# Inverse Scattering at a Fixed Energy

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**Abstract**

We prove that the averaged scattering solutions to the Schrödinger equation with short-range electromagnetic potentials $(V, A)$ where $V(x) = O(|x|^{-\rho}), A(x) = O(|x|^{-\rho}), |x| \to \infty, \rho > 1$, are dense in the set of all solutions to the Schrödinger equation that are in $L^2(K)$ where $K$ is any connected bounded open set in $\mathbb{R}^n, n \geq 2$, with smooth boundary. We use this result to prove that if two short-range electromagnetic potentials $(V_1, A_1)$ and $(V_2, A_2)$ in $\mathbb{R}^n, n \geq 3$, have the same scattering matrix at a fixed positive energy and if the electric potentials $V_j$ and the magnetic fields $F_j := \mathrm{curl} A_j, j = 1, 2$, coincide outside of some ball they necessarily coincide everywhere.
In a previous paper of Weder and Yafaev the case of electric potentials and magnetic fields in $\mathbb{R}^n, n \geq 3$, that are asymptotic sums of homogeneous terms at infinity was studied. It was proven that all these terms can be uniquely reconstructed from the singularities in the forward direction of the scattering amplitude at a fixed positive energy.
The combination of the new uniqueness result of this paper and the result of Weder and Yafaev implies that the scattering matrix at a fixed positive energy uniquely determines electric potentials and magnetic fields that are a finite sum of homogeneous terms at infinity, or more generally, that are asymptotic sums of homogeneous terms that actually converge, respectively, to the electric potential and to the magnetic field.

We consider the Schrödinger operator in $\mathbb{R}^n$,

$$H := (i\nabla + A)^2 + V = H_0 + Q,$$

where the free Hamiltonian, $H_0 := -\Delta$ is a self-adjoint operator with domain the Sobolev space $\mathbb{H}^2$ and

$$Q := 2iA \cdot \nabla + i\,\mathrm{Div} A + A^2 + V$$

is the perturbation.

For the purpose of this talk we assume, for simplicity, that $A, V$ are $C^\infty$ and that,

$$|\partial_\alpha A(x)| + |\partial_\alpha V(x)| \leq C(1 + |x|)^{-\rho-|\alpha|}, \rho > 1,$$

but if fact many of the results that I will discuss hold under weaker conditions.

The Schrödinger operator $H$ is self-adjoint and bounded below with domain $\mathbb{H}^2$. It has no singular–continuous spectrum , its absolutely–continuous spectrum is $[0, \infty)$ and it has no positive eigenvalues. The negative spectrum consists of eigenvalues with finite multiplicity and they can only accumulate at zero.

We state below standard results in the limiting absorption principle and in scattering theory that we need [1, 2, 3]

To state the limiting absorption principle we introduce weighted $L^2$ spaces for $s \in \mathbb{R}$.

$$L_s^2 := \left\{ f : (1+|x|^2)^{s/2} f(x) \in L^2 \right\}, \|f\|_{L_s^2} := \|(1+|x|^2)^{s/2} f(x)\|_{L^2},$$

and for any $\alpha, s \in \mathbb{R}$,

$$\mathbb{H}^{\alpha,s} := \left\{ f(x) : (1+|x|^2)^{s/2} f(x) \in \mathbb{H}^{\alpha} \right\}, \|f\|_{\mathbb{H}^{\alpha,s}} := \|(1+|x|^2)^{s/2} f(x)\|_{\mathbb{H}^{\alpha}}.$$

$\mathbb{H}^{\alpha}$ denotes the standard Sobolev spaces. $\mathbb{C}^{\pm}$ designates, respectively, the upper, lower, complex half-plane.

The limiting absorption principle is the following statement. For $z$ in the resolvent set of $H$ let $R(z) := (H-z)^{-1}$ be the resolvent. Then, for every $E \in (0,\infty)$ the following limits,

$$R(E \pm i0) := \lim_{\epsilon \downarrow 0} R(E \pm i\epsilon),$$

exist in the uniform operator topology in $\mathcal{B}\left(L_s^2, \mathbb{H}^{\alpha,-s}\right), s > 1/2, |\alpha| \leq 2$. The functions,

$$R_{\pm}(E) := \begin{cases} R(E), & \operatorname{Im} E \neq 0, \\ R(E \pm i0) & , E \in (0,\infty), \end{cases}$$

defined for $E \in \mathbb{C}^{\pm} \cup (0,\infty)$ with values in $\mathcal{B}\left(L_s^2, \mathbb{H}^{\alpha,-s}\right)$ are analytic for $\operatorname{Im} E \neq 0$ and locally Hölder continuous for $E \in (0,\infty)$ with exponent $\gamma$ satisfying $\gamma < 1, \gamma < s - 1/2$.

