Two-body scattering at low energies

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- Zero energy asymptotics of the resolvent for a class of slowly decaying potentials.

  (Joint work with S. Fournais.)

- The zero energy scattering problem for $V(x) \approx -\gamma r^{-\mu}$; $\mu < 2$.

  (Joint work with J. Dereziński.)

Reference:
S. Fournais and E. Skibsted, *Zero energy asymptotics of the resolvent for a class of slowly decaying potentials*, Math. Z. **248** (2004), 593–633.

Consider

$$H = -\Delta + V \text{ on } L^2(\mathbb{R}^d_x).$$

Conditions (stated here imprecisely): For some $\mu \in (0,2)$

$$-Cr^{-\mu} \leq V(x) \leq -\epsilon r^{-\mu};\ r = |x| > R,$$

and similarly for a certain "virial expression".
For fixed $0 < \theta < \pi$

$$\Gamma_\theta := \{z \in \mathbb{C} \setminus \{0\} \,\big|\, |z| \leq 1, \arg z \in (0,\theta)\}.$$

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Main result: Complete asymptotic expansions (in weighted spaces)

$$R(\zeta) = (H - \zeta)^{-1} \asymp \sum_{n=0}^{\infty} R_n^{+(-)} \zeta^n;$$

$$\Gamma_\theta \ni \zeta \to 0 \text{ (or } \Gamma_\theta \ni \overline{\zeta} \to 0).$$

Intuition: WKB-ansatz for stationary solutions to the Schrödinger equation $-\psi'' + V\psi = E\psi$ in dimension $d = 1$

$$\psi \approx C_+(E-V)^{-\frac{1}{4}} e^{i\int (E-V)^{\frac{1}{2}}\,dx} + C_-(E-V)^{-\frac{1}{4}} e^{-i\int (E-V)^{\frac{1}{2}}\,dx}.$$

Suggestive for weights: Let $\langle x\rangle = \sqrt{1+x^2}$. Then

$$\langle x\rangle^{-s}\psi \in L^2 \text{ uniformly in } E > 0$$
$$\Leftrightarrow s > \tfrac{1}{2} + \tfrac{\mu}{4}.$$

Condition (1)

Let $V(x) = V_1(x) + V_2(x)$, $x \in \mathbb{R}^d$, *be a real-valued potential. There exists* $0 < \mu < 2$ *such that* V *satisfies the requirements:*

- *There exists* $\epsilon_1 > 0$ *such that* $V_1(x) \le -\epsilon_1 \langle x \rangle^{-\mu}$; $\langle x \rangle = \sqrt{1 + r^2}$.
- *For all* $\alpha \in (\mathbb{N} \cup \{0\})^d$ *there exists* $C_\alpha > 0$ *such that*
$$\langle x \rangle^{\mu + |\alpha|} |\partial^\alpha V_1(x)| \le C_\alpha.$$
- *There exists* $\epsilon_2 > 0$ *such that*
$$-r^{-2} \left(x \cdot \nabla (r^2 V_1) \right) \ge -\epsilon_2 V_1.$$
- $V_2(-\Delta + i)^{-1}$ *is a compact operator on* $L^2(\mathbb{R}^d)$.
- *There exists* $\delta, C, R > 0$ *such that*
$$|V_2(x)| \le C r^{-1-\mu/2-\delta},$$
for $|x| > R$.
- V *satisfies unique continuation at infinity.*

Theorem (Limiting absorption principle)

Suppose the above Condition (1). Then for all $s \in (\frac{1}{2} + \frac{\mu}{4}, \infty)$ *and all* $0 < \theta < \pi$ *the family of operators*

$$B(\zeta) = \langle x \rangle^{-s} R(\zeta) \langle x \rangle^{-s}$$

is uniformly Hölder continuous in Γ_θ*. In particular there exists* $C_{s,\theta} > 0$ *such that*

$$\sup_{\zeta \in \Gamma_\theta} \|B(\zeta)\| \leq C_{s,\theta},$$

and the limits

$$B(0 + \mathrm{i}0) := \lim_{\zeta \to 0, \zeta \in \Gamma_\theta} B(\zeta),$$

$$B(0 - \mathrm{i}0) := \lim_{\zeta \to 0, \zeta \in \Gamma_\theta} B(\zeta)^*$$

exist in $B(L^2(\mathbb{R}^d))$.

