,\rho)\geq 0} p[H_\Lambda^{(2)}(q,\rho)]\ ,$$

while with $u \leq 0$ (BCS repulsion) it has the form

$$p := \lim_\Lambda p_\Lambda[H_\Lambda] = \inf_{q\geq 0} \inf_{\rho\,:\,\sigma(q,\rho)\geq 0} p[H_\Lambda^{(2)}(q,\rho)]\ ,$$

$$H_\Lambda^{(2)}(q,\rho) := T_\Lambda + v\rho N_\Lambda - \frac{1}{2}u(Q_\Lambda^* q + Q_\Lambda q^*) - \frac{V}{2}v\rho^2 + \frac{V}{2}u|q|^2\ ,$$

where $q \in \mathbb{C}$, $\rho \in \mathbb{R}_+$ and the function:
$\sigma(q,\rho) := \inf_{k\in\mathbb{R}^\nu}(f(k,\rho) - |h(k,q)|) = v\rho - \mu - |u|q$.

2.1 CCR and Quasi-Free States

- Let $f \mapsto b(f) := a(f) + a^*(f)$ on the Fock space $\mathfrak{F}(L^2(\mathbb{R}^\nu)$ and (*Weyl operators*) $W(f) :=$ exp$(ib(f))$. Then CCR take the form

$$W(f)W(g) = e^{-i\sigma(f,g)}W(f+g)\,,\ \sigma(f,g) = \mathfrak{Im}(f,g).$$

- *Truncated states* on $\mathfrak{A}(b)$ are defined recursively:

$$\omega(b(f))_t = \omega(b(f)),\ \omega(b(f)b(g)) = \omega(b(f)b(g))_t + \omega(b(f))_t\omega(b(g))_t,$$
$$\omega(b(f_1)...b(f_n)) = \sum \omega(b(f_k)...)_t...\omega(...b(f_l))_t,$$

- *Quasi-Free states(QF)* on the Weyl algebra $\mathfrak{A}(b)$ are :

$$\omega(W(f)) = \text{exp}\{i\omega(b(f)) - \frac{1}{2}\omega(b(f)b(f))_t\}.$$

For them all truncated functions of order $n > 2$ vanish: $\phi(f) = \omega(a^*(f))$, $\langle f, Rg\rangle = \omega(a^*(f)a(g)) - \omega(a^*(f))\omega(a(g))$, $\langle f, Sg\rangle = \omega(a(f)a(\overline{g})) - \omega(a(f))\omega(a(\overline{g}))$.

2.2 Equilibrium States of the BCS-Bose Hamiltonian

- **Theorem 2:** *Pure* Gibbs state $\omega_\alpha(-) = \lim_\Lambda \omega_{H_\Lambda}(-)$ generated by the BCS-Bose Hamiltonian is *Quasi-Free* with parameters defined by Theorem 1: $c = \sqrt{\rho_0} e^{i\alpha} = \lim_\Lambda \omega_{H_\Lambda}(a_{k=0}/\sqrt{V})$

$$\omega(a_k^* a_k) = \langle f_k, R f_k \rangle + |c|^2 V \delta_{k,0}, \quad \omega(a_k a_{-k}) = \langle \phi_k, S \phi_k \rangle + c^2 V \delta_{k,0},$$

for $V \to \infty$ with $\rho(k) = r(k)$, $\overline{\rho} = \int \mathrm{d}k\, \rho(k) + \rho_0$ and $\sigma = \int \mathrm{d}k\, \lambda(k) s(k)$.

- Since $f(k) = \epsilon(k) - \mu + v\overline{\rho}$, $E(k) = \left\{ f^2(k) - u^2 |\lambda(k)|^2 (\rho_0 + |\sigma|)^2 \right\}^{1/2}$.

$$\overline{\rho} = \frac{1}{2} \int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} \left\{ \frac{f(k)}{E(k)} \coth \frac{1}{2} \beta E(k) - 1 \right\} + \rho_0 ,$$

$$(|\sigma| + \rho_0) = \frac{u\,(|\sigma| + \rho_0)}{2} \int_{\mathbb{R}^\nu} \frac{d^\nu k}{(2\pi)^\nu} \frac{|\lambda(k)|^2}{E(k)} \coth \frac{1}{2} \beta E(k) + \rho_0 .$$

- Chemical potential and the "spectrum":

$$\mu = v\overline{\rho} + u(\rho_0 + |\sigma|) \Leftrightarrow E(k = 0) = 0.$$

3. Condensations, Pairing and Type III BEC

3.1 Let $u > 0$ (BCS-Bose **attraction**), $v > 0$ and $v - u > 0$ (**stability**).

- For *large* u there is $\mu_c^-(u) < \mu_c$: $\sigma \neq 0$, $\rho_0 = 0$ and "gap", when $\mu_c^-(u) < \mu < \mu_c = v\bar{\rho}_c^{PBG}$.
- For $\mu_c \leq \mu$: $\rho_0 > 0$(**one-mode BEC**) $\Rightarrow \sigma \neq 0$ and no "gap".

3.2 Let $u < 0$ (BCS-Bose **repulsion**).

- Pressure for BCS-Bose = Mean-Field model ($v > 0$).
- Condensations: $\sigma = 0$ and $\rho_0 = 0$ **!**
- BEC $\neq 0$ for $\mu > \mu_c = v\bar{\rho}_c^{PBG}$ and is type III.