The wave operators,

$$W_{\pm} := \mathrm{s} - \lim_{t \to \pm\infty} e^{itH} \, e^{-itH_0}$$

exist as strong limits and are complete, i.e., Range $W_{\pm} = \mathcal{H}_{ac}$ where $\mathcal{H}_{ac}$ denotes the subspace of absolute continuity of $H$. Moreover, they have the intertwining property, $HW_{\pm} = W_{\pm}H_0$.

The scattering operator,

$$\mathrm{S} := W_{+}^{*} \, W_{-}$$

is unitary.

Let us denote by $T_0(E)$ the following trace operator,

$$(T_0(E)\phi)(\omega) := 2^{-1/2} \, E^{(n-2)/4} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iE^{1/2} x \cdot \omega} \, \phi(x) \, dx,$$

that is bounded from $L^2_s$, $s > 1/2$, into $L^2(\mathbb{S}^{n-1})$, and furthermore, the operator valued function $E \to T_0(E)$ from $(0, \infty)$ into $\mathcal{B}(L^2_s, L^2(\mathbb{S}^{n-1}))$ is locally Hölder continuous with exponent $\gamma < 1, \gamma < s - 1/2$. Moreover, the operator,

$$\left(\mathcal{F}_0 \phi\right)(E, \omega) := \left(T_0(E)\phi\right)(\omega),$$

extends to a unitary operator from $L^2$ onto $\hat{\mathcal{H}} := L^2((0, \infty); L^2(\mathbb{S}^{n-1}))$ that gives a spectral representation for $H_0$, i.e.,

$$\mathcal{F}_0 H_0 \mathcal{F}_0^* = E,$$

the operator of multiplication by $E$ in $\hat{\mathcal{H}}$.

The perturbed trace operators are defined as follows,

$$\left(T_\pm(E)\phi\right)(\omega) := T_0(E)(I - QR_\pm(E))\phi,$$

for $E \in (0, \infty)$. They are bounded from $L^2_s$, $s > 1/2$, into $L^2(\mathbb{S}^{n-1})$, and furthermore, the operator valued functions $E \to T_\pm(E)$ from $(0, \infty)$ into $\mathcal{B}(L^2_s, L^2(\mathbb{S}^{n-1}))$ are locally Hölder continuous with exponent $\gamma < 1, \gamma < s - 1/2$. The operators,

$$\left(\mathcal{F}_\pm \phi\right)(E, \omega) := \left(T_\pm(E)\phi\right)(\omega),$$

extend to unitary operators from $\mathcal{H}_{ac}$ onto $\hat{\mathcal{H}}$ and they give spectral representations for the restriction of $H$ to $\mathcal{H}_{ac}$,

$$\mathcal{F}_\pm H \mathcal{F}_\pm^* = E,$$

the operator of multiplication by $E$ in $\hat{\mathcal{H}}$. Furthermore, the stationary formulae for the wave operators hold,

$$W_\pm = \mathcal{F}_\pm^* \mathcal{F}_0.$$

**As S commutes with $H_0$, we have that,**

$$\left(\mathcal{F}_0 S \mathcal{F}_0^* \phi\right)(E, \omega) = S(E)\phi,$$

**where $S(E), E > 0$, is unitary on $L^2(\mathbb{S}^{n-1})$. The operator $S(E)$ is the scattering matrix.**

**This time-dependent definition of the scattering matrix generalizes to general short-range potentials the definition given in terms of scattering solutions that satisfy the Sommerfeld radiation condition.**

**The scattering matrix has the following stationary representation,**

$$S(E) = I - 2\pi i \mathcal{F}_0\, Q\, [I - R_+(E)\, Q]\ \mathcal{F}_0^*, E \in (0, \infty). \tag{1}$$

**The scattering matrix can be represented in terms of averaged scattering solutions as follows. For any $f \in L^2(\mathbb{S}^{n-1})$ let us define the unperturbed averaged scattering solutions as,**

$$\phi_{0,f}(x; E) := \int_{\mathbb{S}^{n-1}} e^{iE^{1/2} x \cdot \omega}\, f(\omega)\, d\omega.$$

**Observe that $\phi_{0,f} \in L^2_{-s}, s > 1/2$, and that $H_0\, \phi_{0,f} = E\, \phi_{0,f}$. The perturbed averaged scattering solutions are defined as,**

$$\phi_{+,f}(x; E) := [I - R_+(E)\, Q]\phi_{0,f},\ E \in (0, \infty),\ f \in L^2(\mathbb{S}^{n-1}).$$

**Then, $\phi_{+,f} \in L^2_{-s}, s > 1/2$, and $H\phi_{+,f} = E\phi_{+,f}$.**

By (1) for $f, g \in L^2(\mathbb{S}^{n-1})$,

$$(S(E)f, g)_{L^2(\mathbb{S}^{n-1})} = (f, g)_{L^2(\mathbb{S}^{n-1})} - i\frac{E^{(n-2)/2}}{2(2\pi)^{n-1}} (Q\,\phi_{+,f}, \phi_{0,g})_{L^2}.$$

If $\rho > (n+1)/2$, $V, A, A^2, \mathrm{Div} A \in L^2_s, s > 1/2$, and we can define the scattering solution,

$$\phi_+(x, \omega; E) := e^{iE^{1/2} x\cdot\omega} - R_+(E)\left(Q e^{iE^{1/2} x\cdot\omega}\right).$$

In this case

$$\phi_{+,f}(x; E) = \int_{\mathbb{S}^{n-1}} \phi_+(x, \omega; E)\, f(\omega)\, d\omega,$$

what justifies the name averaged scattering solutions.