References:
1) J. Dereziński and E. Skibsted, *Classical scattering at low energies*, Aarhus Preprint Series No.: 6, April 2006.

2) J. Dereziński and E. Skibsted, *Quantum scattering at low energies*, in preparation.

Consider henceforth the Hamiltonian

$$H = H_0 + V \text{ on } L^2(\mathbb{R}^d_x),$$

where

$$H_0 = -\Delta/2 = 2^{-1}p^2, \; p = -\mathrm{i}\nabla_x,$$

under the following (simplifying) condition on the potential:

Condition (A)

$$V(x) = -\gamma|x|^{-\mu} + O(|x|^{-\mu-\epsilon}),$$

where $\mu \in (0, 2)$ *and* $\gamma, \epsilon > 0$.
More generally,

$$\partial_x^\beta \left(V(x) + \gamma|x|^{-\mu}\right) = O(|x|^{-\mu-\epsilon-|\beta|}).$$

Compactly supported singularities may be included, cf. Condition (1). Most results hold under weaker conditions.

Basic scattering observables: The asymptotic normalized velocities

$$\omega^{\pm} = \pm \lim_{t \to \pm\infty} e^{itH} \hat{x} e^{-itH} 1_{\{H \geq 0\}}; \ \hat{x} = \frac{x}{r}.$$

Wave operators: Unitary operators

$$W^{\pm} : \mathcal{H}_{diag} \to \mathcal{H}_{pos};$$

$$\mathcal{H}_{diag} = L^2([0,\infty); L^2(S^{d-1}_{\omega}), \mathrm{d}\lambda) = \int_0^{\infty} \oplus L^2(S^{d-1}_{\omega}) \, \mathrm{d}\lambda,$$

$$\mathcal{H}_{pos} = 1_{\{H \geq 0\}} L^2(\mathbb{R}^d_x).$$

Properties:

$$\omega^{\pm} W^{\pm} = W^{\pm} \omega, \ HW^{\pm} = W^{\pm} \lambda.$$

Introduce $S = (W^+)^* W^-$, and decompose in terms of S–matrices

$$S = \int_0^{\infty} \oplus S(\lambda) \, \mathrm{d}\lambda;$$

here $S(\lambda)$ is unitary on $L^2(S^{d-1}_{\omega})$.

Consider the classical Hamiltonian $h(x,\xi) := \frac{1}{2}\xi^2 + V(x)$ on the phase space $\mathbb{R}^d \times \mathbb{R}^d$, using $h_0(x,\xi) := \frac{1}{2}\xi^2$ as the free Hamiltonian.

For any $\xi \in \mathbb{R}^d$, $\xi \neq 0$, and x in an appropriate outgoing/incoming region the following problem admits a solution:

$$\begin{cases} \ddot{y}(t) = -\nabla V(y(t)), \\ y(\pm 1) = x, \\ \xi = \lim_{t\to\pm\infty} \dot{y}(t). \end{cases} \tag{1}$$

One obtains a family $y^{\pm}(t,x,\xi)$ of solutions depending smoothly on parameters. All (positive energy) scattering orbits, i.e. orbits satisfying $\lim_{t\to\pm\infty} |y(t)| = \infty$, are of this form (with energy $\lambda = \frac{1}{2}\xi^2$).

Using these solutions one constructs a solution $\phi^{\pm}(x,\xi)$ to the eikonal equation

$$\frac{1}{2}\left(\nabla_x \phi^{\pm}(x,\xi)\right)^2 + V(x) = \frac{1}{2}\xi^2 \tag{2}$$

satisfying $\nabla_x \phi^{\pm}(x,\xi) = \dot{y}(\pm 1, x, \xi)$.

References:
1) H. Isozaki, J.Kitada, *Modified wave operators with time-independent modifiers*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **32** (1985), 77–104.

2) H. Isozaki, J.Kitada, *Scattering matrices for two-body Schrödinger operators*, Scientific papers of the College of Arts and Sciences, Tokyo Univ. **35** (1985), 81–107.

