Spectral analysis for convolution operators on groups

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Contents

1 Convolution operators on groups

- X, *locally compact group* (LCG) with a left Haar measure λ,
- $M(X)$, *complex bounded Radon measures*,
- $C_0(X)$, continuous functions on X decaying to 0 at infinity.

For $\mu, \nu \in M(X)$ and $f \in \mathcal{H} := L^2(X, d\lambda)$ we set

$$(\mu * f)(x) := \int_X d\mu(y)\, f(y^{-1}x), \quad x \in X,$$

$$\int_X d(\mu * \nu)(x)\, g(x) := \int_X \int_X d\mu(x) d\nu(y)\, g(xy), \quad g \in C_0(X).$$

Given $\varphi : X \to \mathbb{R}$ (nice enough), $\varphi\mu \in M(X)$ corresponds to the bounded functional

$$C_0(X) \ni g \mapsto \int_X d\mu(x)\, \varphi(x) g(x) \in \mathbb{C}.$$

The *convolution operator* H_μ, $\mu \in \mathsf{M}(\mathsf{X})$, is defined by

$$(\mathsf{H}_\mu f)(x) := (\mu * f)(x) = \int_{\mathsf{X}} d\mu(y)\, f(y^{-1}x), \quad f \in \mathcal{H},\ x \in \mathsf{X}.$$

One has

1. $\|\mathsf{H}_\mu\| \leq |\mu|(\mathsf{X})$,
2. $(\mathsf{H}_\mu)^* = \mathsf{H}_{\mu^*}$, where $\mu^*(E) = \overline{\mu(E^{-1})}$ (E Borel subset of X).

$\Longrightarrow$ H_μ `is bounded and selfadjoint if` $\mu = \mu^* \in \mathsf{M}(\mathsf{X})$.

2 Theorem

- $\mathrm{Hom}(\mathsf{X},\mathbb{R})$, continuous group morphisms $\Phi : \mathsf{X} \to \mathbb{R}$

Definition 2.1. *Let* $\mu = \mu^* \in \mathsf{M}(\mathsf{X})$.

(a) $\Phi \in \mathrm{Hom}(\mathsf{X},\mathbb{R})$ *belongs to* $\mathrm{Hom}_\mu^{\mathrm{ad}}(\mathsf{X},\mathbb{R})$ *if*

$$\begin{aligned} \Phi\mu, \Phi^2\mu, \Phi^3\mu \in \mathsf{M}(\mathsf{X}), &\qquad \textit{(decay assumptions)},\\ (\Phi\mu) * \mu = \mu * (\Phi\mu), &\qquad \textit{(first order commutation)},\\ (\Phi\mu) * (\Phi^2\mu) = (\Phi^2\mu) * (\Phi\mu), &\qquad \textit{(second order commutation)}. \end{aligned}$$

(b) We set

$$\mathcal{K}_\mu^{\mathrm{ad}} := \bigcap_{\Phi \in \mathrm{Hom}_\mu^{\mathrm{ad}}(\mathsf{X},\mathbb{R})} \ker(\mathsf{H}_{\Phi\mu}).$$

Theorem 2.2. *Let* X *be a LCG and let* $\mu = \mu^* \in \mathsf{M}(\mathsf{X})$. *Then*

$$\mathcal{H}_{\text{sing}}(\mathsf{H}_\mu) \subset \mathcal{K}_\mu^{\text{ad}}.$$

Thus:

- H_μ is purely absolutely continuous if $\mathcal{K}_\mu^{\text{ad}} = \{0\}$.

- The spectral structure of H_μ is somehow related to the "size" of the space $\text{Hom}(\mathsf{X}, \mathbb{R})$?

3 Sketch of the proof

Let $\Phi \in \mathrm{Hom}_{\mu}^{\mathrm{ad}}(X, \mathbb{R})$.

1. One has the (first and second order) commutation identities

$$K := i[H_\mu, \Phi] = -iH_{\Phi\mu} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad [H_\mu, K] = 0.$$

2. Set $A := \frac{1}{2}(\Phi K + K\Phi)$ on $\mathcal{D}(A) := \{f \in \mathcal{H} \mid \Phi K f \in \mathcal{H}\}$.

   (by analogy to $D := \frac{1}{2}(QP + PQ) = \frac{1}{2}\big(Qi\big[\frac{P^2}{2}, Q\big] + i\big[\frac{P^2}{2}, Q\big]Q\big)$ in quantum mechanics)

3. One has $i[H_\mu, A] = K^2 \geq 0$, and thus

$$i[H_\mu, A] = K^2 > 0 \quad \text{on } [\ker(K)]^\perp.$$

4. Apply the following Kato-Putnam-type theorem with $\mathcal{H}_0 = [\ker(\mathsf{K})]^\perp$ and

$$\mathsf{A}_0 = \mathsf{A} \restriction \mathcal{H}_0,$$
$$\mathsf{H}_0 = \mathsf{H}_\mu \restriction \mathcal{H}_0.$$

If H_0 is a bounded selfadjoint operator in a Hilbert space $\mathcal{H}_0$, one has:

Definition 3.1. A selfadjoint operator A_0 in $\mathcal{H}_0$ is *weakly conjugate to* H_0 if $\mathrm{i}[\mathsf{H}_0, \mathsf{A}_0]$ is bounded and $\mathrm{i}[\mathsf{H}_0, \mathsf{A}_0] > 0$.

Theorem 3.2. Let A_0 be weakly conjugate to H_0 with $\mathrm{i}[\mathsf{H}_0, \mathsf{A}_0]$ regular w.r.t. A_0. Then the spectrum of H_0 is purely absolutely continuous.

5. Thus

$$\mathcal{H}_{\text{sing}}(\mathsf{H}_\mu) \subset \ker(\mathsf{K}) = \ker(\mathsf{H}_{\Phi\mu}).$$

4 Examples

Example 4.1 (Central groups)**.** *Let* X *be central, take a non compact element* $z \in \mathsf{Z}(\mathsf{X})$*, and set* $\mu := \delta_z + \delta_{z^{-1}}$*. Choose* $\Phi \in \mathrm{Hom}(\mathsf{X}, \mathbb{R})$ *such that* $\Phi(z) = \frac{1}{2}\Phi(z^2) \neq 0$*. Then*

$$\mathcal{H}_{\mathrm{s}}(\mathsf{H}_\mu) \subset \ker(\mathsf{H}_{\Phi\mu}) = \left\{ \mathsf{f} \in \mathcal{H} \mid \mathsf{f}(z^{-1}\cdot) = \mathsf{f}(z\cdot) \right\} = \{0\}.$$

Thus $\mathcal{H}_{\mathrm{ac}}(\mathsf{H}_\mu) = \mathcal{H}$*.*

Other examples can be found on $\mathsf{X} = \mathsf{S}_3 \times \mathbb{Z}$*,* $\mathsf{X} = \mathrm{SU}(2) \times \mathbb{R}$*, etc.*

Example 4.2 (Abelian groups)**.** *Application of the Fourier transform to Theorem 2.2 gives a result on multiplication operators.*

Example 4.3 (Semidirect products)**.** $\mathcal{H}_{\mathrm{ac}}(\mathsf{H}_a) \neq \{0\}$ *for appropriate functions* $a = a^* \in \mathsf{L}^1(\mathsf{X})$ *on* $\mathsf{X} = \mathsf{N} \times_\tau \mathsf{G}$*, where* N, G *are discrete with* G *abelian and* $\tau : \mathsf{G} \to \mathrm{Aut}(\mathsf{N})$ *is a suitable group morphism.*

5 Some references

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