Note that it follows from the definition of the wave operators that the scattering operator S and the scattering matrix $S(E)$ are invariant under the gauge transformation, $A \to A + \nabla\psi$, where $|\psi(x)| \leq C(1+|x|)^{-\mu}, |\nabla\psi(x)| \leq C(1+|x|)^{-1-\mu}, \mu > 0$.

The magnetic field is defined as follows,

$$F := \text{ curl } A, \quad F^{(ij)} = \partial_i A_j - \partial_j A_i.$$

THEOREM 1. (W., CPDE 2007 [4])

Suppose that $n \geq 3$. Let $S_j(E)$ be the scattering matrices corresponding, respectively, to $(V_j, F_j), j = 1, 2$. Then, if some $E > 0$, $S_1(E) = S_2(E)$ and $V_1(x) = V_2(x), F_1(x) = F_2(x)$ for $|x| \geq R > 0$, we have that the electric potentials and the magnetic fields coincide everywhere, i.e. $V_1(x) = V_2(x), F_1(x) = F_2(x), x \in \mathbb{R}^n$.

This theorem extends the result of W. Inv. Prob. 1991 [5], where the case $F = 0$ and $\rho > (n+1)/2$ was considered.

An essential tool on the proof of this theorem is the completeness of the averaged scattering solutions: we prove that the set,

$$\{\phi_{+,f}(x; E)\}_{f \in L^{\mathbb{S}^{n-1}}}$$

is dense in the set of all solutions to

$$H\varphi = E\varphi$$

in $L^2(K)$ for any connected, open, bounded set $K$, with a regular boundary.

In the case that $F = 0, \rho > n$ the completeness of the scattering solutions was proved by Eidus, CPDE 1982 [6], and for $F = 0, \rho > (n+1)/2$ in W., Inv. Prob. 1991 [5]. See also Sylvester and Uhlmann, 1990 [7], Ramm, 1992 [8] and Isakov, 1998 [9].

We know consider a different type of uniqueness result where the asymptotics at infinity of the electric potential and the magnetic field are uniquely reconstructed from the singularity of the scattering amplitude in the forward direction.

The scattering matrix is an integral operator on $L^2(\mathbb{S}^{n-1})$ with integral kernel $s(\omega,\omega';E)$,

$$S(E)\varphi(\omega) = \int_{\mathbb{S}^{n-1}} s(\omega,\omega';E)\,\varphi(\omega')\,d\omega'.$$

$s(\omega,\omega';E)$ is $C^\infty$ away from the diagonal $\omega=\omega'$.

Let us denote by $\dot{\mathcal{S}}^{-\rho}$ the set of $C^\infty(\mathbb{R}^n\setminus\{0\})$-functions $f(x)$ such that $\partial_\alpha f(x) = O(|x|^{-\rho-|\alpha|})$ as $|x|\to\infty$ for all $\alpha$. An important example of functions from the class $\dot{\mathcal{S}}^{-\rho}$ are homogeneous functions $f\in C^\infty(\mathbb{R}^n\setminus\{0\})$ of order $-\rho$ such that $f(\lambda x) = \lambda^{-\rho} f(x)$ for all $x\in\mathbb{R}^n, x\neq 0$, and $\lambda>0$.

Let the functions $f_j\in\dot{\mathcal{S}}^{-\rho_j}$ where $\rho_j\to\infty$ (but the condition $\rho_j<\rho_{j+1}$ is not required). The notation

$$f(x) \simeq \sum_{j=1}^{\infty} f_j(x) \tag{2}$$

means that, for any $N$, the remainder

$$f - \sum_{j=1}^{N} f_j \in \dot{\mathcal{S}}^{-\rho} \quad \text{where} \quad \rho = \min_{j\geq N+1}\rho_j. \tag{3}$$

In particular, if the sum (3) consists of a finite number $N$ of terms, then the inclusion (3) should be satisfied for all $\rho$. A function $f\in C^\infty$ is determined by its asymptotic expansion (2) up to a term from the Schwarz class $\mathcal{S}=\mathcal{S}^{-\infty}$.