Use the functions $\phi^{\pm}(x,\xi)$ to construct appropriate quantum modifiers:

$$J^{\pm}f(x) := (2\pi)^{-d}\int \mathrm{e}^{\mathrm{i}\phi^{\pm}(x,\xi)-\mathrm{i}\xi y}a^{\pm}(x,\xi)f(y)\mathrm{d}y\mathrm{d}\xi. \tag{3}$$

Here $a^{\pm}(x,\xi)$ is an appropriate cut-off supported in the domain of the definition of $\phi^{\pm}$, equal to one in the incoming/outgoing region.

One constructs modified wave operators:

$$W^{\pm}f := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}J^{\pm}\mathrm{e}^{-\mathrm{i}tH_0}f, \tag{4}$$

and the corresponding modified scattering operator

$$S = W^{+*}W^{-}.$$

The free Hamiltonian H_0 can be diagonalized by the direct integral

$$\mathcal{H}_0 = \int_0^\infty \oplus L^2(S^{d-1})\, \mathrm{d}\lambda, \tag{5}$$

and

$$\mathcal{F}_0(\lambda)f(\omega) = (2\lambda)^{(d-2)/4}\hat{f}(\sqrt{2\lambda}\omega); \;\; f \in L^2(\mathbb{R}^d), \tag{6}$$

where $\hat{f}$ refers to the d–dimensional Fourier transform.

Note that indeed,

$$L^2(\mathbb{R}^d) \ni f \to \int_0^\infty \oplus \mathcal{F}_0(\lambda) f \,\mathrm{d}\lambda$$

is a unitary operator.

The operator $\mathcal{F}_0(\lambda)$ can be interpreted as a bounded operator from the weighted space

$$L^{2,s}(\mathbb{R}^d) := \langle x \rangle^{-s} L^2(\mathbb{R}^d),\; s > \frac{1}{2},$$

to $L^2(S^{d-1})$.

Problem (Basic scattering problem)

Show that the wave and scattering operators can be restricted to a fixed energy λ, and study properties of the restricted operators (referred to as wave and scattering matrices, respectively).

This question is conceptually simpler in the case of the scattering operator S. It satisfies $SH_0 = H_0S$, so abstract theory guarantees the existence of a decomposition

$$S \simeq \int_{]0,\infty[} \oplus S(\lambda)\mathrm{d}\lambda,$$

where $S(\lambda)$ are unitary operators on $L^2(S^{d-1})$ defined for almost all λ.

One can prove under Condition (A) (actually under much weaker conditions) that $S(\cdot)$ can be chosen to be a strongly continuous function (which fixes uniquely $S(\lambda)$ for all $\lambda \in]0,\infty[$). $S(\lambda)$ is called the *scattering matrix at the energy* λ.

As for wave operators, they satisfy $W^\pm H_0 = HW^\pm$, so it is natural to use the direct integral decomposition (5) only from the right and the natural question is whether we can give a rigorous meaning to $W^\pm \mathcal{F}_0(\lambda)^*$:

There exists a unique strongly continuous function $]0,\infty[\ni \lambda \mapsto W^\pm(\lambda)$ with values in the space of bounded operators from $L^2(S^{d-1})$ to $L^{2,-s}(\mathbb{R}^d)$ with $s > \frac{1}{2}$ such that for $f \in L^{2,s}(\mathbb{R}^d)$ and $g \in C_c(]0,\infty[)$

$$W^\pm g(H_0) f = \int_{]0,\infty[} g(\lambda) W^\pm(\lambda) \mathcal{F}_0(\lambda) f \,\mathrm{d}\lambda.$$

$W^\pm(\lambda)$ are called *wave matrices at energy* λ. For $\tau \in L^2(S^{d-1})$, $W^\pm(\lambda)\tau$ is an element of $L^{2,-s}(\mathbb{R}^d)$ satisfying

$$\left(-\frac{1}{2}\Delta + V(x) - \lambda\right) W^\pm(\lambda)\tau = 0, \tag{7}$$

which does not belong to $L^2(\mathbb{R}^d)$. We call $W^\pm(\lambda)\tau$ an *averaged generalized eigenfunction of* H.

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which does not belong to $L^2(\mathbb{R}^d)$. We call $W^{\pm}(\lambda)\tau$ an *averaged generalized eigenfunction of* H.

One can also extend the domain of $W^{\pm}(\lambda)$ so that it can act on the delta function at $\omega \in S^{d-1}$, denoted δ_ω. Now $w^{\pm}(\omega,\lambda) := W^{\pm}(\lambda)\delta_\omega$ is an element of $L^{2,-p}(\mathbb{R}^d)$ for $p > \frac{d}{2}$ and it also satisfies

$$\left(-\frac{1}{2}\Delta + V(x) - \lambda\right) w^{\pm}(\omega,\lambda) = 0. \tag{8}$$

It behaves in the outgoing/incoming region as a plane wave. It will be called the *generalized eigenfunction of* H *at a fixed asymptotic direction* ω.

Other possible definitions of wave and scattering operators. In the short-range case, that is $\mu > 1$, the usual definitions are

$$\begin{aligned} W_{\mathrm{sr}}^{\pm} f &:= \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0} f \\ S_{\mathrm{sr}} &:= W_{\mathrm{sr}}^{+*} W_{\mathrm{sr}}^{-} . \end{aligned}$$

The operators $W^{\pm}$ and $W_{\mathrm{sr}}^{\pm}$ differ by a momentum-dependent phase factor:

$$\begin{aligned} W^{\pm} &= W_{\mathrm{sr}}^{\pm} \mathrm{e}^{\mathrm{i}\psi_{\mathrm{sr}}^{\pm}(p)}, \\ S &= \mathrm{e}^{-\mathrm{i}\psi_{\mathrm{sr}}^{+}(p)} S_{\mathrm{sr}} \mathrm{e}^{\mathrm{i}\psi_{\mathrm{sr}}^{-}(p)}. \end{aligned}$$

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$$W^{\pm} = W_{\rm sr}^{\pm} e^{i\psi_{\rm sr}^{\pm}(p)},$$
$$S = e^{-i\psi_{\rm sr}^{+}(p)} S_{\rm sr} e^{i\psi_{\rm sr}^{-}(p)}.$$

Similarly, in the case $\mu > \frac{1}{2}$ one can use the so-called Dollard construction:

$$W_{\rm dol}^{\pm} f := \lim_{t\to\pm\infty} e^{itH} U_{\rm dol}(t) f,$$
$$U_{\rm dol}(t) := e^{-i\int_0^t (p^2/2+V(sp)1_{\{|sp|\geq R_0\}})\,ds}, \quad R_0 > 0,$$
$$S_{\rm dol} := W_{\rm dol}^{+*} W_{\rm dol}^{-}.$$

Analogously, we have

$$W^{\pm} = W_{\rm dol}^{\pm} e^{i\psi_{\rm dol}^{\pm}(p)},$$
$$S = e^{-i\psi_{\rm dol}^{+}(p)} S_{\rm dol} e^{i\psi_{\rm dol}^{-}(p)}.$$

We are motivated to study low-energy classical mechanics by the following

Problem (Basic low-energy scattering problem)

Study asymptotics of the wave and scattering matrices at energy $\lambda = 0$.

To this end we change variables to "blow up" the discontinuity at $\lambda = 0$. This amounts to looking at $\xi = \sqrt{2\lambda}\omega$ as depending on two independent variables $\lambda \geq 0$ and $\omega \in S^{d-1}$.

For any $\omega \in S^{d-1}$, $\lambda \in [0, \infty[$ and x from an appropriate outgoing/incoming region there exists a solution of the problem

$$\begin{cases} \ddot{y}(t) = -\nabla V(y(t)), \\ \lambda = \frac{1}{2}\dot{y}(t)^2 + V(y(t)), \\ y(\pm 1) = x, \\ \omega = \pm \lim_{t\to\pm\infty} y(t)/|y(t)|. \end{cases} \tag{9}$$

One obtains a family $y^{\pm}(t, x, \omega, \lambda)$ of solutions depending smoothly on parameters. All scattering orbits are of this form. Using these solutions one constructs a solution $\phi^{\pm}(x, \omega, \lambda)$ to the eikonal equation

$$\frac{1}{2}\left(\nabla_x \phi^{\pm}(x, \omega, \lambda)\right)^2 + V(x) = \lambda \tag{10}$$

satisfying $\nabla_x \phi^{\pm}(x, \omega, \lambda) = \dot{y}(\pm 1, x, \omega, \lambda)$.