We assume that $V$ and $F$ admit the asymptotic expansions,

$$V(x) \simeq \sum_{j=1}^{\infty} V_j(x), \quad F(x) \simeq \sum_{j=1}^{\infty} F_j(x). \tag{4}$$

THEOREM 2. ( W. and Yafaev, Inv. Prob. 2005 [10]) Suppose that an electric potential $V(x)$ and a magnetic field $F(x)$ that are $C^\infty(\mathbb{R}^n)-$functions, $n \geq 3$, admit the asymptotic expansions (4) and where $V_j(x)$ and $F_j(x)$ are homogeneous functions of orders $-\rho_j$ and $-r_j$, respectively, where $1 < \rho_1 < \rho_2 < \cdots$ and $2 < r_1 < r_2 < \cdots$. Then, the scattering data consisting of the kernel $s(\omega, \omega'; E)$ of the scattering matrix at a fixed positive energy $E$ in a neighborhood of the diagonal $\omega = \omega'$ uniquely determines each one of the $V_j(x)$ and the $F_j(x)$.

We also have formulae for the reconstruction of the $V_j$ and the $F_j, j = 1, 2, \cdots$

For the case of long-range potentials see Weder and Yafaev 2007 [11].

For previous results see: Joshi and Sa Barreto 1988 [12], 1999 [13], and Joshi 2000 [14].

Combining these two theorems we obtain our main result,

THEOREM 3. (W., 2007, [4]) Let the electric potentials $V_j$ and the magnetic fields $F_j$ be $C^\infty(\mathbb{R}^n)-$ functions, $n \geq 3, j = 1, 2$, and assume that they satisfy,

$$|\partial_\alpha V_j(x)| \leq C(1+|x|)^{-\rho-|\alpha|}, |\partial_\alpha F_j(x)| \leq C(1+|x|)^{-1-\rho-|\alpha|}, \quad \rho > 1,$$

for all $\alpha$. Moreover, suppose that they admit the asymptotic expansions

$$V_j(x) \simeq \sum_{l=1}^{\infty} V_{j,l}(x), \quad F_j(x) \simeq \sum_{l=1}^{\infty} F_{j,l}(x), j = 1, 2, \tag{5}$$

where $V_{j,l}$ and $F_{j,l}$ are homogeneous functions of orders, respectively, $-\rho_{j,l}$ and $-r_{j,l}$, with, $1 < \rho_{j,1} < \rho_{j,2} < \cdots$, and $2 < r_{j,1} < r_{j,2} < \cdots, j = 1, 2$. Assume, moreover, that the asymptotic expansions (5) actually converge, respectively, to $V_j$ and $F_j, j = 1, 2$, in pointwise sense, for $|x|$ large enough, or just that the sums in (5) are finite. Let $S_j(E)$ be, respectively, the scattering matrices corresponding to $(V_j, F_j), j = 1, 2$. Then, if for some $E > 0$, $S_1(E) = S_2(E)$, we have that $V_1(x) = V_2(x)$ and $F_1(x) = F_2(x), x \in \mathbb{R}^n$.

It is known since quite some time that the scattering matrix at a fixed positive energy uniquely determines electric potentials and magnetic fields if strong restrictions on the decay at infinity are imposed. Ramm 1987, 1988 [8], Nakamura, Uhlmann and Sun 1995 [15] consider potentials of compact support, and Novikov R G 1994 [16], Eskin and Ralston 1995 [17], and Uhlmann and Vasy 2002 [18], potentials decaying exponentially at infinity. In W. 2004 [19] the uniqueness at a fixed quasi-energy was proven for potentials periodic in time, that decay exponentially at infinity.

On the contrary, for general short-range potentials the scattering matrix at a fixed positive energy does not determine uniquely the potential. Indeed, in Chadan and Sabatier 1989 [20] examples -in three dimensions- are given of non-trivial radial oscillating potentials with decay as $|x|^{-3/2}$ at infinity such that the corresponding scattering amplitude is identically zero at some positive energy. Moreover, in dimension two there are examples by Grinevich 1986 [21] of potentials with a regular decay as $|x|^{-2}$ at infinity that have zero scattering amplitude at some positive energy. W. and Yafaev 2005 [10] give an example in two dimensions of a potential that decays as $|x|^{-2}$ where the leading order of the scattering amplitude is zero for all energies.

Nevertheless, as we discussed above if two general short-range electric potentials and magnetic fields coincide outside of some ball and if they have the same scattering matrix at some positive energy they are equal everywhere.

Theorem 3 shows a new aspect of the inverse scattering problem at a fixed energy. Namely, that uniqueness holds for general short-range electric potentials and magnetic fields without strongly restricting the decay at infinity, provided that the electric potential and the magnetic field have a regular behavior at infinity. Of course, this eliminates the oscillations and hence there is no contradiction with the examples of Chadan and Sabatier 1989 [20].

Furthermore, as we consider three or more dimensions there is no contradiction with the two dimensional examples of Grinevich 1986 [21].