In the quantum case, we replace $\phi^\pm(x,\xi) \to \phi^\pm(x,\omega,\lambda)$ in the definition of modifiers $J^\pm$, which fixes the wave operators $W^\pm$ and hence the scattering operator S. We also improve on the choice of the symbol $a^\pm(x,\xi)$ by assuming that in the incoming/outgoing region it satisfies an appropriate transport equation.
For all $\tau \in L^2(S^{d-1})$ we introduce

$$\begin{aligned}
(J^+(\lambda)\tau)(x) &:= (J^+\mathcal{F}_0(\lambda)^*\tau)(x) \\
&= (2\pi)^{-d/2}\int \mathrm{e}^{\mathrm{i}\phi^+(x,\omega,\lambda)}\tilde{a}^+(x,\omega,\lambda)\tau(\omega)\mathrm{d}\omega; \\
(T^+(\lambda)\tau)(x) &:= (\mathrm{i}(HJ^+ - J^+H_0)\mathcal{F}_0(\lambda)^*\tau)(x) \\
&= (2\pi)^{-d/2}\int \mathrm{e}^{\mathrm{i}\phi^+(x,\omega,\lambda)}\tilde{t}^+(x,\omega,\lambda)\tau(\omega)\mathrm{d}\omega;
\end{aligned}$$

where

$$\begin{aligned}
\tilde{a}^+(x,\omega,\lambda) &:= (2\lambda)^{(d-2)/4}a^+(x,\sqrt{2\lambda}\omega), \\
\tilde{t}(^+x,\omega,\lambda) &= (2\lambda)^{(d-2)/4}t^+(x,\sqrt{2\lambda}\omega).
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\end{aligned}$$

Let Δ_ω denote the Laplace-Beltrami operator on the sphere S^{d-1}. For $n \in \mathbb{R}$ we define the Sobolev spaces on the sphere

$$L^{2,n}(S^{d-1}) := (1-\Delta_\omega)^{-n/2}L^2(S^{d-1}).$$

The following result (stated only for the “plus”-case) is essentially well-known; it includes a formula for the wave matrices.

Theorem

Suppose Condition (A). Let $s > \frac{1}{2}$, $n \geq 0$ *and* $\lambda > 0$. *Then*

$$W^+(\lambda) \quad := \quad J^+(\lambda) + \mathrm{i}R(\lambda - \mathrm{i}0)T^+(\lambda) \tag{11}$$

defines a bounded operator in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,-s-n}(\mathbb{R}^d))$, *which depends strongly continuously on* $\lambda > 0$. *For all* $f \in L^{2,s}(\mathbb{R}^d)$ *and all* $g \in C_c(]0,\infty[)$ *we have*

$$W^+ g(H_0) f = \int_0^\infty g(\lambda) W^+(\lambda) \mathcal{F}_0(\lambda) f \, \mathrm{d}\lambda.$$

If we set

$$w^+(\omega,\lambda) := W^+(\lambda)\delta_\omega,$$

then we obtain a continuous function

$$S^{d-1} \times]0,\infty] \ni (\omega,\lambda) \mapsto w^+(\omega,\lambda) \in L^{2,-p}(\mathbb{R}^d)$$

with $p > d/2$.

The main new results about wave matrices can be summarized as follows.

Theorem

For $s > \frac{1}{2} + \frac{\mu}{4}$ and $n \geq 0$

$$W^+(0) := J^+(0) + \mathrm{i}R(0 - \mathrm{i}0)T^+(0) \tag{12}$$

defines a bounded operator in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,-s-n(1-\mu/2)}(\mathbb{R}^d))$.
If we set

$$w^+(\omega, 0) := W^+(0)\delta_\omega,$$

then we obtain a function in $L^{2,-p}(\mathbb{R}^d)$ with $p > \frac{\mu}{2} + \frac{d}{2} - \frac{d\mu}{4}$ depending continuously on ω.