## Idea of the Proof of Theorem 1

1) Completeness of Averaged Scattering Solutions.

Let $K \subset \mathbb{R}^n$ be connected, open, bounded and with a regular boundary. Suppose that $\varphi \in L^2(K)$ is a solution to

$$H\varphi = \left[(i\nabla + A)^2 + V\right]\varphi = E\varphi$$

that is orthogonal to all the averaged scattering solutions, i.e.,

$$(\varphi, \phi_{+,f})_{L^2(K)} = 0, f \in L^2(\mathbb{S}^{n-1}), \tag{6}$$

and define,

$$\psi := R_+(E)\varphi,$$

where we have extended $\varphi$ by zero to $\mathbb{R}^n \setminus K$.

Using (6) we prove that,

$$T_-(E)\varphi = 0.$$

Then, by the Agmon-Kuroda argument, $\psi \in L^2_{-s}, s < 1/2$, and as

$$H\psi = E\psi + \varphi,$$

with $\varphi(x)=0$, for $x \in \mathbb{R}^n \backslash K$, it follows from unique continuation that $\psi$ is identically zero on $\mathbb{R}^n \setminus K$ and then,

$$\|\varphi\|^2_{L^2(K)} = ((H-E)\psi, \varphi)_{L^2(K)} = (\psi, (H-E)\varphi)_{L^2(K)} = 0,$$

and it follows that $\varphi = 0$.

2) The Identity.

For simplicity we assume that $F_j = 0, j = 1, 2$. We prove that,

$$\phi^{(1)}_{+,f}(x; E) = \phi^{(2)}_{+,f}(x; E), \text{ for } |x| \geq R, f \in L^2(\mathbb{S}^{n-1}),$$

and using this result we obtain that,

$$\int_{B_R} (V_2 - V_1)\, \phi^{(1)}_{+,f}\, \overline{\phi^{(2)}_{+,g}}\, dx = 0,\ f, g \in L^2(\mathbb{S}^{n-1}).$$

Then, by the completeness of the averaged scattering solutions,

$$\int_{B_R} (V_2 - V_1)\, \varphi_1\, \overline{\varphi_2}\, dx = 0 \tag{7}$$

for every $\varphi_j \in \mathcal{H}^2(B_R)$ that are solutions to,

$$(H_0 + V_j)\varphi_j = E\varphi_j, j = 1, 2.$$

3) The Faddeev Solutions.

For every $p \in \mathbb{C}^n \setminus \mathbb{R}^n$, with $p^2 = E$ and $|p|$ large enough we construct the Faddeev's solutions, $\varphi_j(x, p) \in \mathbb{H}^2_{\mathrm{loc}}(\mathbb{R}^n)$ to the equations

$$(H_0 + \chi_{B_R}(x)V_j)\varphi_j(x, p) = E\varphi_j(x, p), j = 1, 2,$$

where $\chi_{B_R}$ is the characteristic function of $B_R$, such that,

$$\varphi_j(x,p) = e^{ip\cdot x}(1+\psi_j(x,p)),$$

where $\|\psi_j(x,p)\|_{\mathbb{H}^1_{-s}} \leq C_s, s > 1/2$, and

$$\text{s}-\lim_{|p|\to\infty} \|\psi_j(x,p)\|_{L^2_{-s}} = 0.$$

Given any $\xi \in \mathbb{R}^n$ take a sequence $p_l^{(j)}$ such that,

$$\int_{B_R} e^{i\xi\cdot x}\,(V_2(x) - V_1(x))\,dx = \lim_{l\to\infty} \int_{B_R} (V_2(x) - V_1(x))\,\varphi_1(x,p_l^{(1)})\,\overline{\varphi_2(x,p_l^{(2)})}\,dx = 0,$$

where we used (7). It follows that $V_1(x) = V_2(x), x \in B_R$.

## Idea of the proof of Theorem 2

$$s(\omega',\omega;E) - \delta(\omega',\omega) = C(E)\,f(\omega',\omega;E),$$

where $C(E)$ is a constant, and $f$ is the scattering amplitude.

We have that,

$$f \approx e^{-\pi i(n-1)/4} k^{(n-1)/2} (2\pi)^{-(n-1)/2} \int_{\Pi_\omega} e^{ik\langle y,\,\omega-\omega'\rangle} R(y,\omega;E;\mathbb{V})\,dy,\ \omega' \to \omega,$$

where, $k = E^{1/2}$, $\mathbb{V} = (V, A)$, and $\Pi_\omega$ is the hyperplane orthogonal to $\omega$, and

$$R(y,\omega;E;\mathbb{V}) := (2ik)^{-1}\int_{-\infty}^{\infty}(V(y+t\omega) - 2k\langle\omega, A(y+t\omega)\rangle)\,dt.$$

We define,

$$R_e(y,\omega;V) := \int_{-\infty}^{\infty} V(y+t\omega)\,dt, \omega \in \mathbb{S}^{n-1},\ y \in \Pi_\omega, \text{ that is even in } \omega, \tag{8}$$

and

$$R_m(y,\omega;A) := \int_{-\infty}^{\infty} \langle\omega, A(y+t\omega)\rangle\,dt, \omega \in \mathbb{S}^{n-1},\ y \in \Pi_\omega, \text{ that is odd in } \omega. \tag{9}$$

If we know $R(y,\omega;E;\mathbb{V}), \omega \in \mathbb{S}^{n-1}$, we uniquely reconstruct $V$ and $F := \mathrm{curl}A$ from the $X$–ray transforms (8) and (9), respectively.

To handle the high-order terms we consider $S(E)$ as a pseudodifferential operator on $L^2(\mathbb{S}^{n-1})$ with symbol $a(y,\omega;E)$, where $y$ is the variable in the hyperplane orthogonal to $\omega$.

$$s(\nu,\omega;\lambda) = (2\pi)^{-d+1}k^{d-1}\int_{\Pi_\omega} e^{-ik\langle y,\nu\rangle}a(y,\omega;\lambda)\,dy$$

so that the scattering matrix $S(\lambda)$ can be regarded as a pseudodifferential operator on $\mathbb{S}^{d-1}$ with right symbol $a(y,\omega;\lambda)$.

Note that the symbol $a(y,\omega;\lambda)$ can be recovered from the kernel $s(\nu,\omega;\lambda)$ by the inversion of the Fourier transform,

$$a(y,\omega;\lambda) = \int_{\Pi_\omega} e^{ik\langle y,\eta\rangle}s(t(\eta),\omega;\lambda)\gamma(\eta)\,d\eta, \quad y \in \Pi_\omega,$$

where $\gamma \in C_0^\infty(\mathbb{R}^{d-1})$ is an arbitrary function such that $\gamma(\eta) = 1$ in some neighbourhood of zero and $\gamma(\eta) = 0$ for, say, $|\eta| \geq 1/2$.

Remark that by eventually adding terms that are identically zero we can always assume that $r_j = \rho_j + 1$.

We prove that $a$ admits the expansion into the asymptotic sum

$$a(y, \omega; E) \simeq 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{j_1, j_2, \ldots, j_n} a_{n,m;j_1,j_2,\ldots,j_n}(y, \omega; E),$$

where $j_k = 1, 2, \ldots$ for all $k = 1, \ldots, n$, $m = 0, 1, \ldots$, the functions $a_{n,m;j_1,j_2,\ldots,j_n}(y, \omega; E)$ only depend on $\mathbb{V}_{j_1}, \mathbb{V}_{j_2}, \ldots, \mathbb{V}_{j_n}$ and are homogeneous of order

$$n - m - \rho_{j_1} - \rho_{j_2} - \ldots - \rho_{j_n}$$

in the variable $y$. In particular,

$$a_{1,0;j}(y, \omega; E) = R(y, \omega; E; \mathbb{V}_j)$$

is a homogeneous function of order $-\rho_j + 1$.

This allows us to uniquely reconstruct each $V_j, F_j$ in a recursive way as follows.

Let us denote by $T$ the mapping that sends $\mathbb{V}$ into the function $a - 1$. Of course, it is defined up to a symbol from the class $\mathcal{S}^{-\infty}$. Thus, we put

$$T(y, \omega; \lambda; \mathrm{V}) = a(y, \omega; \lambda) - 1.$$

Moreover, we distinguish the leading order term, $R$, of the linear part of $T$, and set

$$Q(y, \omega; \lambda; \mathrm{V}) = T(y, \omega; \lambda; \mathrm{V}) - R(y, \omega; \lambda; \mathrm{V}).$$

We need the following simple property of the mapping $Q$.

PROPOSITION 1. *Suppose that* $\mathbb{V}^{(j)} \in \mathcal{S}^{-\rho^{(j)}}$, $j = 1, 2$, *where* $\rho^{(2)} > \rho^{(1)} > 1$. *Then*

$$Q(\mathbb{V}^{(1)} + \mathbb{V}^{(2)}) - Q(\mathbb{V}^{(1)}) \in \mathcal{S}^{-\rho^{(2)}+1-\varepsilon}, \quad \varepsilon = \min\{\rho^{(1)} - 1, 1\} > 0.$$

It is convenient to introduce the following notation. Suppose that some function $f$ admits the expansion in the asymptotic series (2) of homogeneous functions. By taking sums of such functions it cannot be excluded that some terms will be equal to zero. Therefore we define $f^\sharp$ as the highest order homogeneous term $f_k$ in (2) that is not identically zero. For example, $f^\sharp = 0$ if $f \in \mathcal{S}$.