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If we set

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Theorem

Let $n \geq 0$. We have

$$W^+(0) = \mathrm{s}-\lim_{\lambda \searrow 0} W^+(\lambda)$$

in the sense of a map in $\mathcal{B}(L^{2,-n}(S^{d-1}), L^{2,-\tilde{s}_n}(\mathbb{R}^d))$, where
$\tilde{s}_n > \max\left(\frac{1}{2} + n, \frac{1}{2} + \frac{\mu}{4} + n(1 - \frac{\mu}{2})\right)$.
The function

$$S^{d-1} \times [0, \infty[\ni (\omega, \lambda) \mapsto w^+(\omega, \lambda) \in L^{2,-\tilde{p}}(\mathbb{R}^d)$$

is continuous with $\tilde{p} > \frac{d}{2}$ for $d \geq 2$ and $\tilde{p} > \frac{1}{2} + \frac{\mu}{4}$ for $d = 1$.

The existence of the limits of wave matrices at zero energy is made possible not only by appropriate assumptions on the potentials, but also by the use of appropriate modifiers. Wave matrices $W^{\pm}_{\mathrm{sr}}(\lambda)$ defined by the standard short-range procedure, as well as the Dollard modified wave operators $W^{\pm}_{\mathrm{dol}}(\lambda)$ do not have this property. They differ from our $W^{\pm}(\lambda)$ by a momentum dependent phase factor that has an oscillatory behaviour as $\lambda \searrow 0$.

In particular,

$$W^{\pm}_{\mathrm{sr}}(\lambda) = W^{\pm}(\lambda) \exp\left(\mathrm{i}O(\lambda^{\frac{1}{2}-\frac{1}{\mu}})\right), \quad 1 < \mu < 2; \tag{13}$$

$$W^{\pm}_{\mathrm{dol}}(\lambda) = W^{\pm}(\lambda) \exp\left(\mathrm{i}O(\lambda^{-\frac{1}{2}} \ln \lambda)\right), \quad \mu = 1; \tag{14}$$

$$W^{\pm}_{\mathrm{dol}}(\lambda) = W^{\pm}(\lambda) \exp\left(\mathrm{i}O(\lambda^{\frac{1}{2}-\frac{1}{\mu}})\right), \quad \frac{1}{2} < \mu < 1. \tag{15}$$

Theorem

There exists the strong limit of scattering matrices at zero energy:

$$\lim_{\lambda \searrow 0} \|S(\lambda)\tau - S(0)\tau\| = 0,$$

where $S(0)$ is a unitary operator on $L^2(S^{d-1})$.

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The short-range and Dollard scattering matrices have oscillatory behaviour at zero energy.

Example (Attractive Coulomb potential)

In the case of Coulomb potentials, at least in dimension $d \geq 3$,

$$S(0) = \mathrm{e}^{\mathrm{i}c} P,$$

where $(P\tau)(\omega) = \tau(-\omega)$. Moreover in this case one may compute

$$S_{\mathrm{dol}}(\lambda) = \mathrm{e}^{\mathrm{i}\lambda^{-1/2}\{c_1 \ln \lambda + c_2 + o(\lambda^0)\}}(P + o(\lambda^0)). \tag{16}$$

A recurrent idea of scattering theory is the parallel behavior of classical and quantum systems. One of its manifestations is the relationship between scattering orbits at a given energy and the location of singularities of the scattering matrix.

In the case of positive energies the relationship is simple and well-known.

To describe it note that scattering orbits of positive energy have the deflection angle that goes to zero when the distance of the orbit to the center goes to infinity. In the quantum case this corresponds to the fact that the integral kernel of scattering matrices $S(\lambda, \omega, \omega')$ at positive energies λ are smooth for $\omega \neq \omega'$ and has a singularity at $\omega = \omega'$.

This picture changes at zero energy. Under Condition (A) the deflection angle of zero-energy orbits is not small for orbits far from the center. The angle of deflection is small for small μ and goes to infinity as μ approaches 2. In the case of the potential $-\gamma|x|^{-\mu}$ this can be computed exactly and equals $-\frac{\mu\pi}{2-\mu}$. In particular, for attractive Coulomb potentials it equals $-\pi$, which agrees with the well-known fact that in this case zero-energy orbits are parabolas.

A main new result is a quantum analog of this fact:

Theorem (A)

The integral kernel of the zero-energy scattering matrix $S(\lambda = 0, \omega, \omega')$ *is smooth away from* ω, ω' *satisfying* $\omega \cdot \omega' = \cos \frac{\mu\pi}{2-\mu}$.

A main new result is a quantum analog of this fact:

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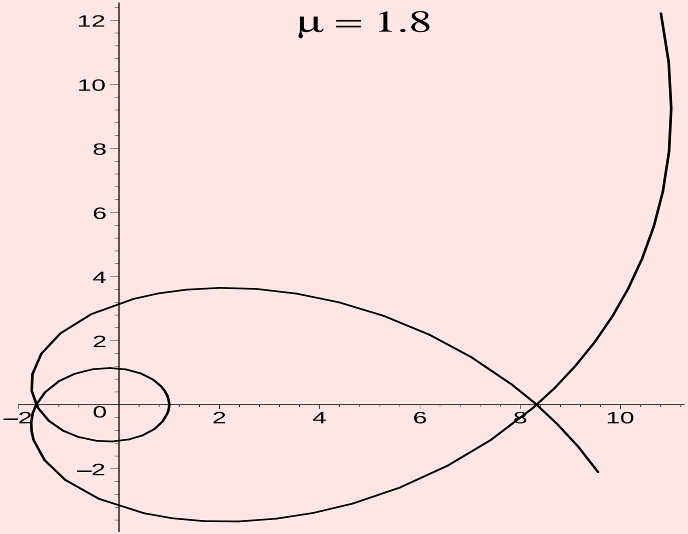
Example (Classical zero energy orbits:)

We look at scattering for the example $V(r) = -\gamma r^{-\mu}$ at energy $\lambda = 0$. For *any* scattering orbit the angle between the outgoing normalized velocity $\omega^+ = \lim_{t\to+\infty} \omega(t)$ and the incoming normalized velocity $\omega^- = -\lim_{t\to-\infty} \omega(t)$ is given by $\frac{\mu}{2-\mu}\pi$. The fact that this angle is independent of the orbit may be seen independently by invoking the scaling and rotational symmetry of Newton's equation; thus there is essentially only ONE scattering orbit at $\lambda = 0$ (see the illustration below for the case $\mu = 1.8$). The implicit equation for this orbit is

$$\frac{2}{1 + \cos\left((2-\mu)\triangle\theta\right)} = r^{2-\mu},$$

where $\triangle\theta$ denotes the angular increment measured from the turning point.


μ = 1.8
12
10
8
6
4
2
−2
−2
0
2
4
6
8
10

Consider the following stronger condition:

Condition (B)

The potential V is spherically symmetric and

$$V(r) = -\gamma r^{-\mu} + O(r^{-1-\mu/2-\epsilon'}),$$

where $\mu \in (0,2)$ and $\gamma, \epsilon' > 0$.
More generally, for all derivatives

$$\frac{\mathrm{d}^k}{\mathrm{d}r^k}\left(V(r) + \gamma r^{-\mu}\right) = O(r^{-1-k-\mu/2-\epsilon'}).$$

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Theorem (B)

Under Condition (B) there exists $c \in R$ *and a compact operator K such that*

$$S(0) = \mathrm{e}^{\mathrm{i}c}\mathrm{e}^{-\ i\frac{\mu\pi}{2-\mu}\sqrt{L^2}} + K,$$

where L^2 *is (minus) the Laplace-Beltrami operator on* S^{d-1}

$$-\Delta_\omega = L^2 = \sum_{1\leq i<j\leq d} L_{ij}^2;\ \mathrm{i}L_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}.$$

1: Classically, the zero-energy orbit for the potential $V_1(r) = -\gamma r^{-\mu}$ is given by the equation

$$\frac{2}{1+\cos\left((2-\mu)\theta\right)} = r^{2-\mu}, \tag{17}$$

where θ denotes the angular increment measured from the turning point.
2: Introduce an incoming scattering phase:

$$\phi_{lead} = -\sqrt{2\gamma}\,\frac{\cos(1-\mu/2)\theta}{1-\mu/2}\,r^{1-\mu/2},$$
$$\cos\theta = -\hat{x}\cdot\omega'.$$