Suppose now that we have found the coefficients $\mathbb{V}_k$ for all $k = 1, \ldots, n-1$, $n \geq 2$. Let us reconstruct $\mathbb{V}_n$. We apply Proposition 1 to the functions $\mathbb{V}^{(1)} = \sum_{j=1}^{n-1} \mathbb{V}_j$, $\mathbb{V}^{(2)} = \mathbb{V} - \mathbb{V}^{(1)}$ where $\rho^{(1)} = \rho_1$, $\rho^{(2)} = \rho_n$. This yields

$$Q(\mathbb{V}) - Q(\sum_{j=1}^{n-1} \mathbb{V}_j) \in \dot{\mathcal{S}}^{-\rho_n+1-\varepsilon}, \quad \varepsilon > 0,$$

and thus, for an arbitrary $\rho_n$, this term can be neglected compared to $R(\mathbb{V}_n)$. All terms $R(\mathbb{V}_j)$, $j \geq n+1$, are also negligible compared to $R(\mathbb{V}_n)$. Therefore,

$$R(\mathbb{V}_n) + \sum_{j \geq n+1} R(\mathbb{V}_j) + (Q(\mathbb{V}) - Q(\sum_{j=1}^{n-1} \mathbb{V}_j)) = a - 1 - T(\sum_{j=1}^{n-1} \mathbb{V}_j)$$

and selecting terms of the highest order, we obtain that

$$R(\mathbb{V}_n) = (a - 1 - T(\sum_{j=1}^{n-1} \mathbb{V}_j))^\sharp.$$

Having found $R(\mathbb{V}_n)$, we recover the functions $V_n$, $F_n$ using the inversion of the two-dimensional Radon transform.

Let me briefly explain how this is done.

For $v \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$, $\rho > 1$, the Radon (or $X$-ray, which is the same in the dimension two) transform is defined by the formula

$$r(\omega, y; v) = \int_{-\infty}^{\infty} v(\omega t + y) dt, \quad \omega \in \mathbb{S}, \quad \langle \omega, y \rangle = 0. \tag{10}$$

Obviously, $r(\omega, y) = r(-\omega, y)$. The Fourier transform $\hat{v}$ of $v$ and hence the function $v$ itself can be recovered in the following way. Let $\omega_\xi$ be one of the two unit vectors such that $\langle \omega_\xi, \xi \rangle = 0$. Then

$$\hat{v}(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i|\xi| s} r(\omega_\xi, s\hat{\xi}; v) ds, \quad \hat{\xi} = \xi |\xi|^{-1}. \tag{11}$$

We apply this method for the reconstruction of a homogeneous function $V \in C^\infty(\mathbb{R}^d \setminus \{0\})$ of order $-\rho < -1$ from its $X$-ray transform

$$R_e(y, \omega; V) := \int_{-\infty}^{\infty} V(y + t\omega)\, dt,$$

known for all $\omega \in \mathbb{S}^{d-1}$ and $y \in \Pi_\omega$, $y \neq 0$. For an arbitrary $x \in \mathbb{R}^d \setminus \{0\}$, consider some two-dimensional plane $\Lambda_x$ orthogonal to $x$ and, for $y \in \Lambda_x$, set $v_x(y) = V(x+y)$. Then for all $\omega \in \Lambda_x$, $|\omega| = 1$, and all $y \in \Lambda_x$, $\langle \omega, y \rangle = 0$,

$$r(\omega, y; v_x) = R_e(\omega, x + y; V). \tag{12}$$

Since $x + y \neq 0$, the function $v_x \in \mathcal{S}^{-\rho}(\mathbb{R}^2)$ so that we can recover the function $v_x$ and, in particular, $v_x(0) = V(x)$ by formula (11). This procedure is used for the reconstruction of the asymptotics of the electric potentials.

In the magnetic case we are given only the integral

$$R_m(y,\omega;A) := \int_{-\infty}^{\infty} \langle \omega, A(y+t\omega)\rangle \, dt, \omega \in \mathbb{S}^{n-1},\ y \in \Pi_\omega, \text{ that is odd in } \omega.$$

for all $\omega \in \mathbb{S}^{d-1}$ and $y \in \Pi_\omega$, $y \neq 0$. Since this integral is zero if $A(x) = \operatorname{grad}\phi(x)$ and $\phi(x) \to 0$ as $|x| \to \infty$, we cannot hope to recover $A$ from this equation. Nevertheless, the corresponding magnetic field $F(x) = \operatorname{curl} A(x)$ can be recovered. Assume again that $A \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a homogeneous vector-valued function of order $-\rho < -1$. Let us consider one of the components of $F(x)$, for example, $F^{(12)}(x)$. We will first show how $F^{(12)}(x)$ can be reconstructed everywhere except the plane $L_{12}$ where $x_3 = \ldots = x_d = 0$. Let $\omega = (\omega_1, \omega_2, 0, \ldots, 0)$ be any unit vector in the plane $L_{12}$, let $\nu = (-\omega_2, \omega_1, 0, \ldots, 0)$ be the unit vector obtained by rotating $\omega$ in the plane $L_{12}$ by the angle $\pi/2$ in the counter-clockwise sense and let $\tilde{x} = (0, 0, x_3, \ldots, x_n) \neq 0$ be an arbitrary vector that is orthogonal to $L_{12}$. Set $f_{\tilde{x}}^{(12)}(y) = F^{(12)}(\tilde{x} + y)$ for $y \in L_{12}$. It is easy to see that

$$r(\omega, y; f_{\tilde{x}}^{(12)}) = -\partial R_m(\omega, s\nu + \tilde{x}; A)/\partial s, \quad y = s\nu. \tag{13}$$