Here $\theta \in [0,\pi)$. The phase solves the eikonal equation for $\lambda = 0$

$$\left|\nabla_x \phi_{lead}\right|^2/2 + V_1 = \lambda(=0).$$

3: Rewrite the classical equations of motion for the zero-energy orbit in terms of a reduced system: We introduce

$$\begin{cases} b = \hat{x} \cdot \frac{\xi}{g};\ g = \sqrt{-2V_1} \\ \bar{c} = (I - |\hat{x}\rangle\langle\hat{x}|)\frac{\xi}{g} \end{cases} .$$

Here ξ signifies momentum. We also introduce a "new time" τ by $\frac{d\tau}{dt} = g/r$. The reduced system is

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{x} = \bar{c} \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\bar{c} = -(1-\frac{\mu}{2})b\bar{c} - c^2\hat{x} \\ \frac{\mathrm{d}}{\mathrm{d}\tau}b = (1-\frac{\mu}{2})\bar{c}^2 \end{cases} . \tag{18}$$

The quantity $k = b^2 + \bar{c}^2$ is conserved for the dynamics (18) on the reduced space $\mathcal{T}^* := T^*(S^{d-1}) \times \mathbb{R}$; the physical relevant value is $k = 1$. (Here we consider $(\hat{x}, \bar{c})$ as belonging to the cotangent bundle $T^*(S^{d-1})$ of S^{d-1}.)

The observable b increases from $-\sqrt{k}$ to $\sqrt{k}$, and $b = 0$ corresponds to the turning point. (The fixed points are given by $\bar{c} = 0$.)

4: Introduce $L^{2,s} = \langle x \rangle^{-s} L^2(\mathbb{R}^d_x)$ and $L^{2,-\infty} = \cup_{s\in\mathbb{R}} L^{2,s}$. Microlocalize in $\mathcal{T}^*$: The "scattering wave front set" $WF^s(u)$ of a distribution $u \in L^{2,-\infty}$ is the subset of $\mathcal{T}^*$ given by the condition

$$\begin{aligned}
& z_0 = (\omega_0, \bar{c}_0, b_0) = (\omega_0, b_0\omega_0 + \bar{c}_0) = (\omega_0, \eta_0) \\
& \notin WF^s(u) \\
& \Leftrightarrow \\
& \exists \text{ neighborhoods } \mathcal{N}_{\omega_0} \ni \omega_0, \ \mathcal{N}_{\eta_0} \ni \eta_0 \\
& \forall \chi_{\omega_0} \in C^\infty(\mathcal{N}_{\omega_0}), \ \chi_{\eta_0} \in C_c^\infty(\mathcal{N}_{\eta_0}) : \\
& \mathrm{Op}^{\mathrm{w}}(\chi_{\omega_0}(\hat{x})\chi_{\eta_0}(b\hat{x} + \bar{c})F(r > 1))u \in L^{2,s}.
\end{aligned} \tag{19}$$

Notice that this quantization is defined by the substitution $b\hat{x} + \bar{c} \to \xi/g$.

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Notice that this quantization is defined by the substitution $b\hat{x} + \bar{c} \to \xi/g$.

Define for $z \in \mathcal{T}^*$

$$L^{2,s}_z = \{u \in L^{2,-\infty} | z \notin WF^s(u)\}.$$

The (maximal) solution of the system (18) that passes z at $\tau = 0$ is denoted by $\gamma(\tau, z)$.

Proposition (Propagation of scattering regularities)

Suppose $u, v \in L^{2,-\infty}$, $Hu = v$, $s \in \mathbb{R}$ *and* $z \notin WF^s(u)$. *Define*

$$\tau_0 = \sup\{\tau > 0 | u \in L^{2,s}_{\gamma(\tilde{\tau},z)} \text{ for all } \tilde{\tau} \in [0,\tau]\}.$$

Then either $\tau_0 = \infty$, *or* τ_0 *is finite and*

$$\gamma(\tau_0, z) \in WF^{s+1+\frac{\mu}{2}}(v).$$

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Very rough idea of proof of Theorem (A):

Apply the above proposition to

$$u = R(0 + \mathrm{i}0)v; \; v \approx e^{\mathrm{i}\phi_{lead}}.$$