Indeed, since $F^{(12)}$ is invariant under rotations in the plane $L_{12}$, it suffices to check (13) for the case $\omega_1 = 1$, $\omega_2 = 0$ when (13) reads as

$$\int_{-\infty}^{\infty} F^{(12)}(t, s, x_3, \ldots, x_n) dt = -\frac{\partial}{\partial s} \int_{-\infty}^{\infty} A^{(1)}(t, s, x_3, \ldots, x_n) dt.$$

For the proof of this relation, we have only to use the definition $F^{(12)} = \partial A^{(2)}/\partial t - \partial A^{(1)}/\partial s$ and the fact that the integral of $\partial A^{(2)}/\partial t$ is zero. Finding expression (13), we can recover $F^{(12)}(y+\tilde{x})$ for all $y \in L_{12}$ by formula (11). By virtue of the homogeneity of the function $F^{(12)}$, this yields $F^{(12)}(x)$ everywhere except the plane $L_{12}$. Thus, the magnetic field $F(x)$ is reconstructed for all $x \neq 0$.

Formula (13) can of course be rewritten in the invariant way. For example, in the case $d = 3$ it means that, for arbitrary $\omega, n \in \mathbb{S}^2$, $\langle \omega, n \rangle = 0$,

$$\int_{-\infty}^{\infty} \langle n, \operatorname{curl} A(t\omega + x) \rangle dt = \langle \omega \wedge n, \nabla_x \int_{-\infty}^{\infty} \langle \omega, A(t\omega + x) \rangle dt \rangle,$$

where the symbol " $\wedge$ " means the vector product.

Finally, if only the combination

$$R(y, \omega; E; \mathbb{V}) := (2ik)^{-1} \int_{-\infty}^{\infty} (V(y + t\omega) - 2k\langle \omega, A(y + t\omega) \rangle)\, dt.$$

is known, then using that $R_e$ and $R_m$ are even and odd functions of $\omega \in \mathbb{S}^{d-1}$, respectively, we obtain that

$$\begin{aligned} R_e(y, \omega; V) &= ik(R(y, \omega; E; \mathrm{V}) + R(y, -\omega; E; \mathrm{V})), \\ R_m(y, \omega; A) &= i2^{-1}(R(y, -\omega; E; \mathrm{V}) - R(y, \omega; E; \mathrm{V})). \end{aligned}$$

This allows us to reconstruct the functions $V$ and $F$ by the formulae given above.

An Adapted Gauge.

$$|\partial^\alpha F(x)| \leq C(1 + |x|)^{-r-|\alpha|},\ r > 2.$$

**We define**

$$\boldsymbol{A}^{(i)}_{\text{reg}} := \int_1^\infty s \sum_{j=1}^n \boldsymbol{F}^{(ij)}(sx)\, x_j ds,$$

$$\boldsymbol{A}^{(i)}_{\infty} := -\int_0^\infty s \sum_{j=1}^n \boldsymbol{F}^{(ij)}(sx)\, x_j ds.$$

**We have that,**

$$\boldsymbol{A}_\infty(\lambda x) = \lambda^{-1}\, \boldsymbol{A}_\infty(x),\ \text{curl}\boldsymbol{A}_\infty(x) = 0, x \neq 0.$$

**Moreover,**

$$\boldsymbol{A}_{\text{trans}} = \boldsymbol{A}_{\text{reg}} + \boldsymbol{A}_\infty.$$

**Define,**

$$U(x) := \int_{\Gamma_{x_0 x}} \langle \boldsymbol{A}_\infty(y), dy \rangle,\ x \neq 0, 0 \neq \Gamma_{x_0 x}.$$

$$\text{grad}U(x) = \boldsymbol{A}_\infty(x).$$

**Then,**

$$\boldsymbol{A}(x) := \boldsymbol{A}_{\text{reg}}(x) + \text{grad}\,((1-\eta(x))U(x)) = \boldsymbol{A}_{\text{reg}}(x) + (1-\eta(x))\boldsymbol{A}_\infty(x) -$$

$$U(x)\text{grad}\eta(x),$$

$$\eta \in C^\infty(\mathbb{R}^3), \eta(x) = 0, |x| \leq \epsilon, \eta(x) = 1, |x| \geq 1.$$

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